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ON MEDIAL BCI-ALGEBRAS

Abstract. *In this paper we introduce so-called medial BCI-algebras. These BCI-algebras form a variety which is defined by the independent axioms system: (3), (4) and (7). All congruences of these BCI-algebras are uniquely determined by their subalgebras. We give also some charactrizations of quotient medial BCI-algebras. Direct products are described, too.*

1. Introduction

Let G be non-empty set with a binary operation $*$ and suppose there is a constant 0. Then $(G,*,0)$ is called *a BCI-algebra* if the following conditions hold:

(1) $((x*y)*(x*z))*(z*y)=0$,

(2) $x*y=y*x=0$ *implies* $x=y$,

(3) $x*0=x$.

As it is well known (see [1] or [2]) *in this algebra we have*

(4) $x*x=0$,

(5) $(x*y)*z=(x*z)*y$.

If a BCI-algebra $(G,*,0)$ satisfies

(6) $0*x=0$,

then it is called *a BCK-algebra*.

A BCI-algebra is called *a medial BCI-algebra,* if it satisfies *the entropy law,* i.e. if

(7) $(x*y)*(z*u)=(x*z)*(y*u)$

for all $x, y, z, u \in G$.

Observe that (7) and (3) imply (5). Moreover, *a medial BCK-algebra has only one element.* Indeed, by (3), (4) and (6) we have

$$x=(x*0)*0=(x*0)*(x*x)=0*(0*x)=0 \text{ for all } x\in G.$$

Now, we prove the fundamental properties of these BCI-algebras.

Theorem 1.1. *Medial BCI-algebras form a variety. An equentional base is given by the independent axioms system: (3), (4) and (7).*

Proof. A medial BCI-algebra satisfies (3), (4) and (7). On the other hand, if an algebra $(G,*,0)$ of type $(2,0)$ satisfies these conditions, then

$$x*y=(x*y)*0=(x*y)*(x*x)=(x*x)*(y*x)=0*(y*x),$$

which implies (1), because

$$((x*y)*(x*z))*(z*y)=((x*x)*(y*z))*(z*y)=(0*(y*z))*(z*y)=$$
$$=(z*y)*(z*y)=0.$$

To prove (4), assume that $x*y=y*x=0$. Then

$$x=(x*0)*0=(x*0)*(x*y)=(x*x)*(0*y)=0*(0*y)=(y*y)*(0*y)=$$
$$=(y*0)*(y*y)=(y*0)*0=y,$$

i.e. $x=y$. Therefore $(G,*,0)$ is a medial BCI-algebra. Hence the class of all medial BCI-algebras is a variety determined by axioms (3), (4) and (7).

To prove that these axioms form an independent system we consider algebras given by Table 1, 2 and 3.

*	0	1
0	0	0
1	0	0

Table 1

*	0	1
0	0	0
1	1	1

Table 2

*	0	1
0	0	0
1	1	0

Table 3

The algebra defined by Table 1 satisfies (4) and (7), but $1*0\neq 1$. Hence (3) is independent. If the algebra is given by Table 2, then (3) and (7) holds, but $1*1=1$, i.e. (4) is not true. The algebra defined by Table 3 is a BCK-algebra, but it is not medial. This completes our proof.

As an immediate consequence we obtain

Corollary 1.1. *Every medial BCI-algebra satisfies the following identities:*

(8) $x*y=0*(y*x)$,

(9) $x*(x*y)=y$,

(10) $0*(0*x)=x$.

Proof. The first condition follows from the above proof. If we put $y=0$ in (8), we obtain (10). This implies $x*(x*y)=(x*0)*(x*y)=(x*x)*(0*y)=$

$=0*(0*y)=y$, which concludes the proof.

Using (8) we obtain

Corollary 1.2. *Let $(G,*,0)$ be a medial BCI-algebra. If $(A,*,0)$ is a subalgebra of $(G,*,0)$, then $x*y\in A$ if and only if $y*x\in A$.*

Now we give simple examples of medial BCI-algebras.

Direct computation shows that quasigroups defined by Table 4 and by Table 5 are medial BCI-algebras.

*	0	1	2	3
0	0	1	3	2
1	1	0	2	3
2	2	3	0	1
3	3	2	1	0

Table 4

*	0	1	2	3	4	5
0	0	2	1	4	3	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	3	1	5	0	4	2
4	4	5	2	3	0	1
5	5	3	4	1	2	0

Table 5

Also *every Boolean group is a medial BCI-algebra.* Generally, *if $(G,\cdot,1)$ is a commutative group and if x^{-1} is the inverse of x in this group, then $(G,*,1)$, where $x*y=x\cdot y^{-1}$, is a medial BCI-algebra.*

From these examples follows that *for every natural n there exists a medial BCI-algebra which has n elements.*

2. Quotient algebras and homomorphisms

A non – empty subset A of a BCI-algebra $(G,*,0)$ is called *an ideal,* if $0\in A$ and if x, $y*x\in A$ implies $y\in A$.

Lemma 2.1. *A subalgebra of a medial BCI-algebra is an ideal.*

Proof. Let $(A,*,0)$ be a subalgebra of a medial BCI-algebra $(G,*,0)$. If $x\in A$ and $y*x\in A$, then x, $x*y\in A$ (Corollary 1.2) and $y=x*(x*y)\in A$ by (9), i.e. A is and ideal.

Note that the converse is not true, i.e. *there exists a medial BCI-algebra and its ideal which is not a subalgebra.* For example, in a medial BCI-algebra $(R,*,0)$ of all real numbers with the operation $x*y=x-y$ the set of all non – negative real numbers is an ideal, but it is not closed under this operation.

Theorem 2.1. *Let $(G,*,0)$ be a medial BCI-algebra. The relation $\sim$ is a congruence in G if and only if there exists a subalgebra A of G such that $x\sim y\Leftrightarrow x*y\in A$.*

Proof. Let A be a subalgebra of G and let $\sim$ be defined as follows:

$$x\sim y \Leftrightarrow x*y\in A.$$

Then $\sim$ is a congruence on G. Indeed, since $0\in A$, it is obvious that $\sim$ is reflexive, i.e. $x\sim x$. By Corollary 1.2, if $x\sim y$, then $y\sim x$, which shows that $\sim$ is symmetric. To prove that this relation is transitive, let $x\sim y$ and $y\sim z$. Then $x*y, y*z\in A$. Since A is a subalgebra, then by (9) and (5) we have $z*x=(y*(y*z))*x=(y*x)*(y*z)\in A$. Hence $x\sim z$. The Substitution Property follows from (7).

On the other hand, let $\sim$ be a congruence on $(G,*,0)$ and let

$$A=\{x\in G: x\sim 0\}.$$

It is clear that $0\in A$. If $x,y\in A$, then $x\sim 0$ and $y\sim 0$. This implies $x*y\sim 0$, i.e. $x*y\in A$. Hence A is a subalgebra of a BCI-algebra $(G,*,0)$. Now we prove that $x\sim y\Leftrightarrow x*y\in A$. If $x\sim y$, then $x*y\sim y*=0$ because $\sim$ is reflexive. Therefore $x*y\in A$. Conversely, if $x*y\in A$, then $x*y\sim 0$ and $x\sim x$. Hence $x*(x*y)\sim x*0$, i.e. $y\sim x$. This finish the proof.

From the above result follows that G is decomposed by the equivalence relation $\sim$ into disjoint classes. The class containing x is denoted by C_x. Let $x*A=\{x*a: a\in A\}$ be *the left coset of A in G* and let $A*x=\{a*x: a\in A\}$ be *the right coset.* We shall consider the structure of $G/_\sim$.

Theorem 2.2. *If the congruence $\sim$ is defined by a subalgebra A, then*

(i) $C_a=C_0=A$ *for all* $a\in A$,

(ii) $C_x=x*A$,

(iii) $x*A=A*(0*x)$,

(iv) $0*(x*A)=A*x$,

(v) $x*A=y*A$ *if and only if* $x*y\in A$.

Proof. (i) Since $\sim$ is an equivalence relation, then $C_x=C_y$ or $C_x\cap C_y=\emptyset$. If $a\in A$, then $a\sim 0$, i.e. $a\in C_a\cap C_0$. Hence $C_a=C_0$. From the proof of Theorem 2.1 follows $C_0=A$.

(ii) If $y\in C_x$, then $x\sim y$. Hence $x*y\in A$, i.e. $x*y=a$ for some $a\in A$. This implies $y=x*(x*y)=x*a\in x*A$, i.e. $C_x\subset x*A$. Conversely, if $y\in x*A$, then $y=x*a$ for some $a\in A$. Therefore

$$(y*x)*(0*a)=(y*0)*(x*a)=y*(x*a)=0\in A.$$

Since $0*a\in A$ and A is an ideal (Lemma 2.1), then $y*x\in A$. Hence $y\sim x$, i.e. $x*A\subset C_x$, which completes the proof.

(iii) If $y\in x*A$, then $y=x*a$ for some $a\in A$. Hence

$$y=x*a=0*(a*x)=(0*0)*(a*x)=(0*a)*(0*)\in A*(0*x)$$

by (8) and (7). Thus $x*A\subset A*(0*x)$. If $y\in A*(0*x)$, then $y=a*(0*x)$ for some $a \in A$. Hence, by (5) and (10), we have

$$(y*x)*a = (y*a)*x = (y*a)*(0*(0*x)) =$$
$$= (y*0)*(a*(0*x)) = y*(a*(0*x)) = 0 \in A.$$

Since A is an ideal, then $y*x \in A$, i.e. $y \sim x$. Thus $A*(0*x)$ is contained in $x*A$.
(iv) is a simple consequence of (8).
(v) follows from (ii).

In the sequel the set $^{G}/_{\sim}$ will be denoted by $^{G}/_{A}$. The operation in $^{G}/_{A}$ is defined by the formula

$$(x*A)*(y*A) = (x*y)*A.$$

This formula is independent of the choice of elements x and y. It is clear that $^{G}/_{A}$ with the above operation is a medial BCI-algebra.

As a simple consequence we have
Corollary 2.1. *If* $(G, *, 0)$ is a medial BCI-algebra and if $^{G}/_{A}$ has only two elements, then $x*A = A*x$ *for all* $x \in G$.

Now we prove the classical result due to Lagrange.
Theorem 2.3. *If $(G, *, 0)$ is a finite medial BCI-algebra, then for every subalgebra A of G we have $Card(G) = Card(A) \cdot Card(^{G}/_{A})$.*
Proof. First we prove that $Card(A) = Card(x*A)$ for all $x \in G$. Let $f: A \to x*A$ be defined as $f(a) = x*a$. If $f(a) = f(b)$, then $x*a = x*b$, which implies

$$0 = (x*a)*(x*b) = (x*(x*b))*a = b*a.$$

In the same manner we prove $0 = a*b$. Hence $a = b$ by (2), i.e. f is one – to – one. Similarly we can prove that $Card(A) = Card(A*x)$.

Suppose now that G has n elements and A has k elements. We can decompose the set G into a union of a finite number of disjoint left cosets of A:

$$G = x_1*A \cup x_2*A \cup x_3*A \cup \ldots \cup x_t*a.$$

Since each of the t cosets in the above decomposition has k elements, the set G itself must have tk elements. Hence $n = tk$, which completes our proof.

From the above result, we are able to conclude that *if a medial BCI-algebra has p elements, where p is prime, then it has no non-trivial subalgebras.*

Direct computation shows that if $f: G_1 \to G_2$ is a homomorphism of medial BCI-algebras G_1 and G_2, then $\ker f = \{x \in G_1 : f(x) = 0\}$ is a subalgebra of G_1. A homomorphism f is one – to – one if and only if ker f has only one element. Moreover, if A is a subalgebra of G_1, then f(A) is a subalgebra of G_2. Conversely, if B is a subalgebra of G_2, then $f^{-1}(B)$ is a subalgebra of G_1.

If A is a subalgebra of G_1, then the mapping $\Phi: G_1 \to {}^{G_1}/_A$ defined by $\Phi(x)=x*A$ is a homomorphism of G_1 onto the quotient BCI-algebra ${}^{G_1}/_A$ and the kernel of Φ is A.

Lemma 2.2. *Let G_1 and G_2 be medial BCI-algebras and let f be a homomorphism from G_1 onto G_2. If A is a subalgebra of G_1 such that ker $f \subset A$, then $A=f^{-1}(f(A))$.*

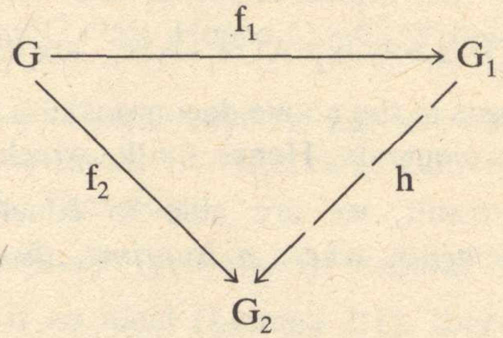
Proof. Let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$, i.e. $f(x)=f(a)$ for some $a \in A$. Hence $f(x*a)=f(x)*f(a)=0$, $x*a \in$ ker $f \subset A$. Since a subalgebra A is an ideal, then $x \in A$, i.e. $f^{-1}(f(A)) \subset A$. But the reverse inclusion always holds and so the desired equality follows.

Theorem 2.4. *If G_1 and G_2 are medial BCI-algebras and if f is a homomorphism from G_1 onto G_2, then there exists a one – to – one correspondence between the subalgebras A of G_1 such that ker $f \subset A$ and the set of all subalgebras B of G_2; specifically B is given by $B=f(A)$.*

Proof. Starting with an arbitrary subalgebra B of G_2, we must produce some subalgebra A of G_1, with ker $f \subset A$, such that $f(A)=B$. The set $A=f^{-1}(B)$ certainly meets these requirements. Indeed, $f^{-1}(B)$ is a subalgebra of G_2 and ker $f=f^{-1}(0)$ is contained in $f^{-1}(B)$. Since f is an onto mapping, then $f(f^{-1}(B))=B$.

To finish off the proof, we argue that the correspondence in question is one – to – one. Suppose then that A_1 and A_2 are subalgebras of G_1, with ker $f \subset A_1$, ker $f \subset A_2$ and $f(A_1)=f(A_2)$. Using the lemma above, we get $A_1=f^{-1}(f(A_1))=f^{-1}(f(A_2))=A_2$, as desired.

Theorem 2.5. *Let G, G_1, G_2 be medial BCI-algebras and let f_1 and f_2 be homomorphisms from G onto G_1 and G_2, respectively. If ker $f_1 \in$ ker f_2, then there exists a unique epimorphism $h: G_1 \to G_2$ such that the following diagram*



is commutative. Moreover,f is an isomorphism if ker $f_1=$ker f_2.

Proof. For any element $f_1(x) \in G_1$ we define $h: G_1 \to G_2$ by $h(f_1(x))=f_2(x)$. If $f_1(x)=f_1(y)$, then $0=f_1(x)*f_1(y)=f_1(x*y)$. Hence $x*y \in$ ker $f_1 \subset$ ker f_2. The implication is that $f_2(x)=f_2(x)*0=f_2(x)*f_2(x*y)=f_2(x*(x*y))=f_2(y)$, i.e. h is a well – defined mapping. Direct computation shows that h is an epimorphism and $f_2=h\circ f_1$. It remains to establish the uniqueness of h. Suppose also that $f_2=g\circ f_1$ for some other function $g:G_1\to G_2$. Then $h(f_1(x))=f_2(x)= =(g\circ f_1)(x)=g(f_1(x))$ for all $f_1(x)\in G_1$, and so h=g.

If ker f_1=ker f_2, then h is an isomorphism. Indeed, if $h(f_1(x))=h(f_1(y))$, then $f_2(x)=f_2(y)$ and $0=f_2(x)*f_2(y)=f(x*y)$, i.e. $x*y\in$ ker f_2=ker f_1. Since ker f_1 is a subalgebra, then $x*y$, $y*x\in$ ker f_1, $0=f_1(x*y)=f_1(x)*f_1(y)$ and $0=f_1(y)*f_1(x)$, which implies (by (2)) $f_1(x)=f_1(y)$. Thus h is a one – to – one mapping.

We finish this section with the following corollaries.

Corollary 2.2. *Let G and H be medial BCI-algebras and let A be a subalgebra of G. If $f:G\to H$ is an epimorphism such that $A\subset$ ker f, then there exists a unique epimorphism $h:{}^{G}/_{A}\to H$ satisfying $f=h\circ g$, where $g:G\to {}^{G}/_{A}$ is a natural homomorphism. Moreover, if A=ker f, then ${}^{G}/_{A}$ and H are isomorphic.*

Corollary 2.3. *Let G and H be medial BCI-algebras and let A be a subalgebra of G. If $f:G\to H$ is an epimorphism such that ker $f\subset A$, then ${}^{G}/_{A}$ and ${}^{H}/_{f(A)}$ are isomorphic.*

Proof. Since f is a homomorphism, then f(A) is a subalgebra of H. Putting $g(x)=f(x)*f(A)$ we obtain an epimorphism $g:G\to {}^{H}/_{f(A)}$ such that ker g= $=f^{-1}(f(A))$. The hypothesis that ker $f\to A$ allows us to appeal to Lemma 2.2, from which $A=f^{-1}(f(A))$, i.e. ker g=A, which completes our proof by Corollary 2.2.

Corollary 2.4. *Let G and H be medial BCI-algebras and let B be a subalgebra of H. If $f:G\to H$ is an epimorphism, then ${}^{G}/_{f^{-1}(B)}$ and ${}^{H}/_{B}$ are isomorphic.*

Corollary 2.5. *If $A\subset B$ are subalgebras of a medial BCI-algebra G, then $({}^{G}/_{A})/({}^{B}/_{A})$ and ${}^{G}/_{B}$ are isomorphic.*

Corollary 2.6. *If A and B are subalgebra of a medial BCI-algebra G, then ${}^{A}/_{(A\cap B)}$ and ${}^{(A*B)}/_{B}$ are isomorphic.*

Proof. Using (4) and (7) we prove that $A*B=\{a*b:a\in A,b\in B\}$ is a subalgebra of G. Moreover, B is a subalgebra of A*B by (10). Then the mapping f defined by $f(a)=a*B$ for all $a\in A$ is an epimorphism from A onto ${}^{(A*B)}/_{B}$. Direct computation shows that ker $f=A\cap B$. Hence ${}^{A}/_{(A\cap B)}$ and ${}^{(A*B)}/_{B}$ are isomorphic by Corollary 2.2.

3. Direct products

If (A,∗,0) and (B,∗,0) are subalgebras of a medial BCI-algebra (G,∗,0), then (G,∗,0) is called *the direct product* of (A,∗,0) and (B,∗,0), if $G=A*B$ and $A\cap B=\{0\}$. The direct product of (A,∗,0) and (B,∗,0) is denoted by $A\circledast B$.

Remark that some medial BCI-algebras cannot be expressed as the direct product of two non – trivial subalgebras. As an example of this situation, we mention the BCI-algebra defined by Table 4.

In the sequel we prove some criteria for a medial BCI-algebra to be a direct product of this subalgebras.

Theorem 3.1. *Let $(A,*,0)$ and $(B,*,0)$ be subalgebras of a medial BCI-algebra $(G,*,0)$. Then $(G,*,0)$ is a direct product of $(A,*,0)$ if and only if each element $x\in G$ can be uniquely expressed in the form $x=a*b$, where $a\in A$ and $b\in B$.*

Proof. If $G=A\circledast B$, then $G=A*B$, i.e. for any $x\in G$ there exists $a\in A$ and $b\in B$ such that $x=a*b$. We prove that this representation is uniquely determined by x. If $x=a*b=c*d$ for some $a,c\in A$ and $b,d\in B$, then

$$(a*c)*(b*d)=(a*b)*(c*d)=x*x=0\in A\cap B\subset B.$$

Since $(B,*,0)$ is a subalgebra, then elements $b*d$, $a*c$, $d*b$ and $c*a$ are in B. In the same manner, we prove that these elements are also in A. Hence these elements are in $A\cap B$. Since $A\cap B=\{0\}$, then $a*c=c*a=b*d=$ $=d*b=0$, which implies $a=c$ and $b=d$. This concludes the first part of our proof.

If each $x\in G$ has an unique representation $x=a*b$, where $a\in A$ and $b\in B$, then $G=A*B$. To prove $A\cap B=\{0\}$, observe that if $x\in A\cap B$, then $x=0*(0*x)$ (by (10)) and $x=x*0$. The uniqueness of representation implies $x=0$. Hence $A\cap B=\{0\}$.

Theorem 3.2. *If A and B are subalgebras of a medial BCI-algebra G such that $G=A\circledast B$, then $G/_A$ and B (also $G/_B$ and A) are isomorphic.*

Proof. Consider the mapping $f:G\to B$ defined as follows:

$$\text{if } x=a*b, \text{ with } a\in A \text{ and } b\in B, \text{ then } f(x)=b.$$

It is to be noted that the uniqueness of this representation for x assume that f is well-defined. We also see at once that f carries G onto B. For given two elements $x=a*b$ and $y=c*d$ of G, the product

$$x*y=(a*b)*(c*d)=(a*c)*(b*d),$$

where the last equality makes use of (7). This entails that

$$f(x*y)=b*d=f(x)*f(y),$$

which shows that f is a homomorphism. Since ker $f=A$, then $G/_A$ B are isomorphic by Corollary 2.2. In the same way, the function g defined by $g(x)=a$ gives rise to the isomorphism from $G/_B$ onto A.

Using the above result and Theorem 2.3, we obtain

Corollary 3.1. *Let G be a finite medial BCI-algebra and let A and B be* *subalgebras of G such that $A \cap B = \{0\}$. Then $G = A \circledast B$ if and only if $Card(G) = Card(A) \cdot Card(B)$.*

Let $(G_1, *, 0_1)$ and $(G_2, \bullet, 0_2)$ be two ,medial BCI-algebras. Direct computation shows that the Cartesian product $G_1 \times G_2$ with the operation

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1 * x_2, y_1 \bullet y_2)$$

is a medial BCI-algebra. This BCI-algebra is denoted by $G_1 \otimes G_2$.
It is clear that

$$H_1 = \{(x, 0_2) : x \in G_1\}$$

and

$$H_2 = \{(0_1, y) : y \in G_2\}$$

are subalgebras of $G_1 \otimes G_2$ which are isomorphic copies of G_1 and G_2, respectively. Moreover, $H_1 * H_2$ and $G_1 \times G_2$ are equal as sets and $H_1 \cap H_2 = \{(0_1, 0_2)\}$. Hence $G_1 \otimes G_2$ is a direct product of H_1 and H_2. On the other hand, if $G = A \circledast B$, then every element $x \in G$ is uniquely expressible in the form $x = a * b$ with $a \in A$, $b \in B$. The function $f(x) = f(a * b) = (a, b)$ is well – defined and carries G onto $A \times B$. If $x = a * b$ and $y = c * d$ be two members of G, then $x * y = (a * c) * (b * d)$ (by (7)) and

$$f(x * y) = (a * c, b * d) = (a, b) \otimes (c, d) = f(x) \otimes f(y),$$

which proves that f is a homorophism. Since ker $f = \{0\}$, then f is an isomorphism. Altogether, we have proved the following theorem.

Theorem 3.3. *If G is a medial BCI-algebra with subalgebras A and B such that $G = A \otimes B$, then G and $A \otimes B$ are isomorphic.*

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STRESZCZENIE

W tej pracy wprowadzamy tzw. medialne BCI-algebry. Te BCI-algebry tworzą rozmaitość, która jest zdefiniowana przez niezależny układ aksjomatów: $x0 = x$, $xx = 0$ i $(xy)(zu) = (xz)(yu)$. Wszystkie kongruencje tych BCI-algebr są jednoznacznie wyznaczone przez podalgebry. Podajemy także pewne charakteryzacje ilorazowych medialnych BCI-algebr. Sumy proste są także opisane.