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ON THE LATTICE OF TOLERANCES FOR A FINITE CHAIN

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Abstract

We provide some description of the lattice of tolerances for a finite chain, pointing to the skeleton tolerance as a special element of this lattice. In particular, we prove that the lattice of all glued tolerances of an n-element chain is isomorphic to the lattice of all tolerances of an $n-1$-element chain nad at the same time is a principal filter of the lattice of an n-element chain.

1. Preliminary

Tolerances, introduced in the sixties of the last century, as a natural generalization of a notion of congruence, have become a very important and powerful tool of a universal algebra. They have been thoroughly investigated since then and some interesting results have been provided, in particular as concerns tolerance relations for lattices.

A *tolerance relation* of a lattice L is a reflexive and symmetric relation compatible with the operations of L. Equivalently, a tolerance of L is the image of a congruence by a surjective lattice homomorphism onto L (Czédli and Grätzer [5]).

Let L be a lattice. All tolerances of the lattice L, ordered by inclusion, form an algebraic lattice denoted by $\mathrm{Tol}(L)$ (see [2]). It is clear that every congruence of a lattice L is a tolerance on L. Therefore, the lattice of all congruences on L, which will be denoted by $\mathrm{Con}(L)$, is a subposet of $\mathrm{Tol}(L)$, but it is not necessarily its sublattice.

In [3], the authors provided some description of the tolerance lattice for a finite chain. Our main goal is to simplify and deepen their characterization.

Let L be a lattice, $T \in \mathrm{Tol}(L)$ and $X \subseteq L$, $X \neq \emptyset$. If every two elements of X are in the relation T, then we call X a *preblock* of T. *Blocks* are maximal preblocks (with respect to inclusion). It is easy to observe that in the case when T is a congruence, blocks coincide with congruence classes of

T, which means that they are pairwise disjoint. The situation is different if a tolerance T is not a congruence, since then some of its blocks overlap.

If α and β are blocks of $T \in \mathrm{Tol}(L)$, then $\{a \vee b \mid a \in \alpha, b \in \beta\}$ and $\{a \wedge b \mid a \in \alpha, b \in \beta\}$ are preblocks of T. Czédli in [4] it is proved that blocks containing these preblocks are uniquely determined and they are, respectively, the join and the meet of blocks α, β in the lattice of all blocks of T (see also Grätzer and Wenzel [8]). This lattice, denoted by L/T, is called *the factor lattice of L modulo T*.

Any tolerance $T \in \mathrm{Tol}(L)$ of a finite lattice L can be represented by the system of its blocks. Blocks of a tolerance $T \in \mathrm{Tol}(L)$ for a finite lattice L are intervals of L ([1]).

Applying some results by Czedli and Klukovits [7] we gave in [9] a simple characterization of the collection of tolerance blocks in the case of a finite chain:

Lemma 1. *(1) A collection $\mathcal{C}$ of subsets of the chain $L_n = \langle\{0, \dots, n-1\}, \leq\rangle$ is the set of all blocks of some tolerance of L iff $\mathcal{C}$ is of the form $\{\alpha_i = [n_i, m_i] \colon i = 1, \dots, k\}$ for some $1 \leq k \leq n-1$, where $n_1 = 0$, $m_k = n-1$ and $n_i < n_{i+1} \leq m_i + 1$ and $m_i < m_{i+1}$ for all $i = 1, \dots, k$.*

(2) A collection $\mathcal{C}$ of subsets of the chain $L_n = \langle\{0, \dots, n-1\}, \leq\rangle$ is the set of all blocks of some congruence of L iff $\mathcal{C}$ is of the form $\{\alpha_i = [n_i, m_i] \colon i = 1, \dots, k\}$ for some $1 \leq k \leq n-1$, where $n_1 = 0$, $m_k = n-1$ and $n_i < n_{i+1} = m_i + 1$ and $m_i < m_{i+1}$ for all $i = 1, \dots, k$.

A tolerance T of a lattice L is called *glued*, see [11], if its transitive closure is the total relation L^2. All glued tolerances of L form a sublattice of $\mathrm{Tol}(L)$ denoted by $\mathrm{Glu}(L)$. The zero of $\mathrm{Glu}(L)$ – the (unique) smallest glued tolerance of L is called the *skeleton tolerance* of L, and it is denoted by $\Sigma(L)$. It is known from Bandelt [1] that the skeleton tolerance of a finite lattice L is generated by the set of all its prime quotients, i.e., pairs $(a, b) \in L^2$ such that $a \prec b$. The factor lattice $L/\Sigma(L)$ is called the *skeleton* $S(L)$ of L. Note that $|S(L)| < |L|$ if $|L| > 1$ by Czédli, Grygiel, and Grygiel [6].

By definition of glued tolerances and Lemma 1, we can observe that

Corollary 1 ([9])**.** *A collection $\mathcal{C}$ of subsets of the chain $L_n = \langle\{0, \dots, n-1\}, \leq\rangle$ is the set of all blocks of some glued tolerance of L iff $\mathcal{C}$ is of the form $\{\alpha_i = [n_i, m_i] \colon i = 1, \dots, k\}$ for some $1 \leq k \leq n-1$, where $n_1 = 0$, $m_k = n-1$ and $n_i < n_{i+1} \leq m_i < m_{i+1}$ for all $i = 1, \dots, k$.*

In [10], the authors introduced the fitting relation $\sqsubseteq$ on the set $\mathrm{Tol}(L)$ that can be defined as follows: $T \sqsubseteq S$ iff every block of S is the union of blocks of T included in it. They proved that the relation is a partial order on $\mathrm{Tol}(L)$ and if $T \sqsubseteq S$, then S/T is a tolerance on L/T. It is easy to observe that $T \sqsubseteq S$ implies $T \subseteq S$ for any $T, S \in \mathrm{Tol}(L)$ but not conversely. The inverse implication is not true even for finite chains, as we can see in Example 1. The authors proved that these two orderings coincide for distributive lattices only in the case of Boolean algebras.

Example 1. *Let* $\{[0,2],[2,4]\}$ *be the family of blocks of* T *and* $\{[0,2],[1,4]\}$ *be the family of blocks of* S*. Then, according to Lemma 1,* $T, S \in \mathrm{Tol}(L_5)$ *and* $T \subseteq S$*. However, it is not true that* $T \sqsubseteq S$*.*

Furthermore,

Theorem 1 ([10])**.** *For every finite lattice* L *and every* $T \in \mathrm{Tol}(L)$ *we have* $\mathrm{Tol}(L/T) \cong [T)_{\sqsubseteq}$*, where* $[T)_{\sqsubseteq} = \{S \in \mathrm{Tol}(L) \colon T \sqsubseteq S\}$.

2. Main results

Let $L_n = \langle \{0, \ldots, n-1\}, \leq \rangle$ be a chain of the length n and let us denote $\Sigma_n = \Sigma(L_n)$. Moreover, let $[\Sigma_n)$ denote the principal filter generated in $\mathrm{Tol}(L_n)$ by Σ_n. Then,

Theorem 2. *For every* $n > 1$*,*

$$[\Sigma_n) \cong \mathrm{Tol}(L_{n-1}).$$

Proof. Let us observe that $\mathcal{C} = \{[i, i+1] \colon i = 0, \ldots, n-2\}$ is the family of blocks of Σ_n. Then, by definition and Corollary 1, we conclude that $\Sigma_n \sqsubseteq T$ for every $T \in [\Sigma_n)$ and hence $[\Sigma_n)_{\sqsubseteq} = [\Sigma_n)$. Therefore, according to Theorem 1, we get $\mathrm{Tol}(L_n/\Sigma_n) \cong [\Sigma_n)$.

On the other hand, it is easy to notice that $\mathrm{Tol}(L_n/\Sigma_n) \cong L_{n-1}$, so we obtain the thesis. □

As $[\Sigma_n) = \mathrm{Glu}(L_n)$, we proved that

Corollary 2. *For every* $n > 1$*,*

$$\mathrm{Glu}(L_n) \cong \mathrm{Tol}(L_{n-1}).$$

Now, let B_n denote an 2^n-element Boolean algebra and $(\Sigma_n]$ the principal ideal generated in $\mathrm{Tol}(L_n)$ by Σ_n. Then

Theorem 3. *For every* $n > 1$*,*

$$(\Sigma_n] \cong B_{n-1}.$$

Proof. Let $\mathcal{C}_i = \{\{j\}\}_{j=0,\ldots,n-1, j \neq i} \cup \{[i, i+1]\}$ for $i = 0, \ldots, n-2$. We can observe that $\mathcal{C}_i$ for $i = 0, \ldots, n-2$ are families of blocks of all tolerances being atoms of $\mathrm{Tol}(L_n)$. Of course, all of them are congruences. Let us denote the tolerance with blocks $\mathcal{C}_i$ by T_i, where $i = 0, \ldots, n-2$.

It is clear that the smallest tolerance including all T_i for $i = 0, \ldots, n-2$ is Σ_n, so

$$\bigcup_{i=0}^{n-2} T_i = \Sigma_n.$$

Now, let $T \subseteq \Sigma_n$ and $T \in \mathrm{Tol}(L)$. Since $\mathcal{C} = \{[i, i+1] \colon i = 0, \ldots, n-2\}$ is the family of blocks of Σ_n, blocks of T can be singletons or two-element intervals of the form $[i, i+1]$, according to Lemma 1. Therefore, $T = \bigcup_{i \in I} T_i$, where I is the the set of all $i \in \{0, \ldots, n-2\}$ such that the interval $[i, i+1]$ is a block of T.

On the other hand, for $I, J \subseteq \{0, \ldots, n-2\}$ such that $I \neq J$ we have

$$\bigcup_{i \in I} T_i \neq \bigcup_{i \in J} T_i.$$

Indeed, suppose $k \in I \setminus J$. Then $[k, k+1]$ is a block of the tolerance $\bigcup_{i \in I} T_i$ but not a block of the tolerance $\bigcup_{i \in J} T_i$.

Thus, the ideal $(\Sigma_n]$ of the lattice $\mathrm{Tol}(L_n)$ is a Boolean algebra generated by $n-1$ atoms T_i, where $i = 0, \ldots, n-2$. The identity is the smallest element of the Boolean algebra and the skeleton tolerance Σ_n is the biggest one. □

3. Conclusions

By Theorem 2 and Theorem 3 we conclude that for any chain L_n for $n > 1$ all tolerances including Σ_n form a lattice isomorphic to $\mathrm{Tol}(L_{n-1}$ and all tolerances included in Σ_n form a Boolean algebra B_{n-1}. Therefore, the lattice $\mathrm{Tol}(L_n)$ for $n > 2$ looks like in Figure 1.

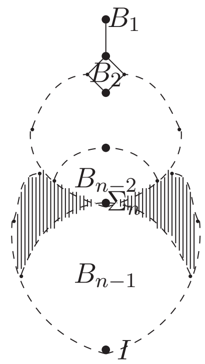

Figure 1

The part colored in grey contains the tolerances incomparable with Σ_n.

Example 2. *In Figure 2 there are depicted all lattices of tolerances* $\mathrm{Tol}(L_n)$ *for n up to 4. The black points denote congruences.*

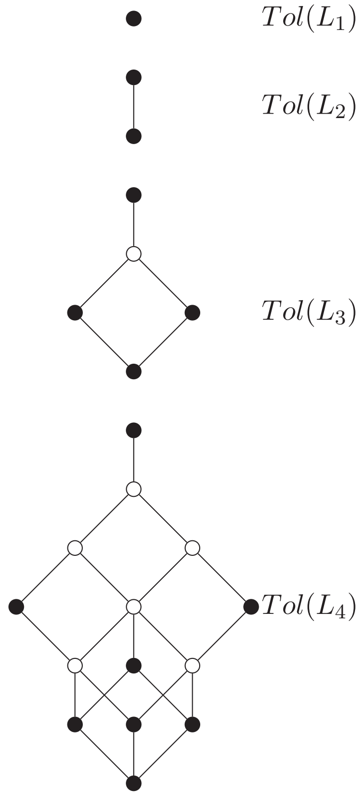

FIGURE 2

It can be shown that for $n > 1$ it holds $\mathrm{Con}(L_n) \cong B_{n-1}$, so $\mathrm{Con}(L_n) \cong (\Sigma_n]$. However, it is not true that $\mathrm{Con}(L_n) = (\Sigma_n]$. In fact $\mathrm{Con}(L_n)$ for $n > 2$ is not a sublattice of $\mathrm{Tol}(L_n)$ as can be easily seen in Example 2.

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