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ALL SPLITTING LOGICS IN THE LATTICE $NEXT(KTB.3'A)$

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Abstract

We examine a special modal logic which is a normal extension of the Brouwer modal logic. It is determined by linearly ordered chains of clusters and the relation between clusters is reflexive and symmetric. The appropriate axiomatization of this logic is proposed in the papers [11] and [12]. There is also proved that all normal extensions of the investigated logic are Kripke complete and have f.m.p. Unfortunately, the cardinality of this family is continuum [13]. One may imagine that the structure of the lattice of these extensions is immensely complex. Then we use the technics of splitting to characterize this lattice and to describe some quite simple fragments. We characterize all the logics that split the lattice.

1. Introduction

The Brouwer logic **KTB** is defined as a normal extension of the minimal normal modal logic **K**. We get $\mathbf{KTB} := \mathbf{K} \oplus T \oplus B$ where:

$$T := \Box p \to p$$
$$B := p \to \Box\Diamond p$$

Semantically, it is determined by Kripke frames with the accessibility relation being reflexive and symmetric. On the other side, logics determined by reflexive and symmetric Kripke frames are called Brouwerian. Also, they are called intransitive, since the relation does not have to be transitive. The absence of transitivity involves many difficulties in studying these logics.

One approach to intransitive logics is to add the weak transitivity property $(4_n) := \Box^n p \to \Box^{n+1} p$ for $n > 1$. If $n = 1$ then, of course, we get just transitivity. In 1964 Thomas defined the following family of logics:

$$\mathbf{T_n^+} := \mathbf{KTB} \oplus (4_n)\,.$$

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He also proved that for different n the logics $\mathbf{T_n^+}$ are different; see [21]. Logics $\mathbf{T_n^+}$ have quite strong algebraic characterization; see [20]. Recently, such logics (especially $\mathbf{T_2^+}$) were intensively examined and some important facts concerning the existence of their Kripke incomplete extensions were established; see [7] and [8].

Anyway, the purely intransitive logics still need examination. Our proposition in this field is to study a subfamily of $NEXT(\mathbf{KTB})$, which is determined by frames with a clear semantical characterization. Such semantical feature is linearity. The motivation for such a choice has two sources. First, is the logic $\mathbf{S4.3}$, which is complete with respect to linearly quasi ordered frames (xRy or yRx for any distinct $x, y \in W$). They are usually presented as chains of clusters. Below, we remind two famous results for its normal extensions due to [1] and [5], respectively.

Theorem 1. *Every normal modal logic extending* $\mathbf{S4.3}$ *has finite model property.*

Theorem 2. *Every normal modal logic extending* $\mathbf{S4.3}$ *is finitely axiomatizable.*

The second source for our motivation comes from the logic $\mathbf{KTB} \oplus alt_3$, where

$$(alt_3) := \Box p \vee \Box(p \to q) \vee \Box((p \wedge q) \to r)) \vee \Box((p \wedge q \wedge r) \to s).$$

This logic is determined by the class of reflexive and symmetric frames forming chains of points. Byrd and Ullrich proved in 1970's that all logics from $NEXT(\mathbf{KTB} \oplus alt_3)$, have f.m.p. and are finitely axiomatizable (and hence - decidable).

It seems to be interesting to compare the above result with Bull's and Fine's. Anyway, we need to be careful in this comparison. For logics above $\mathbf{S4.3}$, frames are uniquely represented as chains of disjoint clusters, whereas for logics extending $\mathbf{KTB}$, clusters do not have to be disjoint in the appropriate frame. In a reflexive and symmetric Kripke frame, some clusters may have non-empty intersection. The logics studied by Byrd and Ullrich are determined by reflexive and symmetric frames forming chains of points; each two points being in relation form, in fact, two-element cluster. Two neighboring clusters have a common point. In the paper we will consider a more general condition of linearity in reflexive and symmetric frames. We accept the existence of n-element clusters for any $n \in \mathbb{N}$. The property of linearity for reflexive and symmetric frames is characterized as follows:

(1) Each cluster has a non-empty intersection with at most two others (similarly as two-element clusters in frames for $NEXT(\mathbf{KTB} \oplus alt_3)$),

(2) If some cluster has non-empty intersections with two other clusters, then each point of the cluster belongs to one of the intersections.

Below, we prepare some tools for dealing with splitting of lattices. As it was said in Introduction the Brouwer logic **KTB** is determined by the class of reflexive and symmetric Kripke frames (symb. KTB-frames). Note that for an arbitrary KTB-frame $\mathfrak{F} = \langle W, R\rangle$ the transitive closure of R is universal on W. For our purpose we will consider only connected frames.

Definition 1. *Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame. Then $\mathfrak{F}$ is connected if for any $x, y \in W$ there is a number $n \in \mathbb{N}$ such that $xR^n y$.*

In a frame $\mathfrak{F} = \langle W, R\rangle$, the point $r \in W$ is called a root if for any $x \in W$ there exists a number n such that $rR^n x$. In a connected KTB-frame each its point $x \in W$ is a root. Moreover, KTB-frame is connected iff each its point is its root.

Logics determined by the class of frames $\mathcal{K}$ is defined as usual:

$$L(\mathcal{K}) := \{\alpha \in Form : \ \mathfrak{F} \models \alpha \ \text{ for each } \ \mathfrak{F} \in \mathcal{K}\}\,.$$

The class $\mathcal{K}$ may consist of one frame only. To compare strength of logics determined by classes of Kripke frames p-morphisms are used.

Definition 2. *Let $\mathfrak{F}_1 = \langle W_1, R_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2\rangle$ be Kripke frames. A map $f : W_1 \to W_2$ is a p-morphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$, if it satisfies the following conditions:*

(p1) f *is from* W_1 *onto* W_2,

(p2) *for all* $x,\, y \in W_1,\ xR_1y$ *implies* $f(x)R_2f(y)$,

(p3) *for each* $x \in W_1$ *and for each* $a \in W_2,$ *if* $f(x)R_2a$ *then there exists* $y \in W_1$ *such that* xR_1y *and* $f(y) = a$.

Each p-morphism is also called a reduction. If there is a p-morphims from $\mathfrak{F}_1$ onto $\mathfrak{F}_2$ then we say that $\mathfrak{F}_1$ is reducible to $\mathfrak{F}_2$ and $\mathfrak{F}_2$ is a p-morphic image of $\mathfrak{F}_1$.

By $R(x)$ we mean a set of neighboring points of $x \in W$ for $\mathfrak{F} = \langle W, R\rangle$. Formally:

$$R(x) := \{y \in W : xRy\}.$$

One may notice that the conditions ($p1$) and ($p2$) are equivalent to the following one:

$$f(R_1(x)) = R_2(f(x)) \quad \text{for any } \ x \in W_1. \tag{1}$$

The next lemma is a logical folklore:

Lemma 1. *Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be Kripke frames. If there exists a p-morphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ then $L(\mathfrak{F}_1) \subseteq L(\mathfrak{F}_2)$.*

Let us also remind two basic algebraic notions such as modal algebra and KTB-algebra.

Definition 3. *An algebra* $\mathfrak{A} = \langle A, \cap, \cup, -, I, 0, 1\rangle$ *is a modal algebra if* $\langle A, \cap, \cup,$
$-, 0, 1\rangle$ *is a Boolean algebra and the unary operator* I *satisfies the conditions:*

(1): $I(1) = 1$,
(2): $I(a \cap b) = I(a) \cap I(b)$ *for any* $a, b \in A$.

Definition 4. *A modal algebra* $\mathfrak{A} = \langle A, \cap, \cup, -, I, 0, 1\rangle$ *is called a* KTB*-algebra if the unary operator* I *satisfies the following conditions for any* $a \in A$:

(3): $I(a) \leq a$,
(4): $a \leq I(-I(-a))$.

There is a nice duality between Kripke frames and modal algebras. It is easy to describe in the finite case. For a finite modal algebra $\mathfrak{A}$ we define the dual frame $\mathfrak{A}_* = \langle W_*, R_*\rangle$ where W_* is the set of atoms of algebra $\mathfrak{A}$ and R_* is a binary relation defined for any $x, y \in W_*$ as follows:

$$xR_*y \quad \text{iff} \quad \forall_{z \in A}\, (x \leq I(z) \quad \Rightarrow \quad y \leq z).$$

It is known that both $\mathfrak{A}$ and $\mathfrak{A}_*$ validate the same formulas. Conversely, for each finite Kripke frame $\mathfrak{F} = \langle W, R\rangle$ we define its dual algebra $\mathfrak{F}^* = \langle 2^W, \cap, \cup, -, I, \emptyset, W\rangle$ where for any $X \subseteq W$

$$I(X) = \{x \in W : \forall_y (xRy \quad \Rightarrow \quad y \in X)\}.$$

Similarly, both frame $\mathfrak{F}$ and its dual algebra $\mathfrak{F}^*$ validate the same modal formulas. For more details see [4]. Moreover, for finite cases we have

$$(\mathfrak{F}^*)_* \cong \mathfrak{F} \quad \text{and} \quad (\mathfrak{A}_*)^* \cong \mathfrak{A}.$$

For infinite case there is only the isomorphism $(\mathfrak{A}_*)^* \cong \mathfrak{A}$.

For some special Kripke frames Lemma 1 may be strengthened to an equivalence.

Lemma 2. *Let* $\mathfrak{F}_1$, $\mathfrak{F}_2$ *be finite Kripke frames such that their dual algebras are simple. Then* $L(\mathfrak{F}_1) \subseteq L(\mathfrak{F}_2)$ *iff there exists a p-morphism from* $\mathfrak{F}_1$ *to* $\mathfrak{F}_2$.

Proof. It is proven by Jónsson's lemma, the congruence extension property of modal algebras, finiteness and simplicity of the dual algebra for $\mathfrak{F}_1$. For details, see for example [18] or [9]. □

In the paper [18] it is also proven:

Lemma 3. *Let $\mathfrak{F} = \langle W, R \rangle$ be a finite KTB-frame and $\mathfrak{A}$ a finite KTB-algebra. Then, $\mathfrak{A}$ is subdirectly irreducible iff $\mathfrak{A}$ is simple. Moreover*

(i): *$\mathfrak{F}^*$ is simple iff $\mathfrak{F}$ is connected,*
(ii): *$\mathfrak{A}$ is simple iff $\mathfrak{A}_*$ is connected.*

Then we get:

Corollary 1. *Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be finite and connected KTB-frames. Then $L(\mathfrak{F}_1) \subseteq L(\mathfrak{F}_2)$ iff there exists a p-morphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$.*

2. Main results

2.1. **Linear Brouwer systems.** We start this section with recalling some basic definitions from [11] and [12].

Definition 5. *Let $\mathfrak{F} = \langle W, R \rangle$ be a KTB-Kripke frame (R is reflexive and symmetric relation on W). Then R is called a tolerance on $\mathfrak{F}$.*

Definition 6. *A non-empty subset $U \subseteq W$ is called a block of the tolerance R, if U is a maximal subset with $U \times U \subseteq R$ (if $U \subseteq V$ and $V \times V \subseteq R$, then $U = V$).*

The term: block of tolerance has the same meaning as the term: cluster. But we prefer to use the first one because, in KTB-Kripke frames, clusters may have non-empty intersections. Hereafter, instead of 'a block of tolerance' we simply use a shorter name 'a block'.

Definition 7. *We say that a frame $\langle W, R \rangle$ consists of linearly ordered blocks if the following two conditions hold:*

$(L1)$ $B_1 \cap B_2 \cap B_3 = \emptyset$,
$(L2)$ $(B_1 \cap B_2 \neq \emptyset \ \&\ B_2 \cap B_3 \neq \emptyset) \ \Rightarrow\ (B_1 \cap B_2) \cup (B_2 \cap B_3) = B_2$
for any three distinct blocks B_1, B_2, B_3

It occurred that the generalized notion of linearity for reflexive and symmetric structures has an adequate syntactic characterization [11], [12]. The following formulas are given there:

$$(3') \quad := \quad \Box p \vee \Box(\Box p \to \Box q) \vee \Box((\Box p \wedge \Box q) \to r),$$
$$(A) \quad := \quad \Box((\Box p \wedge q) \to r) \vee \Box((\Box q \wedge r) \to s) \vee \Box((\Box r \wedge s \wedge \Diamond\neg s) \to p) \vee$$
$$\vee \Box((\Box s \wedge p \wedge \Diamond\neg p) \to q).$$

and the logic: $\mathbf{KTB.3'A} := \mathbf{KTB} \oplus (3') \oplus (A)$ is considered. In [11], it is also proven that:

Theorem 3. *Logic* $\mathbf{KTB.3'A}$ *is complete with respect to the class of reflexive and symmetric frames with linearly ordered blocks. Logic* $\mathbf{KTB.3'A}$ *has f.m.p.*

One may asked question if all Kripke frames validating formulas **T**, **B**, $(3')$ and (A) are these frames (reflexive and symmetric) fulfilling the conditions $(L1)$ and $(L2)$? The answer to this question is: yes.

Lemma 4. *Let $\mathfrak{F}$ be a Kripke frame such that $\mathfrak{F} \models \mathbf{T}, \mathbf{B}, (3'), (A)$. Then $\mathfrak{F}$ is reflexive and symmetric and the conditions (L1) and (L2) hold.*

Proof. Obviously, if in a given Kripke frame there exists a point which is irreflexive, then the axiom **T** is falsified. Also, if in a frame exist two points being in a relation which is not symmetric, then axiom **B** is falsified. Suppose that the condition $(L1)$ does not hold in some reflexive and symmetric Kripke frame. Then there are at least four points x_1, x_2, x_3 and x_4 belonging to three different blocks, i.e. $\{x_1, x_2\} \subset B_1$, $\{x_2, x_3\} \subset B_2$, $\{x_2, x_4\} \subset B_3$ and $x_2 \in B_1 \cap B_2 \cap B_3$. Also $\neg x_1 R x_3$, $\neg x_1 R x_4$ and $\neg x_3 R x_4$. See Fig. 2 from [11]. We define valuation:

$$\{x_2, x_3, x_4\} \subseteq V(p) \text{ and } x_1 \notin V(p),$$
$$\{x_2, x_4\} \subseteq V(q) \text{ and } \{x_1, x_3\} \not\subseteq V(q), \text{ and } x_4 \notin V(r).$$

Then we get:

$$x_3 \models_V \Box p, \ \ x_4 \models_V \Box p \wedge \Box q, \text{ and } \ x_3 \not\models_V \Box p \to \Box q,$$
$$x_4 \not\models_V (\Box p \wedge \Box q) \to r.$$

Hence $x_2 \not\models_V \Box p$, $x_2 \not\models_V \Box(\Box p \to \Box q)$ and $x_2 \not\models_V \Box[(\Box p \wedge \Box q) \to r]$. And $x_2 \not\models_V (3')$.

Suppose, on the contrary, that the condition $(L2)$ does not hold in some Kripke frame $\mathfrak{F} = \langle W, R \rangle$. Hence there exists at least five points x_1, x_2, x_3, x_4, x_5 belonging to three different blocks, i.e. $\{x_1, x_2\} \subset B_1$, $\{x_2, x_3, x_4\} \subset B_2$, $\{x_4, x_5\} \subset B_3$. Then $x_3 \notin (B_1 \cap B_2) \cup (B_2 \cap B_3)$ and $(B_1 \cap B_2) \cup (B_2 \cap B_3) \neq B_2$. See Fig. 4 from [11].

We define valuation:

$$\{x_2, x_3, x_4, x_5\} \subseteq V(p) \text{ and } x_1 \notin V(p),$$
$$\{x_2, x_3, x_4\} \subseteq V(q) \text{ and } x_5 \notin V(q), \text{ and } x_3 \notin V(r).$$

Since the points x_1, x_3, x_5 belong to three different blocks then they are not in relation R. Hence we get:

$$x_3 \models_V \Box p \wedge \Box q, \text{ and } x_3 \not\models_V (\Box p \wedge \Box q) \to r, \text{ and } x_4 \models \Box p, \text{ and}$$
$$x_4 \not\models_V \Box p \to \Box q.$$

Point x_2 sees x_1, x_3 and x_4 then we get:

$$x_2 \not\models_V \Box p, \ x_2 \not\models_V \Box(\Box p \to \Box q), \ x_2 \not\models_V \Box[(\Box p \wedge \Box q) \to r].$$

Hence: $x_2 \not\models_V (3')$.

□

Similarly to logics from $NEXT(\mathbf{S4.3})$ and $NEXT(\mathbf{KTB} \oplus alt_3)$ we also have:

Theorem 4. *All logics from $NEXT(\mathbf{KTB.3'A})$ are Kripke complete and have f.m.p.*

In contrast to logics from $NEXT(\mathbf{S4.3})$ or $NEXT(\mathbf{KTB} \oplus alt_3)$ it occurred that (see [13]):

Theorem 5. *The cardinality of the family $NEXT(\mathbf{KTB.3'A})$ is continuum.*

For further research we need to specify vocabulary concerning details of the structure of frames with linearly ordered blocks. First of all, we denote the whole class of such frames by $\mathcal{LOB}$ and any Kripke frame from this class by LOB-frame. We say that a block B_1 sees B_2 if $B_1 \cap B_2 \neq \emptyset$ and $B_1 \neq B_2$. In a LOB-frame any block sees at most two others. If it saw more than two, then (L1) or (L2) would not hold. Let $\mathfrak{F} = \langle W, R \rangle$ be a connected LOB-frame. Then if some block B sees no other blocks then $B = W$. Such a frame is called a trivial one. Hence, in non-trivial LOB-frames at least two blocks exist. Second, in a LOB-frame we may distinguish two kinds of blocks: an external block sees one block and the internal block sees two blocks. If $\mathfrak{F}$ does not contain external blocks then an arbitrary fixed block B_0 gives rise to the sequences of blocks: $B_1, B_2, B_3, ...$ and $B_{-1}, B_{-2}, B_{-3}, ...$such that any B_i sees B_{i-1} and B_{i+1}. This is an infinite $(\omega * +\omega)$-chain of blocks (if all of them are distinct) or a finite circle of blocks of a length $n \geq 4$ (if $B_i = B_j$ for $i < j$). The case $n = 2$ is impossible to occur (two blocks are necessarily external) as well as $n = 3$ (a circle which consists of 3 blocks is trivial). The class of closed LOB-frames will be denoted by $\mathcal{CLOB}$ and its members as $CLOB$-frames. Third, if $\mathfrak{F}$ contains an external block B_0 then again we obtain a chain of internal blocks: $B_1, B_2, B_3, ...$, which can be an infinite ω-chain or a finite chain of a length $n \geq 2$ (if another external block stops the construction). Frames having external blocks will be called open (denoted as $OLOB$-frames) and their class will be denoted by $\mathcal{OLOB}$.

Let us add that the trivial frames will be treated by us as open ones. Then we assume that an open frame consists of at least one external blocks.

Then we introduce the notion of a cell.

Definition 8. *Let $\mathfrak{F} \in \mathcal{OLOB}$ be a connected frame consisting of $n \geq 2$ blocks. Suppose they are numerated as B_1, B_2,...,B_n,... accordingly with the order. The internal cell is defined: $C_i = B_{i-1} \cap B_i$ for $i = 2, ..., n, ...$. The external cell is defined as $C_1 = B_1 \setminus B_2$. For finite open frames consisting of n blocks, the other external block is defined as $C_{n+1} = B_n \setminus B_{n-1}$. Internal and external cells will be called just cells.*

One may observe that a finite and connected $OLOB$-frame having n blocks is divided into $n+1$ nonempty cells. In the trivial case for $n=1$, we treat the whole block as one cell. If $n=2$ then we may divide the $OLOB$-frame into two external cells and one internal. And so on.

Definition 9. *Let $\mathfrak{F} \in \mathcal{CLOB}$ be a connected frame consisting of at least four blocks. Suppose they are numerated as $..., B_{-n},...,B_{-2},B_{-1},B_0,\ B_1,\ B_2,...,B_n,...,\ n \geq 2$. Then we consider the intersections $C_i = B_{i-1} \cap B_i$ for $i \in \mathbb{Z}$ and called them also internal cells (or simply - cells).*

A finite and connected $CLOB$-frame with n blocks is also divided into $n+1$ nonempty cells. Let us observe that points belonging to the same cell are modally indistinguishable. If $x, y \in C_i$ and $C_i = B_{i-1} \cap B_i$ then $R(x) = R(y) = B_{i-1} \cup B_i$. Hence if $x \models \Box\alpha$ then $z \models \alpha$ for any $z \in R(x)$. But $R(x) = R(y)$. Then $y \models \Box\alpha$.

Frames from $\mathcal{OLOB}$ will be called chains and denoted by $\mathfrak{Ch}$; frames from $\mathcal{CLOB}$ – circles and denoted by $\mathfrak{C}$. Additionally, by Ch_n^k we mean a family of chain frames having n cells and such that the number of points in each cell is less or equal to k. Any frame from this family will be denoted as $\mathfrak{Ch}(k_1, k_2, ..., k_n)$, where the $k_i \leq k$ for $i = 1, 2, ..., n$ is the number of points in the i-th cell. We allow that $n = 1$. Then we get we get one cell frame, which is a trivial one. For $n = 2$ we get $\mathfrak{Ch}(k_1, k_2) = \mathfrak{Ch}(k_1 + k_2)$, which is, actually one cell frame. So, in fact, we do not have two cell chains. But we will allow that the symbol $\mathfrak{Ch}(k_1, k_2)$ anyway has sense. An infinite chain frame is denoted $\mathfrak{Ch}(k_1, k_2, ...)$. Similarly, by C_n^k we mean a family of circle frames having n cells such that the number of points in each cell is less or equal to k. The appropriate frame is denoted $\mathfrak{C}(k_1, k_2, ..., k_n)$. In infinite case we simply write $\mathfrak{C}(k_1, k_2, ...)$. We see that $\mathfrak{C}(k_1) = \mathfrak{Ch}(k_1)$, $\mathfrak{C}(k_1, k_2) = \mathfrak{Ch}(k_1, k_2) = \mathfrak{Ch}(k_1 + k_2)$. Also $\mathfrak{C}(k_1, k_2, k_3) = \mathfrak{Ch}(k_1 + k_2 + k_3)$.

We introduce the abbreviations: instead of $\mathfrak{Ch}\underbrace{(1, 1, ..., 1)}_{n}$ (or $\mathfrak{C}\underbrace{(1, 1, ..., 1)}_{n}$) we shall write $\mathfrak{Ch}_n$ (or $\mathfrak{C}_n$). Formal definitions are given below.

Definition 10. *Let $\mathfrak{Ch}_n = \langle W_n, R \rangle$ be a frame defined as follows: $W_n = \{x_i :\ 1 \leq i \leq n\}$, the relation R is the following:*

$$x_i R x_j \quad \textit{iff} \quad |i - j| \leq 1; \quad \textit{for every} \quad 1 \leq i, j \leq n.$$

The frame $\mathfrak{Ch}_n$ is called a chain frame of depth n.

Definition 11. *The circle frame $\mathfrak{C}_n$, $n > 3$ of depth n is defined as follows. $\mathfrak{C}_n = \langle W_n, R \rangle$, where $W_n = \{x_i :\ 1 \leq i \leq n\}$. The relation R is defined as follows:*

$$x_i R x_j \quad \textit{iff} \quad |i - j|[mod(n-1)] \leq 1; \quad \textit{for every} \quad 1 \leq i, j \leq n.$$

Further we distinguish two subclasses from $\mathcal{OLOB}$ and $\mathcal{CLOB}$:

- $\mathcal{OLOB}(1)$ - subclass of open frames with linearly ordered blocks, whose each cell has one point.
- $\mathcal{CLOB}(1)$ - subclass of closed frames with linearly ordered blocks, whose each cell has one point.

Example 1. *In Fig. 1 there is presented a frame from $\mathcal{OLOB}(1)$ having 5 cells: $C_1 = B_1 \backslash B_2$, $C_2 = B_1 \cap B_2$, $C_3 = B_2 \cap B_3$, $C_4 = B_3 \cap B_4$, $C_5 = B_5 \backslash B_4$. Symbolically we write it as $\mathfrak{Ch}_5$.*

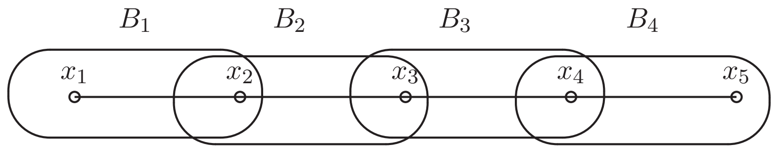

FIGURE 1. Frame from $\mathcal{OLOB}(1)$ having 5 cells written as $\mathfrak{Ch}_5$

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Example 2. *In Fig. 2 there is presented a frame from $\mathcal{OLOB}$ having 4 cells: $C_1 = B_1 \backslash B_2$, $C_2 = B_1 \cap B_2$, $C_3 = B_2 \cap B_3$, $C_4 = B_3 \backslash B_2$,. Symbolically we write it as $\mathfrak{Ch}(1,1,2,1)$. Additionally, we see that $x_3 \models \Box\alpha \;\Leftrightarrow\; x_4 \models \Box\alpha$ for any formula α.*

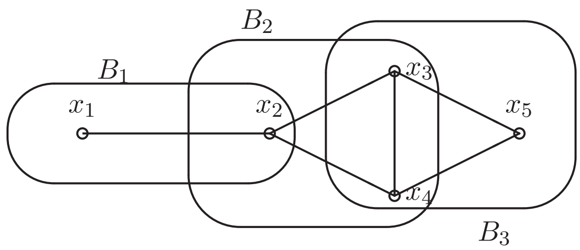

FIGURE 2. Frame from $\mathcal{OLOB}$ having 4 cells written as $\mathfrak{Ch}(1,1,2,1)$

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Example 3. *In Fig. 3 there is presented a frame from $\mathcal{CLOB}(1)$ having 9 cells: $C_i = B_{i-1} \cap B_i$, for $i = 2, ..., 8$ and $C_9 = B_8 \cap B_1$. Symbolically we write it as $\mathfrak{C}_9$.*

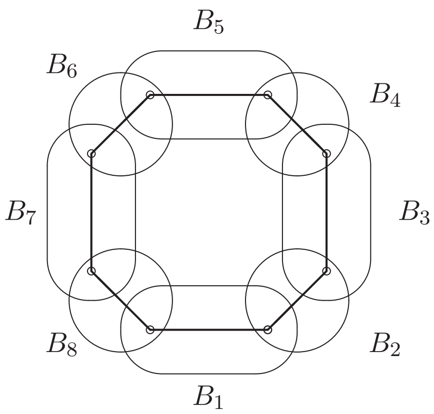

FIGURE 3. Frame from $\mathcal{CLOB}(1)$ having 9 cells written as $\mathfrak{C}_9$.

2.2. **Reductions in $\mathcal{LOB}$.** To compare strength of logics determined by Kripke frames from $\mathcal{LOB}$, we described the possible p-morphism between them. We start with reductions in $\mathcal{OLOB}$.

2.2.1. *Reductions in $\mathcal{OLOB}$.* In this part of the paper we describe possible reductions between chain frames. Some of the presented proofs are similar to the proofs of reductions between parasol frames from [9]. We start with reduction in the class $\mathcal{OLOB}(1)$.

Lemma 5. *If $n > m \geq 1$, then $L(\mathfrak{Ch}_m) \not\subseteq L(\mathfrak{Ch}_n)$.*

Proof. Obviously, there is no p-morphism from $\mathfrak{Ch}_m$ to $\mathfrak{Ch}_n$. From Corollary 1 we conclude that $L(\mathfrak{Ch}_m) \not\subseteq L(\mathfrak{Ch}_n)$. □

Suppose that $m \geq n$. The case with $m = n$ is trivial. Thus, we shall consider only the cases with $m > n$. We prove the existence of a p-morphism between $\mathfrak{Ch}_{2n}$ and $\mathfrak{Ch}_n$.

Lemma 6. *Let $\mathfrak{Ch}_{2n} = \langle W_{2n}, R\rangle$, with $W_{2n} = \{x_1, \ldots, x_{2n}\}$ and $\mathfrak{Ch}_n = \langle W_n, R'\rangle$, with $W_n = \{x'_1, \ldots, x'_n\}$ be two chain frames. The relation R and R' are defined in the appropriate way (see Definition 10). The following function:*

$$\begin{array}{llll} f(x_i) = x'_i & \textit{for each} & i \leq n\,, \\ f(x_i) = x'_{2n-(i-1)} & \textit{for each} & n < i \leq 2n \end{array}$$

is a p-morphism from $\mathfrak{Ch}_{2n}$ to $\mathfrak{Ch}_n$.

Proof. The mapping f defines the operation on the chain frame $\mathfrak{Ch}_{2n}$ which could be described as folding up the frame on half. As a result we obtain the frame $\mathfrak{Ch}_n$. The appropriate folding for $n = 3$ is shown in Fig. 4.

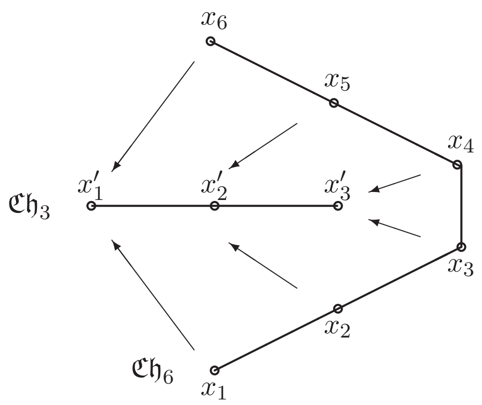

FIGURE 4. The diagram of p-morphism from $\mathfrak{Ch}_6$ to $\mathfrak{Ch}_3$

The function f is onto. The condition (1) also holds. This is because the function f maps x_1 to x'_1, then it moves along $\mathfrak{Ch}_n$ with a short (1-step) stop at the final point x'_n. The final point of the whole journey is x'_1.

□

We may generalize the above lemma by proving the existence of a suitable p-morphism from $\mathfrak{Ch}_{kn}$ to $\mathfrak{Ch}_n$, for each $k \geq 1$. In this case the frame with kn points are folded up k-times.

Lemma 7. *Let* $\mathfrak{Ch}_{kn} = \langle W_{kn}, R\rangle$, $W_{kn} = \{x_1, \ldots, x_{kn}\}$, $k \geq 1$ *and* $\mathfrak{Ch}_n = \langle W_n, R'\rangle$, $W_n = \{x'_1, \ldots, x'_n\}$ *be two chain frames. Then the following function:*

$$\begin{aligned}
&f(x_i) = f(x_{i+2n}) = \ldots = f(x_{i+2sn}) = x'_i \quad \textit{for any} \quad i \leq n \\
&\quad \textit{and} \quad i + 2sn \leq kn\,, \\
&f(x_{i+n}) = f(x_{i+3n}) = \ldots = f(x_{i+(2s+1)n}) = x'_{n-(i-1)} \quad \textit{for any} \\
&\quad i \leq n \quad \textit{and} \quad i + (2s+1)n \leq kn
\end{aligned}$$

is a p-morphism from $\mathfrak{Ch}_{kn}$ *to* $\mathfrak{Ch}_n$.

Proof. Analogous to the proof of Lemma 6. □

The p-morphism between chain frames, which is described above, is not the unique one. Below we describe another one.

Lemma 8. *Let* $\mathfrak{Ch}_{2n-1} = \langle W_{2n-1}, R\rangle$, $W_{2n-1} = \{x_1, \dots, x_{2n-1}\}$ *and* $\mathfrak{Ch}_n = \langle W_n, R'\rangle$, $W_n = \{x'_1, \dots, x'_n\}$ *be two chain frames. Then the following function:*

$$\begin{array}{rcll} g(x_i) & = & x'_i & \textit{for any} \quad i \le n \\ g(x_i) & = & x'_{2n-i} & \textit{for any} \quad n < i \le 2n-1 \end{array}$$

is a p-morphism from $\mathfrak{Ch}_{2n-1}$ *to* $\mathfrak{Ch}_n$.

Proof. This time, the map g is another kind of folding up on half the chain frame $\mathfrak{Ch}_{2n-1}$. Now, the point x_n laying in the middle is the point of folding up. For $n = 3$ the folding looks like in Fig. 5.

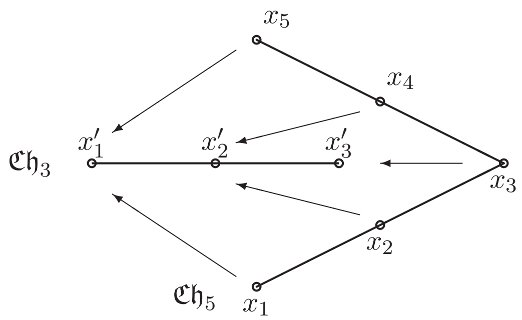

FIGURE 5. The diagram of the p-morphism from $\mathfrak{Ch}_5$ to $\mathfrak{Ch}_3$.

The function g maps $\mathfrak{Ch}_{2n-1}$ to $\mathfrak{Ch}_n$ similarly as the function f described in Lemma 7, but now without the intermediate stop. □

We may also generalize the above lemma; we take, for instance, two frames: $\mathfrak{Ch}_{n+k(n-1)}$ and $\mathfrak{Ch}_n$, for any $k \ge 1$. The first frame may be folded up k-times to get $\mathfrak{Ch}_n$.

Lemma 9. *Let* $\mathfrak{Ch}_{n+k(n-1)} = \langle W_{n+k(n-1)}, R\rangle$, $W_{n+k(n-1)} = \{x_1, \dots, x_{n+k(n-1)}\}$, $k \ge 1$ *and* $\mathfrak{Ch}_n = \langle W_n, R'\rangle$, $W_n = \{x'_1, \dots, x'_n\}$ *be two chain-frames. Then the following function:*

$$\begin{array}{rcl} g(x_i) & = & g(x_{i+2(n-1)}) = \dots = g(x_{i+2s(n-1)}) = x'_i \\ & & \textit{for any} \quad i \le n \quad \textit{and} \quad i + 2s(n-1) \le n + k(n-1)\,, \\ g(x_{i+(n-1)}) & = & g(x_{i+3(n-1)}) = \dots = g(x_{i+(2s+1)(n-1)}) = x'_{n-(i-1)} \\ & & \textit{for any} \quad i \le n \quad \textit{and} \quad i + (2s+1)(n-1) \le n + k(n-1) \end{array}$$

is a p-morphism from $\mathfrak{Ch}_{n+k(n-1)}$ *to* $\mathfrak{Ch}_n$.

Proof. Analogous to the proof of Lemma 8 □

One may observe that the two ways of folding up chain frames presented above, can be mixed up. For example, for $\mathfrak{Ch}_8$ and $\mathfrak{Ch}_3$ there is the following p-morphism:

$$p(x_i) = \begin{cases} x'_i & \text{for} \quad i \leq 3, \\ x'_{6-i} & \text{for} \quad 3 < i \leq 5, \\ x'_{i-5} & \text{for} \quad 6 \leq i \leq 8. \end{cases}$$

The first folding up is made according to the p-morphism g, the second one – to f, see Fig. 6.

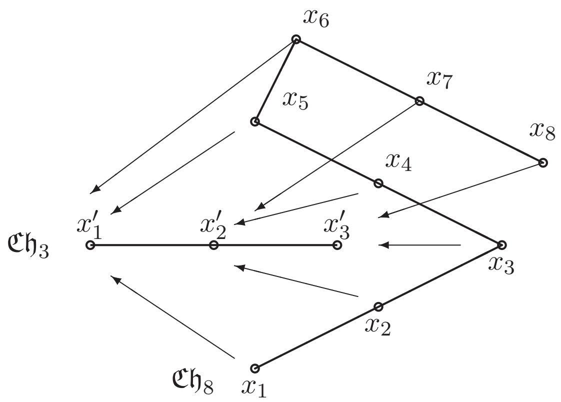

FIGURE 6. The diagram of p-morphism from $\mathfrak{Ch}_8$ to $\mathfrak{Ch}_3$

One may notice that there are plenty of reductions in $\mathcal{OLOB}(1)$. For example, the chain frame $\mathfrak{Ch}_3$ is a p-morphic reduct of the following ones:

$$\mathfrak{Ch}_5,\ \mathfrak{Ch}_6,\ \mathfrak{Ch}_7,\ \mathfrak{Ch}_8,\ \mathfrak{Ch}_9,\ \mathfrak{Ch}_{10},\ \ldots$$

what means, in fact, that almost all chain frames $\mathfrak{Ch}_m$ are reducible to $\mathfrak{Ch}_3$. Now, we are ready to prove the main theorem concerning the reduction between frames from $\mathcal{OLOB}(1)$.

Theorem 6. *Let $m = kn + (n-1)l$ for some $k \geq 1$ and $l \geq 0$. Then $\mathfrak{Ch}_m$ is reducible to $\mathfrak{Ch}_n$.*

Proof. Let us observe that for $k = 1$ the required p-morphism is the function g defined in Lemma 9. Let $k \geq 2$. The idea of the proof combines both previous constructions: we 'move' along $\mathfrak{Ch}_n$ back and forth $k+l$ times, and 'make stop' $k-1$ times and 'pass' l intermediate endpoints without a stop.

The exact definition of the required p-morphism may be found in [9], p.66–67.

□

Corollary 2. *The chain frame $\mathfrak{Ch}_n$ is a reduct of the following chain frames with $k \in \mathbb{N}$:*

$$
\begin{aligned}
&\mathfrak{Ch}_{2n-1},\ \mathfrak{Ch}_{2n}\,,\\
&\mathfrak{Ch}_{3n-2},\ \mathfrak{Ch}_{3n-1},\ \mathfrak{Ch}_{3n}\,,\\
&\mathfrak{Ch}_{4n-3},\ \mathfrak{Ch}_{4n-2},\ \mathfrak{Ch}_{4n-1},\ \mathfrak{Ch}_{4n}\,,\\
&\ldots\\
&\mathfrak{Ch}_{kn-(k-1)},\ \mathfrak{Ch}_{kn-(k-2)},\ \ldots,\ \mathfrak{Ch}_{kn}\,,\\
&\ldots\ .
\end{aligned}
\tag{2}
$$

Let us notice that, for a given n, the numbers $kn-(k-p)$ with $1 \leq p \leq k$ cover an infinite segment of $\mathbb{N}$. Indeed, for each given n one can take $k := n-1,\ p := n-1$ and then $k := n,\ p := 1$. For such a choice we get two natural numbers which are consecutive: $(n-1)n$ and $n^2-(n-1)$. The first number is the last index in some line of (2) and the second one is the first index in the next line of (2). The line from which the infinite segment starts is the one with the index $(n-1)n-(n-2) = (n-1)^2+1$. Then we may reformulate the above result in the following way:

Corollary 3. *For any number n and any $m \geq (n-1)^2+1$, the chain frame $\mathfrak{Ch}_m$ is reducible to $\mathfrak{Ch}_n$.*

Theorem 6 may be strengthened to an equivalence.

Theorem 7. *Let $n < m$ and $n \geq 3$. The frame $\mathfrak{Ch}_m$ is reducible to $\mathfrak{Ch}_n$ iff $m = kn + (n-1)l$ for some $k \geq 1$ and $l \geq 0$.*

Proof. The proof of the simple implication proceeds analogously to the one for parasol frames, which may be found in the paper [9], pp. 66-71 or in [10]. We however sketch the proof. We describe the possible p-morphism from $\mathfrak{Ch}_m = \langle W_m, R\rangle$ to $\mathfrak{Ch}_n = \langle W_n, R'\rangle$. Let $W_m = \{x_1, x_2, ..., x_m\}$ and $W_n = \{y_1, y_2, ..., y_n\}$. First, we prove that any function f gluing two neighboring points in $\mathfrak{Ch}_m$ and mapping them into some inner point from $\mathfrak{Ch}_n$ is not a p-morphism. Suppose, on the contrary, that $x_i R x_{i+1} R x_{i+2} R x_{i+3}$ for $1 \leq i \leq m-3$ and $f(x_{i+1}) = f(x_{i+2}) = y_j,\ j \neq 1$ and $j \neq n$. Then $R(x_{i+1}) = \{x_i, x_{i+1}, x_{i+2}\},\ f(R(x_{i+1})) = \{y_j, y_{j+1}\}$ (or $f(R(x_{i+1})) = \{y_j, y_{j-1}\}$ - it depends on the chosen direction of the mapping). See Fig. 7. But then $R'(f(x_{i+1})) = \{y_{j-1}, y_j, y_{j+1}\}$ and the condition (1) of p-morphism does not hold. A contradiction. contradiction. So, we see that if we glue any two neighboring points, then we have to map them onto an external point from $\mathfrak{Ch}_n$. But then we get the p-morphism f described in Lemma 6.

Similarly, one may prove that any function gluing more than two neighboring points in $\mathfrak{Ch}_m$ and mapping them into some inner point from $\mathfrak{Ch}_n$ is not a p-morphism.

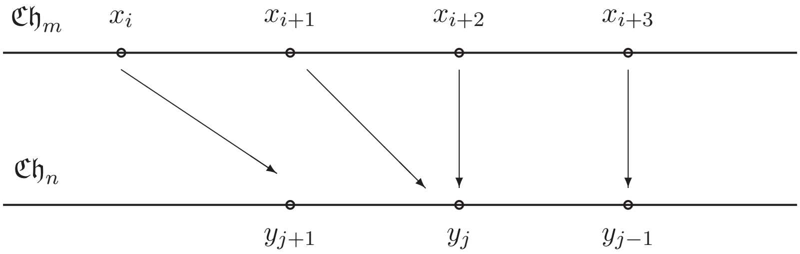

FIGURE 7

Then we will prove that it is impossible to glue three (or more) points and map them onto some external point from $\mathfrak{Ch}_n$; such a function, if exists is not a p-morphism. Suppose, on the contrary, that it is. Hence we get: that $x_iRx_{i+1}Rx_{i+2}Rx_{i+3}Rx_{i+4}$ for $1 \leq i \leq m-3$ and $f(x_{i+1}) = f(x_{i+2}) = f(x_{i+3}) = y_n$, (if they are mapped to y_1 then the proof is analogous). Then $R(x_{i+2}) = \{x_{i+1}, x_{i+2}, x_{i+3}\}$ and $f(R(x_{i+2})) = \{y_n\}$. But then $R(f(x_{i+2})) = \{y_{n-1}, y_n\}$. The condition (1) does not hold. A contradiction.

Then we see that any p-morphism between chain frames must be a combination of these two described in Lemmas 6 and 8.

□

Let us add that if the frame $\mathfrak{Ch}$ is a trivial one, then any other frame (also from $\mathcal{OLOB}(1)$, but not necessarily) is reducible to it.

We should shortly discuss the infinite case. Suppose we have an infinite chain frame $\mathfrak{Ch}_\infty$. Suppose it has the beginning (if not, then we may treat the infinite frame as an infinite circle).

Lemma 10. *The frame $\mathfrak{Ch}_\infty$ is reducible to $\mathfrak{Ch}_n$ for any $n \geq 1$.*

Proof. We map the 'first' point of $\mathfrak{Ch}_\infty$ onto the external point of $\mathfrak{Ch}_n$. Then the next point of $\mathfrak{Ch}_\infty$ onto the next one of $\mathfrak{Ch}_n$, and so on. If we reach the other external point of $\mathfrak{Ch}_n$ then we turn back. The mapping takes an infinite number of times. Such function is a p-morphism. □

Now, we describe reduction between classes $\mathcal{OLOB}$ and $\mathcal{OLOB}(1)$. Indeed, the following holds:

Lemma 11. *Let $\mathfrak{F} \in Ch_n^k$, $k \in \mathbb{N}$ and $\mathfrak{Ch}_n \in \mathcal{OLOB}(1)$ be two chain frames having n-cells each, with $n \geq 3$. Then $\mathfrak{F}$ is reducible to $\mathfrak{Ch}_n$.*

Proof. Let $\mathfrak{F} = (k_1, k_2, ..., k_n)$, $1 \leq k_i \leq k$ for $i = 1, 2, ..., n$. Appropriately, $\mathfrak{Ch}_n = \underbrace{(1, 1, ..., 1)}_{n}$. Then the p-morphism from $\mathfrak{F}$ to $\mathfrak{Ch}_n$ is simply a gluing of the points from each cell from the first frame and mapping them onto the appropriate one-point cells from $\mathfrak{Ch}_n$. Such a gluing is the needed p-morphism. □

On the other side any function mapping points from one cell onto two points from distinct cells is not a p-morphism.

Lemma 12. *Let $\mathfrak{F} \in Ch^k_n$, $k \in \mathbb{N}$ and $\mathfrak{Ch}_{n+1} \in \mathcal{OLOB}(1)$, $n \geq 3$ be two chain frames such that at least one cell in the $\mathfrak{F}$ has two points. Then $\mathfrak{F}$ is not reducible to $\mathfrak{Ch}_{n+1}$.*

Proof. In order to set attention suppose that the frame $\mathfrak{F}$ has $n-1$ one-point cells and exactly one cell has two elements. Then both the frames has the same number of points. Let x_i, x_{i+1} belong to the same cell and there are x_{i-1}, x_{i+2} from $R(x_i, x_{i+1})$ but from different cells. We map these points onto four points from $\mathfrak{Ch}_{n+1}$: $f(x_k) = y_k$ for $k = i-1, i, i+1, i+2$ and such that $y_{i-1}Ry_iRy_{i+1}Ry_{i+2}$. Obviously Obviously $\neg y_{i-1}Ry_{i+1}$ and $\neg y_iRy_{i+2}$ and $\neg y_{i-1}Ry_{i+2}$. But then we obtain $\neg f(x_i)Rf(x_{i+2})$ although x_iRx_{i+2}. This contradicts the condition (p2) of p-morphism.

This reasoning may be generalized for frames having cells with a larger number of points. □

We conclude, that the p-morphism described in Lemma 11, is a unique one (up to isomorphism of p-morphic images of frames from Ch^k_n). As a conclusion of Lemmas 11 and 12 we get:

Corollary 4. *Let $m \geq 3$, $n \geq m$, $k \in \mathbb{N}$. Then any frame $\mathfrak{F} \in Ch^k_n$ is reducible to $\mathfrak{Ch}_m$ iff the appropriate frame $\mathfrak{Ch}_n \in \mathcal{OLOB}(1)$ is reducible to $\mathfrak{Ch}_m$.*

By combining Lemma 11 and Corollary 3 we obtain:

Corollary 5. *Let $k \in \mathbb{N}$, $n \geq 3$. For any number $m \geq (n-1)^2 + 1$, the chain frame $\mathfrak{F} \in Ch^k_m$ is reducible to $\mathfrak{Ch}_n$.*

From Lemmas 10 and 11 we conclude:

Corollary 6. *red2'Let $\mathfrak{F} \in Ch^k_\infty$. Then $\mathfrak{F}$ is reducible to $\mathfrak{Ch}_n$ for any $n \geq 1$.*

2.2.2. *Reductions in $\mathcal{CLOB}$.* First, we describe reductions in $\mathcal{CLOB}(1)$.

Lemma 13. *Let $\mathfrak{C}_{kn} = \langle W_{kn}, R\rangle$, $W_{kn} = \{x_1, \ldots, x_{kn}\}$, $k \geq 2$ and $\mathfrak{C}_n = \langle W_n, R'\rangle$, $W_n = \{y_1, \ldots, y_n\}$ be two circular frames. Then the following function:*

$$f(x_j) = y_i \quad \textit{iff} \quad i = j[mod(n)];\ \textit{for any} \quad i \leq n\,, j \leq kn$$

is a p-morphism from $\mathfrak{C}_{kn}$ *to* $\mathfrak{C}_n$*.*

Proof. The function f may be described as winding $\mathfrak{C}_{kn}$ to $\mathfrak{C}_n$ k-times. It is onto. Let x_jRx_{j+1}. Obviously, $f(x_j)R'f(x_{j+1})$ since $f(x_j) = y_{j[mod(n)]}$, $f(x_{j+1}) = y_{j+1[mod(n)]}$ and $y_{j[mod(n)]}R'y_{j+1[mod(n)]}$. Now, we check the condition (p3) of p-morphism. Let $f(x_j)R'x_i$. If $f(x_j) = x_i$ then the thesis is trivial. Let $f(x_j) \neq x_i$ and suppose $f(x_j) = y_{i+1}$ (it could be also $f(x_j) = y_{i-1}$, but it is analogous). Then we take the point x_{j-1} if $j \geq 2$ (or x_{kn} if $j = 1$). We get x_jRx_{j-1} and $f(x_{j-1}) = y_i$ (or x_1Rx_{kn} and $f(x_{kn}) = y_1$, appropriately). □

The above Lemma may be strengthened to an equivalence.

Theorem 8. *Let* $n \geq 5$*. The frame* $\mathfrak{C}_m$ *is reducible to* $\mathfrak{C}_n$ *iff* $n|m$*.*

Proof. The proof of the simple implication proceeds analogously to the one for wheel frames, which may be found in the paper [18]. Below, we present its shortcut. In the circle $\mathfrak{C}_m$ all cells are internal and have one point. Any function gluing more than two neighboring points from $\mathfrak{C}_m$ and mapping them onto some point from $\mathfrak{C}_n$ is not a p-morphism (see proof of Theorem 7). Hence, let $f : \mathfrak{C}_m \to \mathfrak{C}_n$, $\mathfrak{C}_m = \langle W_m, R\rangle$, $W_m = \{x_1, \ldots, x_m\}$, $m \geq 5$ and $\mathfrak{C}_n = \langle W_n, R'\rangle$, $W_n = \{y_1, \ldots, y_n\}$. We may suppose that $f(x_1) = y_1$ and $f(x_2) = y_2$ (up to a re-enumeration and re-orientation of $\mathfrak{C}_n$). Now, again, if $f(x_i) = y_j$ and $f(x_{i-1}) = y_{j-1}$ then $f(x_{i+1}) = y_{j+1}$, etc. Hence we wind wind $\mathfrak{C}_m$ to $\mathfrak{C}_n$ and finish at $f(x_m) = y_n$. If not then we get a contradiction with (1).

□

Similarly to Lemma 11 we may describe reductions from $\mathcal{CLOB}$ to $\mathcal{CLOB}(1)$.

Lemma 14. *Let* $\mathfrak{F} \in C_n^k$ *and* $\mathfrak{C}_n \in \mathcal{CLOB}(1)$*. Then* $\mathfrak{F}$ *is reducible to* $\mathfrak{C}_n$*.*

Proof. Let $\mathfrak{C}(k_1, k_2, ..., k_n)$, $1 \leq k_i \leq k$ for $i = 1, 2, ..., n$. To reduct $\mathfrak{C}_n^k$ to $\mathfrak{C}_n$ we glue points from each cell from the first frame and map them onto the one-point cells from $\mathfrak{C}_n$. Such a gluing is the p-morphism. □

Let us remind (see the proof of Lemma 12) that any function mapping points from one cell onto two points from distinct cells is not a p-morphism. We notice again, that the described in Lemma 14 p-morphism, is a unique one (up to isomorphism of $\mathfrak{C}_n$). As a conclusion of Lemma 14 (and a counterpart of Lemma 12 for circles) we get:

Corollary 7. *Let* $k, n, m \geq 1$*. Then any frame* $\mathfrak{F} \in C_n^k$ *is reducible to* $\mathfrak{C}_m$ *iff the appropriate frame* $\mathfrak{C}_n \in \mathcal{CLOB}(1)$ *is reducible to* $\mathfrak{C}_m$*.*

2.2.3. *Reductions between $\mathcal{CLOB}$ and $\mathcal{OLOB}$.* Now, we describe p-morphisms from $\mathcal{CLOB}$ to $\mathcal{OLOB}$. We start with the classes $\mathcal{CLOB}(1)$ and $\mathcal{OLOB}(1)$.

Lemma 15. *Let $\mathfrak{C}_{2n}$, $n \geq 1$ be a circular frame from $\mathcal{CLOB}(1)$ and $\mathfrak{Ch}_n$ – a chain frame from $\mathcal{OLOB}(1)$. Then $\mathfrak{C}_{2n}$ is reducible to $\mathfrak{Ch}_n$.*

Proof. Let $\mathfrak{C}_{2n} = \langle W_{2n}, R\rangle$, where $W_{2n} = \{x_1, x_2, ..., x_{2n}\}$ and $\mathfrak{Ch}_n = \langle W'_n, R'\rangle$ where $W'_n = \{x'_1, x'_2, ..., x'_n\}$. We define the p-morphism as follows:

$$\begin{aligned} f(x_i) &= x'_i && \text{for each} && i \leq n\,, \\ f(x_i) &= x'_{2n-(i-1)} && \text{for each} && n < i \leq 2n. \end{aligned}$$

We notice that the p-morphism is very similar to the one defined in the proof of Lemma 6. It is holding up of the circle in half. See Fig. 8. The proof proceeds similarly to the earlier one. □

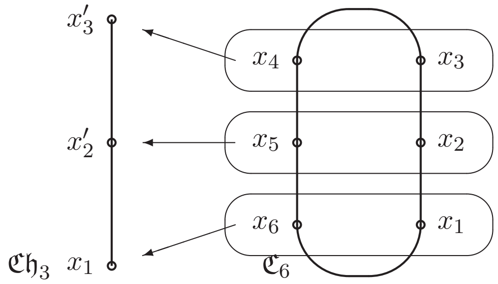

FIGURE 8. The diagram of p-morphism from $\mathfrak{C}_6$ to $\mathfrak{Ch}_3$.

For circle frames with an even number of points there is possible to define another kind of p-morphism.

Lemma 16. *Let $\mathfrak{C}_{2n}$, $n \in \mathbb{N}$ be a circular frame from $\mathcal{CLOB}(1)$ and $\mathfrak{Ch}_{n+1}$ – a chain frame from $\mathcal{OLOB}(1)$. Then $\mathfrak{C}_{2n}$ is reducible to $\mathfrak{Ch}_{n+1}$.*

Proof. Let $\mathfrak{C}_{2n} = \langle W_{2n}, R\rangle$, where $W_{2n} = \{x_1, x_2, ..., x_{2n}\}$ and $\mathfrak{Ch}_{n+1} = \langle W'_{n+1}, R'\rangle$ where $W'_{n+1} = \{x'_1, x'_2, ..., x'_n, x'_{n+1}\}$. We define the p-morphism as follows:

$$\begin{aligned} f(x_i) &= x'_i && \text{for each} && i \leq n+1\,, \\ f(x_i) &= x'_{2n-(i-2)} && \text{for each} && n+1 < i \leq 2n. \end{aligned}$$

We notice that the p-morphism is very similar to the one defined in the proof of Lemma 8. It is another holding up in half of the circle and gluing

$2n-2$ points. Two points are not glued and they are mapped onto x'_1 and x'_{n+1}, respectively. $\square$

Circle frame with an odd number of points is also reducible to a chain frame.

Lemma 17. *Let $\mathfrak{C}_{2n-1}$, $n \in \mathbb{N}$ be a circular frame from $\mathcal{CLOB}(1)$ and $\mathfrak{Ch}_n$ – a chain frame from $\mathcal{OLOB}(1)$. Then $\mathfrak{C}_{2n-1}$ is reducible to $\mathfrak{Ch}_n$.*

Proof. Let $\mathfrak{C}_{2n-1} = \langle W_{2n-1}, R\rangle$, where $W_{2n-1} = \{x_1, x_2, ..., x_{2n-1}\}$ and $\mathfrak{Ch}_n = \langle W'_n, R'\rangle$ where $W'_n = \{x'_1, x'_2, ..., x'_n\}$. We define the p-morphism as follows:

$$\begin{array}{rcll} g(x_i) & = & x'_i & \text{for any} \quad i \leq n \\ g(x_i) & = & x'_{2n-i} & \text{for any} \quad n < i \leq 2n-1 \end{array}$$

The p-morphism is also very similar to the one defined in Lemma 8 and the proof is actually analogous to the earlier one. $\square$

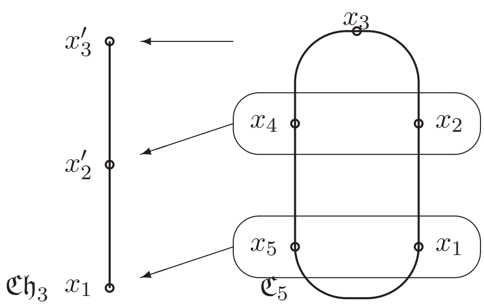

FIGURE 9. The diagram of p-morphism from $\mathfrak{C}_5$ to $\mathfrak{Ch}_3$

To describe more precisely the reductions from $\mathcal{CLOB}(1)$ to $\mathcal{OLOB}(1)$ we also get:

Lemma 18. *For each $n > 2$, the chain frame $\mathfrak{Ch}_n$ is a reduct of the following circle frames: $\mathfrak{C}_{2n-2}$, $\mathfrak{C}_{2n-1}$ and $\mathfrak{C}_{2n}$. There is no smaller circle frame reducible to $\mathfrak{Ch}_n$.*

Proof. The first part follows from Lemmas 15-17 The second one is a consequence of the fact that there is no p-morphism from $\mathfrak{C}_m$, $m \geq 3$ to $\mathfrak{Ch}_n$ such that $2n > m + 2$. $\square$

To complete the description of reductions from $\mathcal{CLOB}$ to $\mathcal{OLOB}$ we notice that:

Corollary 8. *For each $n > 2$ the chain frame $\mathfrak{Ch}_n$ is a reduct of a circle frame $\mathfrak{F} \in C^k_m$ if $m \in \{2n-2, 2n-1, 2n\}$ and any $k \geq 1$.*

Let us observe that there are more possible reductions $\mathcal{CLOB}(1)$ to $\mathcal{OLOB}(1)$. They will be obtained by a superposition of the reduction described above and the other described in two previous subsections.

Now, we discuss the case of reduction of infinite circle $\mathfrak{C}_\infty$.

Lemma 19. *The frame $\mathfrak{C}_\infty$ is reducible to any $\mathfrak{Ch}_n$, $n \geq 1$.*

Proof. We choose the 'first' point of $\mathfrak{C}_\infty$ quite arbitrarily and map it onto the external point of $\mathfrak{Ch}_n$. Then the next point (from 'left side') of $\mathfrak{C}_\infty$ is mapped onto the next one of $\mathfrak{Ch}_n$, and and so on. Analogously for points from 'right side' of the chosen point from $\mathfrak{C}_\infty$. If we reach the other external point of $\mathfrak{Ch}_n$ then we turn back. The mapping takes an infinite number of times. Such function is a p-morphism. □

Theorem 9. *Let $\mathfrak{Ch}_n$ be a chain frame from $\mathcal{OLOB}(1)$. Then for any $k \geq 1$ and any $m \geq 2(n-1)^2 + 2$ the frames $\mathfrak{F} \in Ch^k_m$ and $\mathfrak{G} \in C^k_m$ are reducible to $\mathfrak{Ch}_n$.*

Proof. From Corollary 5 we conclude that for any $k \geq 1$ and any $m \geq (n-1)^2 + 1$ frame $\mathfrak{F} \in Ch^k_m$ is reducible to $\mathfrak{Ch}_n$. Then it also holds for $m \geq 2(n-1)^2 + 2$.

For a circle frame $\mathfrak{G} \in C^k_m$ we reduce it first to the circle frame $\mathfrak{C}_m$ (see Lemma 14). If m is even then by Lemma 16 we reduce $\mathfrak{C}_n$ to $\mathfrak{Ch}_{\frac{m}{2}}$; if m is odd, then $\mathfrak{C}_m$ is reducible to $\mathfrak{Ch}_{\frac{m+1}{2}}$ (see Lemma 15). Superposition of two reductions is a reduction. We apply Corollary 3 to frames: $\mathfrak{Ch}_{\frac{m}{2}}$ and $\mathfrak{Ch}_{\frac{m+1}{2}}$. They are reducible to $\mathfrak{Ch}_n$ for any m such that $\frac{m}{2} \geq (n-1)^2 + 1$ (and $\frac{m+1}{2} \geq (n-1)^2 + 1$). By a simple calculation we get that for any $m \geq 2(n-1)^2 + 2$, the frame $\mathfrak{G} \in C^k_m$, $k \geq 1$ is reducible to $\mathfrak{Ch}_n$.

□

From Lemmas 19 and 14 we also get:

Corollary 9. *Let $\mathfrak{F} \in C^k_\infty$. Then $\mathfrak{F}$ is reducible to $\mathfrak{Ch}_n$ for any $n \geq 1$.*

On the other side, one may noticed that there are no reductions from $\mathcal{OLOB}(1)$ to $\mathcal{CLOB}(1)$ neither from $\mathcal{OLOB}$ to $\mathcal{CLOB}$.

2.3. **Splittings.** Since the cardinality of the family $NEXT(\mathbf{KTB.3'A})$ is continuum, then we shall find and describe some interesting proper sublattices of the whole investigated lattice. A powerful method of division of a complete lattice into two special parts is the method of splitting. By splitting we get a sharp line of division of the whole lattice.

Definition 12. *Let $\mathfrak{L} := \langle L, \wedge, \vee\rangle$ be a lattice and $a \in L$. Then a splits $\mathfrak{L}$ if there exists $b \in L$ such that for any $x \in L$, either $x \leq a$ or $b \leq x$, but not both. The pair (a, b) is called a splitting pair of the lattice $\mathfrak{L}$. The element a splits the lattice, whereas the element b is called the splitting partner of a.*

Sometimes, a trivial splitting holds when $a < b$.In such a case a is a unique cocover of b and b is a unique cover of a.

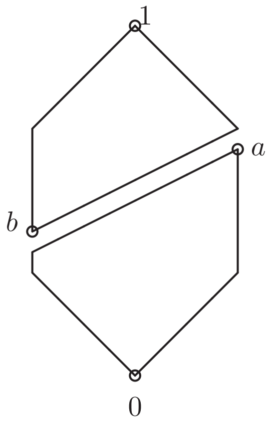

FIGURE 10. A splitting of a lattice $\mathfrak{L}$.

To deal with splittings we need the notion of a characteristic formula. It was first introduced for intuitionistic logic (and Heyting algebras) by [6], but later the notion was adopted to modal logics as well. Then, by the theory of duality, characteristic formulas are also used for Kripke frames. For each finite frame $\mathfrak{F} = \langle W, R\rangle$ we define its diagram $\Delta_{\mathfrak{F}}$ as follows:

- for each element $a \in W$ we fix a distinct propositional variable p_a.
- $\Delta_{\mathfrak{F}} := \{p_a \to \Diamond p_b :\ aRb\} \cup \{p_a \to \neg\Diamond p_b :\ \neg(aRb)\} \cup \{p_a \to \neg p_b : a \neq b\} \cup \left\{\bigvee_{x\in W} p_x\right\}$

The characteristic formula for the frame $\mathfrak{F}$ is defined $\delta_{\mathfrak{F}} := \bigwedge \Delta_{\mathfrak{F}}$. We say that a KTB-frame $\mathfrak{F} = \langle W, R\rangle$ has a finite depth n if for any $y \in W$ it holds that $xR^n y$, for any root x of $\mathfrak{F}$. The depth of $\mathfrak{F}$ is a minimal such an n. Let us remind that in a connected KTB-frame each point is a root.

If the frame $\mathfrak{F}$ determines the logic $L(\mathfrak{F})$ that splits the given lattice, then we say that $\mathfrak{F}$ splits the lattice. Let $\mathfrak{F}$ be a frame of finite depth n, that splits $NEXT(L_0)$. For the given point x we take the formula $\kappa_{\mathfrak{F},x} := \Box^n \delta_{\mathfrak{F}} \to \neg p_x$. Obviously, for the given valuation V we get

$$x \not\models_V \kappa_{\mathfrak{F},x}\ .$$

Similarly, for any other point y (which may be also treated as a root) we get

$$y \not\models_V \kappa_{\mathfrak{F},y}.$$

The formulas $\kappa_{\mathfrak{F},x}$ and $\kappa_{\mathfrak{F},y}$ are somehow equivalent in the sense that, for any valuation V

$$x \not\models_V \kappa_{\mathfrak{F},x} \quad \Leftrightarrow \quad y \not\models_V \kappa_{\mathfrak{F},y} \ .$$

Hence, actually we may choose any point as a root.

The splitting partner of $L(\mathfrak{F})$ is the logic $L_0 \oplus \kappa_{\mathfrak{F},x}$. It is the smallest logic not verified by $\mathfrak{F}$. It will be also denoted by $L_0/\mathfrak{F}$. Then the splitting pair in $NEXT(L_0)$ is the following:

$$\{L(\mathfrak{F}),\, L_0 \oplus \kappa_{\mathfrak{F},x}\}.$$

The following theorem [15] is called the general splitting theorem:

Theorem 10. *Let $L_0 \in NEXT(\mathbf{K})$ and $\mathfrak{F}$ be a finite Kripke frame with a root r. Then the following conditions are equivalent:*
(i) $\mathfrak{F}$ splits $NEXT(L_0)$.
(ii) There is $n \in \mathbb{N}$ such that for any frame $\mathfrak{G}$ with $\mathfrak{G} \models L_0$, if $\Box^{(n)}\delta_{\mathfrak{F}}\wedge$ is satisfiable in $\mathfrak{G}$, then $\Box^{(m)}\delta_{\mathfrak{F}} \wedge p_r$ is also satisfiable in $\mathfrak{G}$ for any $m > n$.

The symbol $\Box^{(n)}$ is defined as usual: $\Box^{(1)}p = p \wedge \Box p$, $\Box^{(n)}p = p \wedge \Box(\Box^{(n-1)}p)$. In reflexive structures (in KTB-frames as well) the condition (ii) is simplified to the following one:

(ii') There is $n \in \mathbb{N}$ such that for any frame $\mathfrak{G}$ with $\mathfrak{G} \models L_0$, if $\Box^n\delta_{\mathfrak{F}} \wedge p_r$ is satisfiable in $\mathfrak{G}$, then $\Box^m\delta_{\mathfrak{F}} \wedge p_r$ is also satisfiable in $\mathfrak{G}$ for any $m > n$.

Here, we also should remember that the root r may be chosen quite arbitrarily.

Theorem 11. [17] *Let L_0 be a modal logic which has f.m.p. If L splits $NEXT(L_0)$, then there exists a finite subdirectly irreducible algebra $\mathfrak{B}$ such that $L = L(\mathfrak{B})$.*

Finite subdirectly irreducible KTB-algebras are in fact simple algebras and they correspond to finite connected KTB-frames. The following two logics split the lattice $NEXT(\mathbf{KTB})$:

(1) $L(\circ)$, where $\circ$ is the frame of one reflexive point [16].
(2) $L(\circ - \circ)$, where $\circ - \circ$ is the frame of two points with full relation [19].

One may notice that the frame of one reflexive point is the chain $\mathfrak{Ch}_1$, whereas the frame of two points with full relation belongs to the one-element class Ch_1^2. Further, in [14] it is proven that:

Theorem 12. *The logics $L(\circ)$ and $L(\circ - \circ)$ are the only logics that split the lattice $NEXT(\mathbf{KTB})$.*

2.4. **Splitting in $NEXT(\mathbf{KTB.3'A})$.** By Theorem 11 we know, that looking for frames that split the lattice $NEXT(\mathbf{KTB.3'A})$ (which is a sublattice of $NEXT(\mathbf{KTB})$), we may restrict our attention to the class of finite connected frames from $\mathcal{LOB}$.

We notice that:

Remark 1. *Let $\mathfrak{F} \in \mathcal{LOB}$ has $n+1$ of cells, $n \geq 3$. Then its depth is at most n.*

Remark 2. *Let $\mathfrak{F} = \langle W, R\rangle \in \mathcal{LOB}$ consists of $n+1$ cells. Then $\Box^n \delta_{\mathfrak{F}} \wedge p_x$ is satisfiable at $x \in W$, for any $x \in W$.*

Lemma 20. *Let $\mathfrak{F}_1 = \langle W_1, R_1\rangle$, $\mathfrak{F}_2 = \langle W_2, R_2\rangle$ be two connected frames from $\mathcal{LOB}$ and $\mathfrak{F}_1$ - a finite one. Then the following conditions are equivalent*

(i) *there is a p-morphism from $\mathfrak{F}_2$ to $\mathfrak{F}_1$.*
(ii) *the formula $\Box^n \delta_{\mathfrak{F}_1} \wedge p_x$ is satisfiable in $\mathfrak{F}_2$ for any $n \in \mathbb{N}$ and $x \in W_1$.*

Proof. (i)$\rightarrow$ (ii) Let $f : \mathfrak{F}_2 \rightarrow \mathfrak{F}_1$ be a p-morphism. We take the following valuation in $\mathfrak{F}_2$:

$$V:\ f^{-1}(a) \rightarrow p_a, \text{ for all points } f^{-1}(a) \in W_2.$$

Then the formula $\Box^n \delta_{\mathfrak{F}_1} \wedge p_x$ is satisfiable at $f^{-1}(x)$ in $\mathfrak{F}_2$ for any $n \in \mathbb{N}$.
(2) $\rightarrow$ (1). If the formula $\Box^n \delta_{\mathfrak{F}_1} \wedge p_x$ is satisfiable at some point $y \in W_2$ for any $n \in \mathbb{N}$, then it means that we may follow the valuation V from $\mathfrak{F}_1$ which leads to the characteristic formula. V in $\mathfrak{F}_1$ was defined as usual: for each point $a \in W_1$ it assigns a variable p_a. We stretch the valuation V on the whole frame $\mathfrak{F}_2$. Then the needed p-morphism is defined: obviously $f(y) = x$ and also $f(y_i) = x_j$ iff $V(y_i) = p_{x_j}$ for any $x_j \in W_1$ and $y_i \in W_2$. The conditions (p2) and (p3) for p-morphisms are fulfilled. $\Box$

On the base the above equivalence between existence of p-morphisms and satisfiability of characteristic formula we get:

Lemma 21. *Let $\mathfrak{Ch}_n$ with $n \geq 3$, be a finite and connected chain frame from $\mathcal{OLOB}(1)$. Then $L(\mathfrak{Ch}_n)$ splits the lattice $NEXT(\mathbf{KTB.3'A})$.*

Proof. Let $\mathfrak{Ch}_n$ be a chain frame from $\mathcal{OLOB}(1)$. From Theorem 9 we deduce that for any $k \geq 1$ and any $m \geq 2(n-1)^2 + 2$ frames from Ch^k_m and C^k_m are reducible to $\mathfrak{Ch}_n$. From Lemma 20 we see that the formula $\Box^m \delta_{\mathfrak{Ch}_n} \wedge p_x$ is satisfiable in these frames. Hence there is an $m_0 \in \mathbb{N}$ (and $m_0 = 2(n-1)^2 + 2$) such that for any frame $\mathfrak{G} \in \mathcal{LOB}$, if $\Box^{m_0} \delta_{\mathfrak{Ch}_n} \wedge p_r$ is satisfiable in $\mathfrak{G}$, then $\Box^{m'} \delta_{\mathfrak{Ch}_n} \wedge p_r$ is also satisfiable in $\mathfrak{G}$ for any $m' > m_0$. The condition (ii') of the Kracht theorem is fulfilled. Then we conclude that for each $n \geq 3$ the finite frame $\mathfrak{Ch}_n$ splits the lattice $NEXT(\mathbf{KTB.3'A})$. $\Box$

Below, we shall prove that no other finite frame from $\mathcal{LOB}$ splits the investigated lattice. Theorem 10 implies that a finite and connected frame $\mathfrak{F}$ from $\mathcal{LOB}$ does not split $NEXT(\mathbf{KTB.3'A})$ if there is a sequence $(\mathfrak{G}_n)_{n\in\mathbb{N}}$ of frames from $\mathcal{LOB}$ such that for any $n \in \mathbb{N}$ the following conditions are fulfilled:
(I) $\Box^n \delta_{\mathfrak{F}} \wedge p_x$ is satisfiable in $\mathfrak{G}_n$
(II) there is an $m > n$ such that $\Box^m \delta_{\mathfrak{F}} \wedge p_x$ is not satisfiable in $\mathfrak{G}_n$.

Lemma 22. *Let $\mathfrak{F} \in Ch_s^k$, $s \geq 3$ be a frame from $\mathcal{OLOB} \setminus \mathcal{OLOB}(1)$. Then $\mathfrak{F}$ does not split $NEXT(\mathbf{KTB.3'A})$.*

Proof. Let the frame $\mathfrak{F} = \langle W, R\rangle$ be written as the following sequence $(k_1, k_2, ..., k_s)$ of cells, where $k_i \leq k$ for each $i = 1, 2, ..., s$. The number of points in $\mathfrak{F}$ is equal $K = \sum_{i=1}^{s} k_i$. Additionally, the points belonging to the cell C_i will get upper index and be denoted $x_l^{(i)}$. We take K propositional variables and define the suitable map V for any $x_l^{(i)} \in W$ as: $V(x_l^{(i)}) = p_l^{(i)}$, $i = 1, 2, ..., s$. At least one cell has more than 1 element. Let it be the cell C_{i_0}. Suppose that $1 < i_0 < s$. Since C_{i_0} has at least two points, then the following formulas belong to the diagram $\Delta_{\mathfrak{F}}$:

$$\begin{aligned}
&p_1^{(i_0-1)} \to \Diamond p_1^{(i_0)},\ p_1^{(i_0-1)} \to \Diamond p_2^{(i_0)},\ p_1^{(i_0)} \to \Diamond p_2^{(i_0)},\ p_1^{(i_0)} \to \Diamond p_1^{(i_0+1)},\\
&p_2^{(i_0)} \to \Diamond p_1^{(i_0+1)},\ p_1^{(i_0-1)} \to \neg\Diamond p_1^{(i_0+1)},\ p_1^{(i_0-1)} \to \neg p_1^{(i_0)},\ p_1^{(i_0-1)} \to \neg p_2^{(i_0)},\\
&p_1^{(i_0)} \to \neg p_2^{(i_0)},\ p_1^{(i_0)} \to \neg p_1^{(i_0+1)},\ p_2^{(i_0)} \to \neg p_1^{(i_0+1)},\ p_1^{(i_0-1)} \to \neg p_1^{(i_0+1)}.
\end{aligned} \tag{3}$$

If $i_0 = 1$, then the diagram includes the formulas:

$$\begin{aligned}
&p_1^{(2)} \to \Diamond p_1^{(1)},\ p_1^{(2)} \to \Diamond p_2^{(1)},\ p_1^{(1)} \to \Diamond p_2^{(1)},\ ,p_1^{(2)} \to \Diamond p_1^{(3)},\\
&p_1^{(1)} \to \neg\Diamond p_1^{(3)},\ p_2^{(1)} \to \neg\Diamond p_1^{(3)},\ p_1^{(2)} \to \neg p_1^{(1)},\ p_1^{(2)} \to \neg p_2^{(1)},\\
&p_1^{(1)} \to \neg p_2^{(1)},\ p_1^{(1)} \to \neg p_1^{(3)},\ p_2^{(1)} \to \neg p_1^{(3)},\ p_1^{(2)} \to \neg p_1^{(3)}.
\end{aligned} \tag{4}$$

We define the sequence of frames $(\mathfrak{G}_n)_{n\in\mathbb{N}}$ as follows. To the frame $\mathfrak{F}$ we add its copy by gluing to the cell C_s the same cell and then the other part of the whole copy. After that we glue another copy $\mathfrak{F}$ (now at the cell C_1) and so on. Then we add a tail, which is a chain of s one-point cells. The applied method of pasting frames is actually very similar to the one described in [14].

$$\mathfrak{G}_1 := (k_1, k_2, ..., k_s, \underbrace{1, 1, \ldots, 1}_{s}),$$

$$\mathfrak{G}_2 := (k_1, k_2, ..., k_s, k_s, ..., k_2, k_1, \underbrace{1, 1, \ldots, 1}_{s}),$$

$$\mathfrak{G}_3 := (k_1, k_2, ..., k_s, k_s, ..., k_2, k_1, k_1, k_2, ..., k_s, \underbrace{1, 1, \ldots, 1}_{s}),$$

$$\vdots$$

Formally, the sequence is defined as follows. The frame $\mathfrak{G}_n$ has $s(n+1)$ cells ($C_1, C_2, ..., C_{s(n+1)}$) and they have the following numbers of elements:

$$\begin{array}{rl}
|C_i| = |C_{i+2s}| = \ldots = |C_{i+2ps}| = k_i \text{ for} & i = 1, 2, \ldots, s \text{ and } p \in \mathbb{N} \\
& \text{and } \ i + 2ps < sn + 1, \\
|C_{i+s}| = |C_{i+3s}| = \ldots = |C_{i+(2p+1)s}| = k_{s+1-i} \text{ for} & i = 1, 2, \ldots, s \text{ and } p \in \mathbb{N}, \\
\text{and} & i + (2p+1)s < sn + 1, \\
|C_{ns+1}| = |C_{ns+2}| = \ldots = |C_{(n+1)s}| = 1 & .
\end{array}$$

As a root of each frame $\mathfrak{G}_n$, $n \in \mathbb{N}$ we may choose any point. So we take, for example, $x_1^{(1)} \in C_1$. The formula $\Box^n \delta_{\mathfrak{F}} \wedge p_1^{(1)}$ is satisfiable in each frame $\mathfrak{G}_n$ at its root (for the given n). It is because the range of the formula is n and points accessible from the the root in n steps follows the frame $\mathfrak{F}$. Then we repeat the valuation V from $\mathfrak{F}$ in each appropriate cell (but we have to do this only in the distance n). But if we take $m = s(n+1)$ then $\Box^m \delta_{\mathfrak{F}} \wedge p_1^{(1)}$ is not satisfiable at $x_1^{(1)}$ neither at any other point. It is because in the tail each point sees at most two others (excluding itself). Hence the conjunction of formulas (3) (or (4)) can not be satisfiable for any valuation. □

Then we shall prove that no circular frame may split the lattice $NEXT(\mathbf{KTB.3'A})$. First, we shall consider frames from $\mathcal{CLOB}(1)$.

Lemma 23. *Let $\mathfrak{C}_s$ with $s > 3$ be a frame from $\mathcal{CLOB}(1)$. Then $\mathfrak{C}_s$ does not split $NEXT(\mathbf{KTB.3'A})$.*

Proof. The frame $\mathfrak{C}_s = \langle W, R\rangle$, $s > 3$ consists of s points forming a circle. For technical reason we shall numerate them as $x_0, x_1, ..., x_{s-1}$. For each point x_i we take a propositional variable p_i, $i = 0, 1, 2, ..., s-1$ and define as usual the suitable map $V(x_i) = p_{x_i}$ for all $x_i \in W$. One may notice that the following formulas belong to the diagram $\Delta_{\mathfrak{C}_s}$:

$$p_0 \to \Diamond p_1,\ p_1 \to \Diamond p_2, ...,\ p_{s-2} \to \Diamond p_{s-1},\ p_{s-1} \to \Diamond p_0,$$
$$p_0 \to \Diamond p_{s-1},\ p_1 \to \Diamond p_0, ...,\ p_{s-2} \to \Diamond p_{s-3},\ p_{s-1} \to \Diamond p_{s-2},$$
$$p_0 \to \neg p_1,\ p_1 \to \neg p_2, ...,\ p_{s-2} \to \neg p_{s-1},\ p_{s-1} \to \neg p_0. \tag{5}$$

As a sequence $(\mathfrak{G}_n)_{n\in\mathbb{N}}$ we take the sequence of chain frames:

$$\mathfrak{G}_n := \mathfrak{Ch}_{(n+1)s}.$$

In each frame $\mathfrak{G}_n$, $n \in \mathbb{N}$ as a root we choose the point x_{s+n}. The valuation is defined as follows:

$$V'(x_k) = p_{(k-n)[mod(s)]}, \quad \text{for} \quad 0 \leq k \leq (n+1)s - 1.$$

Obviously, in the root we have $V'(x_{s+n}) = p_0$. If $k < n$ then the subtraction $(k-n)[mod(s)]$ is treated as adding the inverse element. For example, for $s := 4$ and $n := 5$ we get $k := 0, 1, ..., 23$ and

$$V'(x_0) = p_{(0-5)[mod(4)]} = p_{(0+3)[mod(4)]} = p_3,$$
$$V'(x_1) = p_{(0-4)[mod(4)]} = p_{(0+0)[mod(4)]} = p_0,\ \dots,$$
$$V'(x_9) = p_{(9-5)[mod(4)]} = p_0,\ \dots, V'(x_{23}) = p_{(23-5)[mod(4)]} = p_2.$$

Let us notice that the chosen root x_{s+n} is distant from the external nodes more than n steps. Then the formula $\Box^n \delta_{\mathfrak{C}_s} \wedge p_0$ is satisfiable in each frame $\mathfrak{G}_n$ at the chosen root. It is because the range of the modality is n, not more. However for $m = ns$ the formula $\Box^m \delta_{\mathfrak{C}_s} \wedge p_0$ is not satisfiable at x_{s+n} neither at any other point. It is because, now, the range of modality covers the entire frame (also its first and last points). In each chain frame $\mathfrak{Ch}_{ns}$ the last point (say x_{ns}) sees only one other point: x_{ns-1}. If, we valuate x_{ns} with some p_k, $k = 1, ..., s-2$, then it must be possible to valuate at x_{ns} also: $p_k \to \Diamond p_{k+1}$ and $p_k \to \Diamond p_{k-1}$, and variables p_{k-1}, p_k and p_{k+1} are distinct. (If $k = 0$ then we take variables: p_{s-1}, p_0 and p_1; if $k = s-1$ then we take variables: p_{s-2}, p_{s-1} and p_0.) But it is impossible since x_{ns} sees only one point. Hence the conjunction of formulas (5) can not be satisfiable. We proved that the condition (II) does not hold for the defined sequence $\mathfrak{G}_n$. □

Now, we shall prove that no other circular frame may split the lattice $NEXT(\mathbf{KTB.3'A})$. Then we consider frames from $\mathcal{CLOB} \setminus \mathcal{CLOB}(1)$.

Lemma 24. *Let $\mathfrak{F} \in C_s^k$, $s > 3$, $k > 1$ be a frame from $\mathcal{CLOB}$. Then $\mathfrak{F}$ does not split $NEXT(\mathbf{KTB.3'A})$.*

Proof. The proof proceeds analogously to the proof of Lemma 23. The number of points in cells is not important in the proof. As the sequence of frames $\mathfrak{G}_n$ we take again a sequence of chains (built up from cells having appropriate to $\mathfrak{C}_s^k$ number of points). The formula $\Box^m \delta_{\mathfrak{F}} \wedge p_1$ with $m = ns$ can not be satisfiable since the first and the last points exist in each $\mathfrak{G}_n$. $\Box$

Theorem 13. *Let $\mathfrak{F}$ be a finite and connected frame from $\mathcal{LOB}$. Logic $L(\mathfrak{F})$ splits the lattice $NEXT(\mathbf{KTB.3'A})$ iff $\mathfrak{F} \in \mathcal{OLOB}(1)$.*

Proof. It follows from Lemmas 21 – 24. Let us add that in the case of $\mathfrak{Ch}_1$ we get the logic $L(\circ)$ that splits also the bigger lattice $NEXT(\mathbf{KTB})$, whereas $\mathfrak{Ch}_2$ is in fact, a chain frame having one two-point cell. But it determines the logic $L(\circ - -\circ)$, (which also splits the bigger lattice $NEXT(\mathbf{KTB})$). $\Box$

2.5. **Structure of the lattice $NEXT(\mathbf{KTB.3'A})$.** Studying the lattice $NEXT(\mathbf{KTB.3'A})$ we want to emphasize that all elements of the lattice are Kripke complete modal logics [11]. The cardinality of lattice is continuum [13]. From Theorem 13 we know all the logics that split the lattice $NEXT(\mathbf{KTB.3'A})$. They are the logics $L(\mathfrak{Ch}_n)$, $n > 0$. For any such logic its appropriate splitting partner is the logic $\mathbf{KTB.3'A} \oplus \Box^n \delta_{\mathfrak{Ch}_n} \to \neg p_x$ which is the smallest logic not verified by $\mathfrak{Ch}_n$.

Example 4. *The logic $L(\mathfrak{Ch}_2) := L(\circ - \circ)$ splits the lattice $NEXT(\mathbf{KTB.3'A})$ trivially. The frame $\mathfrak{Ch}_2$ is, in fact, the two-element cluster. Then its splitting partner must be the trivial logic $L(\circ)$.*

Example 5. *The logic $L(\mathfrak{Ch}_3) := L(\circ - \circ - \circ)$ splits the lattice $NEXT(\mathbf{KTB.3'A})$. See Fig. 11.*

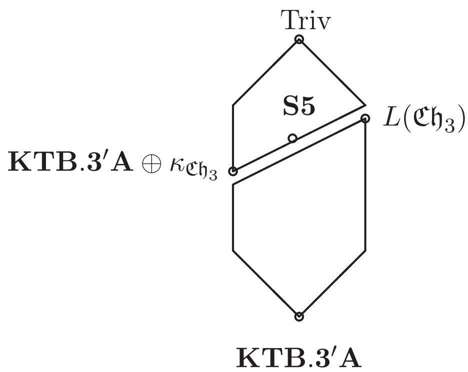

FIGURE 11. The splitting of a lattice $NEXT(\mathbf{KTB.3'A})$ by $L(\mathfrak{Ch}_3)$.

Let us notice that the logic **S5**, which is determined by family of clusters, is not a splitting logic. It is a sup-logic of $\mathbf{KTB.3'A} \oplus \kappa_{\mathfrak{Ch}_3}$. Another sup-logic of $\mathbf{KTB.3'A} \oplus \kappa_{\mathfrak{Ch}_3}$ is the logic $L(\mathfrak{Ch}_4)$ (since it is not sub-logic of $L(\mathfrak{Ch}_3)$).

Analogously, **S5** is a sup-logic of $\mathbf{KTB.3'A} \oplus \kappa_{\mathfrak{Ch}_4}$. Then we will consider a join-splitting of $NEXT(\mathbf{KTB.3'A})$.

Definition 13. *Let $\mathfrak{L} := \langle L, \wedge, \vee \rangle$ be a lattice. Then $b \in L$ is a join-splitting of $\mathfrak{L}$ by $F \subseteq L$ if all $a \in F$ split $\mathfrak{L}$ and $b = \bigvee\{\mathfrak{L}/a : a \in F\}$. Element b is denoted by $\mathfrak{L}/F$.*

One may notice that all chain frames as well as circular frames which have more than 3 cells are reduced to $\mathfrak{Ch}_3$ or to $\mathfrak{Ch}_4$. It follows from Theorem 6, Lemmas 11 and 14, and Lemmas 15 – 17. Then the join-splitting by $\{L(\mathfrak{Ch}_3), L(\mathfrak{Ch}_4)\}$ gets us the logic determined by frames having at most two cells. But such frames are, in fact, clusters. Then we have:

Example 6. *Let $F = \{L(\mathfrak{Ch}_3), L(\mathfrak{Ch}_4)\}$. Then $\mathbf{S5} = \mathbf{KTB.3'A}/F$. See Fig.12.*

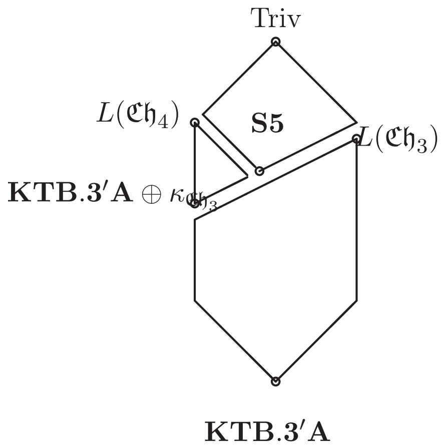

FIGURE 12. A join-splitting of a lattice $NEXT(\mathbf{KTB.3'A})$.

Remark 3. *The logics $L(\mathfrak{Ch}_n)$, $n \geq 1$ have an elegant axiomatization. If $n = 1$ then $L(\circ) = Triv$ and is axiomatized by adding the axiom $p \leftrightarrow \Box p$. For $n \geq 2$ we take the appropriate axiom (4_{n-1}) and the axiom:*

$$alt_3 := \Box p \vee \Box(p \to q) \vee \Box((p \wedge q) \to r) \vee \Box((p \wedge q \wedge r) \to s).$$

And we get $L(\mathfrak{Ch}_n) = \mathbf{KTB.3'A} \oplus alt_3 \oplus (4_{n-1})$, for $n \geq 2$.

Our next aim is to characterize some fragments of the lattice $NEXT(\mathbf{KTB.3'A})$ in details. We would like to describe neighbors of some well described logics, especially $L(\mathfrak{Ch}_n)$, $n \in \mathbb{N}$.

Definition 14. *A modal logic M is a cocover of L in the lattice $NEXT(L_0)$ iff the following two conditions hold:*
1) $M \subset L$
2) for any modal logic M', if $M \subset M' \subset L$ then $M = M'$ or $M' = L$.

Respectively, one may define the notion of a cover $M \supset L$ of a logic L. Obviously, if M is cocover of L, then L is a cover of M.

We know that all normal extensions of $\mathbf{KTB.3'A}$ are Kripke complete and have f.m.p. As a consequences of the Lemma 18, Theorem 7 and Lemma 20 we conclude that cocovers of each $L(\mathfrak{Ch}_n)$ with $n \in \mathbb{N}$ belong to $\mathcal{OLOB}(1)$, $\mathcal{CLOB}(1)$ or are determined by some frames from Ch^2_n, $n \in \mathbb{N}$.

Example 7. *The logic $L(\mathfrak{Ch}_1)$ determined by the trivial chain $\mathfrak{Ch}_1$ has one cocover - the logic $L(\mathfrak{Ch}_2) = L(\circ - -\circ)$.*

Example 8. *The logic $L(\mathfrak{Ch}_2)$ have 3 cocovers - logics: $L(\mathfrak{Ch}(3))$ (which is the logic determined by three element cluster), $L(\mathfrak{Ch}_3)$, $L(\mathfrak{Ch}_4)$.*

Example 9. *The logic $L(\mathfrak{Ch}_3)$ has 8 cocovers: $L(\mathfrak{Ch}_5)$, $L(\mathfrak{Ch}_6)$, $L(\mathfrak{Ch}_7)$, $L(\mathfrak{Ch}_8)$, $L(\mathfrak{C}_5)$, $L(\mathfrak{C}_6)$, $L(\mathfrak{Ch}(1,2,1))$ and $L(\mathfrak{Ch}(1,1,2))$.*

Example 10. *The logic $L(\mathfrak{Ch}_4)$ has 12 cocovers: $L(\mathfrak{Ch}_7)$, $L(\mathfrak{Ch}_8)$, $L(\mathfrak{Ch}_{10})$, $L(\mathfrak{Ch}_{11})$, $L(\mathfrak{Ch}_{12})$, $L(\mathfrak{Ch}_{17})$, $L(\mathfrak{Ch}_{18})$ $L(\mathfrak{C}_6)$, $L(\mathfrak{C}_7)$, $L(\mathfrak{C}_8)$, $L(\mathfrak{Ch}(1,1,2,1))$ and $L(\mathfrak{Ch}(1,1,1,2))$.*

We see that $L(\mathfrak{Ch}_7)$, $L(\mathfrak{Ch}_8)$ and $L(\mathfrak{C}_6)$ are the common cocovers of both the logics: $L(\mathfrak{Ch}_3)$ and $L(\mathfrak{Ch}_4)$. Let us consider Example 9. The splitting partner of logic $L(\mathfrak{Ch}_3)$ is the smallest logic not verified by $\mathfrak{Ch}_3$. We know from Lemmas 15 – 17, Theorem 6 and Lemmas 11 and 14 that the only frames from $\mathcal{LOB}$ which are not reduced to $\mathfrak{Ch}_3$ are the once from the classes: Ch^k_4, C^k_3, and C^k_7, $k \in \mathbb{N}$. Then the splitting partner of the logic $L(\mathfrak{Ch}_3)$ is $\{L(\mathfrak{F}) : \mathfrak{F} \in Ch^k_4 \cap C^{k'}_3 \cap C^{k''}_7\}$ with $k, k', k'' \geq 1$.
Similarly, as in Examples 9 and 10 we may describe cocovers of logics $L(\mathfrak{Ch}_n)$ for $n > 4$. One may notice that their number is always finite.

3. Final remarks

Although the family of $NEXT(\mathbf{KTB.3'A})$ has cardinality continuum, it has occurred that some fragments of its structure are relatively simple. This is because the lattice $NEXT(\mathbf{KTB.3'A})$ has countably many splitting logics. They are splitting partners of the logics determined by chains of points. Moreover, the logic **S5** has a strong position in this lattice being the join-splitting logic by $L(\mathfrak{Ch}_3)$ and $L(\mathfrak{Ch}_4)$.

The lattice of $NEXT(\mathbf{KTB.3'A})$ requires further investigations. For example, we report the problems [1]:

Problem 1. *Let* $L = \mathbf{KTB.3'A} \oplus \kappa_{\mathfrak{Ch}_n}$ *be the splitting partner of* $L(\mathfrak{Ch}_n)$*, for* $n > 2$*. And let* $NEXT(L)$ *be the lattice of its extensions. Is its cardinality finite for the given* n*? Is it possible to describe the structure of this lattice?*

Problem 2. *The logics* $L(\mathfrak{Ch}_n)$*,* $n \geq 1$ *are tabular and form the decreasing sequence*

$$L_1 \supset L_2 \supset L_3 \supset ...$$

where L_{i+1} *is a cocover of* L_i*. We define* i*-th slice as* $\{L : L'_i \subseteq L \subseteq L_i\}$ *where* L'_i *is the splitting partner of* L_{i+1}*. Is this possible to describe such sequences of slices in* $NEXT(\mathbf{KTB.3'A})$*?*

Our future work will also concern the following problems:

(1) Existence of Halldén complete logics in $NEXT(\mathbf{KTB.3'A})$,
(2) Existence of logics with the interpolation property in $NEXT(\mathbf{KTB.3'A})$,
(3) Algebraic counterpart of $\mathbf{KTB.3'A}$.

Let us also notice that the axiom $(3')$ can be generalized (analogously like alt_3 to alt_n). Then our next research will concern also Kripke frames with a higher number of branching.

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