Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XXI (2016)
93–104
DOI http://dx.doi.org/10.16926/m.2016.21.08

X-MAXIMAL CONGRUENCES AND RELEVANT SETS FOR ALGEBRAS

BOGDAN STARUCH AND BOŻENA STARUCH

Abstract

We introduce and investigate X-maximal congruences and relevant sets for a given algebra. We describe interrelations among these concepts and atomicity of the congruence lattice and the number of atoms. We also investigate subdirect decomposition of algebras into subdirectly irreducible factors.

Keywords: congruence lattice, atomic lattice, atom, relevant set, subdirect product, maximal congruence. AMS subject classification: Primary: 08A30; Secondary: 08B26.

1. Introduction

In this paper we introduce and investigate the notion of an X-maximal congruence for a given algebra $\mathbf{A}$, that is, a maximal congruence on $\mathbf{A}$ which does not include any pair of elements from the subset $X \subseteq A$. Basing on this concept we introduce the new notion of a relevant set for $\mathbf{A}$, i.e. a subset $X \subseteq A$ such that every nontrivial congruence on $\mathbf{A}$ contains a pair of elements from X. It means that the identity relation is X-maximal for $\mathbf{A}$. Proposition 6 expresses the relationship among these concepts.

We also consider minimality (under inclusion as well as under cardinality) of relevant sets for a given algebra and we investigate some properties of congruence lattices, mainly atomicity and existence of atoms. For example, we prove that the existence of a finite minimal relevant set for $\mathbf{A}$ implies the finiteness of its congruence lattice and we obtain the estimation of the number of atoms as well.

The studies within the field of congruence lattice properties are still important and worth of effort (see [5], [3]), however our motivation has its source in our previous work on partial algebras (see [6], [7], [8]). We were interested in finding properties of injections of a given partial algebra $\mathbf{B}$ into algebras (in the sense of universal algebra) which were called extensions. As there is a large number of extensions of a given partial algebra $\mathbf{B}$ we had to look for some final objects with a rather simple structure. To get such objects we take quotients of the free extension of $\mathbf{B}$ by maximal congruences which do not contain any pair of elements from B. It means that we take B-maximal congruences for the free extensions of $\mathbf{B}$. The subdirect decomposition into a minimal number of subdirectly irreducible factors of such final extensions plays a crucial role in this researching.

In this paper, we consider subdirect decomposition, too. Birkhoff's Theorem (Theorem 1) states that every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras. We show in Theorem 4 that, under the assumption that the congruence lattice is atomic, it is possible to reduce the number of subdirectly irreducible factors to the number of atoms. Finally, Theorem 5 states that if there exists a finite relevant set with minimal cardinality n, then it is possible to reduce the number of subdirectly irreducible factors to at most $n-1$.

1.1. **Preliminaries.** An algebra $\mathbf{A}$ of type F is an ordered pair (A, F), where A is a nonempty set and F is a finite family of finitary operations on A. An algebra $\mathbf{A}$ is trivial if $|A| = 1$.

A binary relation θ on A is called a congruence on an algebra $\mathbf{A}$ of type F if it is an equivalence relation on A satisfying the compatibility property, i.e. for each n-ary function $f \in F$ and elements $a_i, b_i \in A$ if $(a_i, b_i) \in \theta$ holds for $i = 1, \ldots, n$, then $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \theta$. The set of all congruences on an algebra $\mathbf{A}$ is denoted by $Con\mathbf{A}$. It is known that $(Con\mathbf{A}, \subseteq)$ is a complete algebraic lattice which is called the congruence lattice of $\mathbf{A}$, and is denoted by $\mathbf{ConA}$.

For an algebra $\mathbf{A}$ and $a_1, \ldots, a_n \in A$ let $\theta(a_1, \ldots, a_n)$ denote the congruence generated by $\{(a_i, a_j) : 1 \leq i, j \leq n\}$, i.e. the smallest congruence such that $a_1, \ldots, a_n$ are in the same equivalence class. A congruence $\theta(a_1, a_2)$ is called a principal congruence. For arbitrary $X \subseteq A$, let $\theta(X)$ be the congruence generated by $X \times X$.

Compact elements in $\mathbf{ConA}$ are just finitely generated congruences, and at the same time, joins of a finite number of principal congruences. The least element, denoted by Δ_A, is the identity relation on A and the greatest element, denoted by ∇_A, is the full relation $A \times A$.

A congruence is an atom in the congruence lattice of $\mathbf{A}$ if it covers Δ_A. A nontrivial algebraic lattice is atomic if every element but the least contains an atom. Minimal elements in $Con\mathbf{A} \setminus \Delta_A$ are atoms in $\mathbf{ConA}$. Every atom is a principal congruence.

A congruence is called maximal if it is a maximal element in $Con\mathbf{A} \setminus \nabla_A$.

Some properties of the congruence lattice of $\mathbf{A}$ are correlated to properties of $\mathbf{A}$.

An algebra $\mathbf{A}$ is

(1) trivial iff $\Delta_A = \nabla_A$,

(2) simple iff $\Delta_A \neq \nabla_A$ and $Con\mathbf{A} = \{\Delta_A, \nabla_A\}$,
(3) subdirectly irreducible iff **ConA** is atomic and there is exactly one atom called monolith.

We will use the fact that if $\theta = \bigcap\{\theta_i \colon i \in I\}$, then $\mathbf{A}/\theta$ is a subdirect product of the algebras $\mathbf{A}/\theta_i$, $i \in I$. Hence if $\Delta_A = \bigcap\{\theta_i \colon i \in I\}$, then $\mathbf{A}$ is a subdirect product of algebras $\mathbf{A}/\theta_i$, $i \in I$. Some proofs will need The Correspondence Theorem. Before looking at this theorem let us recall that for any $\theta \in Con\mathbf{A}$ the closed interval $[\theta, \nabla_A] = \{\phi \in Con\mathbf{A} \colon \theta \subseteq \phi\}$ is a sublattice of **ConA**. The Correspondence Theorem states that the mapping h defined on $[\theta, \nabla_A]$, where $\theta \in Con\mathbf{A}$ for some algebra $\mathbf{A}$, by $h(\phi) = \phi/\theta$ is a lattice isomorphism from $[\theta, \nabla_\mathbf{A}]$ to $\mathbf{ConA}/\theta$. We will also use the proof of Birkhoff's Theorem (see [1]) using original notation, so we quote the proof from [2] in details.

Theorem 1. *Every algebra* $\mathbf{A}$ *is isomorphic to a subdirect product of subdirectly irreducible algebras (which are homomorphic images of* A*).*

Proof. As trivial algebras are subdirectly irreducible we only need to consider the case of nontrivial $\mathbf{A}$. For $a, b \in A$ with $a \neq b$, we can find a congruence $\theta_{a,b}$ on $\mathbf{A}$ which is maximal with respect to the property $(a, b) \notin \theta_{a,b}$. Then clearly $\theta(a, b) \vee \theta_{a,b}$ is the smallest congruence in $[\theta_{a,b}, \nabla_A] \setminus \{\theta_{a,b}\}$, so $\mathbf{A}/\theta_{a,b}$ is subdirectly irreducible. As $\bigcap\{\theta_{a,b} \colon a \neq b\} = \Delta_A$ we can show that $\mathbf{A}$ is subdirectly embeddable in the product of the indexed family of subdirectly irreducible algebras $\mathbf{A}/\theta_{a,b}$, $a \neq b$. $\square$

For more facts concerning universal algebra or lattice theory see [2], [4].

2. X-maximal congruences

Definition 1. *Let* $\mathbf{A}$ *be a nontrivial algebra and let* $X \subseteq A$ *be a nonempty subset. We say that a congruence* θ *separates* X *iff* $\theta \cap X^2 = \Delta_X$*. And a congruence* $\theta \in Con\mathbf{A} \setminus \nabla_\mathbf{A}$ *is called* X*-maximal for* $\mathbf{A}$ *iff* θ *is a maximal congruence that separates* X*.*

Notice that if $\mathbf{A}$ is trivial, then $|Con\mathbf{A}| = 1$ and thus there exists no set X such that any congruence is X-maximal.

Notice also that by Zorn's lemma, for any chain Γ of congruences separating X, the sum $\bigcup\Gamma$ is an upper bound for Γ and separates X. It is worth also noting that for every congruence $\theta \in Con\mathbf{A} \setminus \nabla_\mathbf{A}$ there exists a nonvoid set that is separated by θ. Thus an X-maximal congruence is well defined.

Example 1. *Let* $\mathbf{A}$ *be a nontrivial algebra. Then*
(1) Δ_A *is* A*-maximal for* $\mathbf{A}$*,*

(2) *if* $\mathbf{A}$ *is simple, then for any nonempty* $X \subseteq A$, $\Delta_{\mathbf{A}}$ *is* X*-maximal for* $\mathbf{A}$.

Proposition 1. *If* θ *is* X*-maximal for* $\mathbf{A}$ *and* $|X| = 1$, *then* θ *is a maximal element in* $Con\mathbf{A} \setminus \nabla_{\mathbf{A}}$. *Hence* $\mathbf{A}/\theta$ *is simple.*

Proposition 2. *If a congruence* θ *is* $\{a, b\}$*-maximal for a nontrivial* $\mathbf{A}$, *and* $a \neq b$, $a, b \in A$, *then* $\mathbf{A}/\theta$ *is subdirectly irreducible.*

Proof. The proof is based on the observation that θ is a maximal congruence separating $\{a, b\}$, so $\theta = \theta_{a,b}$, where $\theta_{a,b}$ is as in the proof of Theorem 1. □

The next proposition follows directly from maximality:

Proposition 3. *Let* $\mathbf{A}$ *be a nontrivial algebra and let* $X \subseteq A$, $|X| \geq 2$ *and* $\theta \in Con\mathbf{A} \setminus \nabla_{\mathbf{A}}$. *Then the following conditions are equivalent:*

(1) θ *is* X*-maximal for* $\mathbf{A}$,

(2) θ *separates* X *and for any* $\varphi \in Con\mathbf{A}$ *if* $\varphi \supset \theta$, *then* φ *does not separate* X. *It means that there exist* $x \neq y$, $x, y \in X$ *such that* $(x, y) \in \varphi$.

Proposition 4. *Let* $\mathbf{A}$ *be a nontrivial algebra and let* $X, Y \subseteq A$ *be nonempty sets. Then*

(1) *if* θ *is* X*-maximal for* $\mathbf{A}$, $X \subseteq Y$ *and* θ *separates* Y, *then* θ *is* Y*-maximal for* $\mathbf{A}$,

(2) *if* Δ_A *is* X*-maximal for* $\mathbf{A}$, *then* Δ_A *is* Y*-maximal for* $\mathbf{A}$ *for every* $Y \supseteq X$,

(3) *if* θ *is* X*-maximal for* $\mathbf{A}$ *and* $Y \supseteq X$, *then every congruence* $\varphi \supset \theta$ *does not separate* Y.

3. Relevant sets and atoms in congruence lattices

Definition 2. *Let* $\mathbf{A}$ *be a nontrivial algebra and let* $X \subseteq A$ *be a nonempty subset. We say that* $\mathbf{A}$ *is* X*-maximal iff* Δ_A *is* X*-maximal for* $\mathbf{A}$. *The set* X *such that* $\mathbf{A}$ *is* X*-maximal is called a relevant set for* $\mathbf{A}$. *If* X *is a relevant set for* $\mathbf{A}$, *then we define the set* $PCon\,(X) = \{\theta\,(x, y) : x \neq y,\ x, y \in X\}$ *consisting of appropriate principal congruences.*

The following simple fact follows directly from the definition.

Proposition 5. *If* X *is a relevant set for* $\mathbf{A}$ *and* $|X| \geq 2$, *then for every* $\theta \in Con\mathbf{A} \setminus \Delta_{\mathbf{A}}$ *there is* $\theta\,(x, y) \in PCon\,(X)$ *such that* $\theta\,(x, y) \subseteq \theta$.

Proof. As $\Delta_A \subset \theta$ is X-maximal for $\mathbf{A}$ and Proposition 3 holds, we conclude that there exist $x \neq y$, $x, y \in X$ such that $(x, y) \in \theta$. Hence $\theta\,(x, y) \subseteq \theta$. □

The next fact follows from The Correspondence Theorem.

Proposition 6. *Let* $\mathbf{A}$ *be a nontrivial algebra and let* $X \subseteq A$ *be a nonempty subset and* $\theta \in Con\mathbf{A} \setminus \nabla_A$*. Then* θ *is* X*-maximal for* $\mathbf{A}$ *iff* $\mathbf{A}/\theta$ *is* X_θ*-maximal, where* $X_\theta = \{[x]_\theta : x \in X\}$*.*

Notice that Example 1 shows that the carrier set A for any algebra $\mathbf{A}$ is relevant for $\mathbf{A}$, and also, that every nonempty subset of A is relevant for a simple algebra $\mathbf{A}$. Moreover, Proposition 4(2) states that any superset of a relevant set for a given algebra is also relevant. Hence it is worthwhile to consider minimal (under inclusion) sets relevant for $\mathbf{A}$. It turns out that it is possible to obtain minimal relevant sets of different finite cardinalities. On the other side, it is not possible to have a finite minimal relevant set and an infinite minimal relevant set in the same algebra, what follows from Corollary 1 and Theorem 2.

Example 2. *Let* $n = \{0, \dots, n-1\}$ *and let* $\mathbf{C_n} = (n, s)$ *be a unary algebra with* $s \colon n \to n$ *defined as* $s(k) = (k+1) \bmod n$*. Then*
$Con\mathbf{C_n} = \{\theta(0,d) : d \,\text{divides}\, n\}$ *and* $(a,b) \in \theta(0,d)$ *iff* $(a = b) \bmod d$*.*
If $n = p_1^{\alpha_1} \cdot \ldots \cdot p_m^{\alpha_m}$ *is the prime factorization with* $p_1 < \ldots < p_m$*, then* $\mathbf{ConC_n}$ *has exactly* m *atoms:* $\theta(0, n/p_1), \dots, \theta(0, n/p_m)$*. The set* $\{0, n/p_1, \dots, n/p_m\}$ *is a minimal relevant set for* $\mathbf{C_n}$*. Moreover, every minimal relevant set for* $\mathbf{C_n}$ *has at least* $m+1$ *and at most* $2m$ *elements.*

Example 3. (1) *The sets* $\{0,3,5\}$*,* $\{1,2,4,12\}$ *and* $\{1,3,7,8\}$ *are minimal relevant sets for* C_{15}*.*

(2) *Let* $\mathbf{A} = \mathbf{Z_2} \times \mathbf{Z_2}$*, where* $\mathbf{Z_2}$ *is the two-element group. Here every relevant set has at least three elements and* $\{(0,0), (0,1), (1,1)\}$*,* $\{(0,0), (1,0), (1,1)\}$ *are minimal relevant sets for* $\mathbf{A}$*, and* $\mathbf{ConA}$ *has three atoms.*

(3) *If* $\mathbf{A}$ *is subdirectly irreducible, then* $\mathbf{ConA}$ *has exactly one atom* $\theta(a,b)$*. Then the set* $\{a,b\}$ *is a minimal relevant set for* $\mathbf{A}$*.*

The above examples show that there is a relationship (which is not straightforward) between the cardinality of minimal relevant sets for $\mathbf{A}$ and the number of atoms in $\mathbf{ConA}$.

Proposition 7. *If a set* X *is relevant for* $\mathbf{A}$ *and* $\varphi \in Con\mathbf{A}$ *is an atom in* $\mathbf{ConA}$*, then there are* $x \neq y$*,* $x, y \in X$ *such that* $\varphi = \theta(x,y)$*. Moreover,* $\theta(x,y)$ *is a minimal element in* $PCon(X)$*.*

Proof. Let $\varphi \in Con\mathbf{A}$ be an atom in $\mathbf{ConA}$. Thus $\varphi \neq \Delta_A$ and by Proposition 5 there are $x \neq y$, $x, y \in X$ such that $(x,y) \in \varphi$. Hence $\theta(x,y) \in PCon(X)$ and $\theta(x,y) \subseteq \varphi$ and hence $\varphi = \theta(x,y)$ by assumption that φ is an atom in $\mathbf{ConA}$. Obviously, φ as an atom is minimal in $PCon(X)$. $\square$

Proposition 8. *Let* $\mathbf{A}$ *be a nontrivial algebra and let* $X \subseteq A$ *be a relevant set for* $\mathbf{A}$ *and let* $\theta(x,y)$ *be a minimal element in* $PCon(X)$. *Then* $\theta(x,y)$ *is an atom in* $\mathbf{ConA}$.

Proof. Let $\Delta_A \subset \varphi \subseteq \theta(x,y)$. Then $\theta(a,b) \subseteq \varphi$ for some $a \neq b$, $a,b \in X$. Hence by minimality of $\theta(x,y)$ we get that $\theta(a,b) = \varphi = \theta(x,y)$ and hence $\theta(x,y)$ is an atom in $\mathbf{ConA}$. $\square$

Example 4.

1. *Let* $\mathbf{A} = \mathbf{C_{15}}$ *(Example 2). Then* $\theta(0,3)$ *and* $\theta(0,5)$ *are all the atoms in* $\mathbf{ConA}$. *For the relevant set* $X = \{0,3,5\}$ *we see that* $PCon(X) = \{\theta(0,3), \theta(0,5), \theta(3,5)\}$.
   Then $\theta(3,5) = \nabla_{\mathbf{A}}$ *and* $\theta(0,3), \theta(0,5)$ *are minimal in* $PCon(X)$.
   For the relevant set $X = \{1,3,7,8\}$ *we see that*
   $PCon(X) = \{\theta(1,3), \theta(1,7), \theta(1,8), \theta(3,7), \theta(3,8), \theta(7,8)\}$.
   Then $\theta(1,3) = \theta(7,8) = \theta(1,8) = \theta(3,7) = \nabla_{\mathbf{A}}$ *and*
   $\theta(3,8) = \theta(0,5)$, $\theta(1,7) = \theta(0,3)$ *are minimal in* $PCon(X)$.
2. *Let* $\mathbf{A} = \mathbf{Z_2} \times \mathbf{Z_2}$. *Then*
   $\theta((0,0),(0,1))$, $\theta((0,0),(1,0))$, $\theta((0,0),(1,1))$ *are all the atoms in* $\mathbf{ConA}$. *For the relevant set* $\{(0,0),(0,1),(1,1)\}$ *(Example 3 (2)) we see that*
   $PCon(X) = \{\theta((0,0),(0,1)), \theta((0,0),(1,1)), \theta((0,1),(1,1))\}$
   and each of its elements is an atom.
   Notice that $\theta((0,0),(1,0)) = \theta((0,1),(1,1))$.
3. *Let* $\mathbf{A} = \mathbf{C_{30}}$. *Then* $\theta(0,6), \theta(0,10), \theta(0,15)$ *are all the atoms in* $\mathbf{ConA}$. *For the relevant set* $\{0,6,10,15\}$ *we find* $PCon(X) = \{\theta(0,6), \theta(0,10), \theta(0,15), \theta(6,10), \theta(6,15), \theta(10,15)\}$. *Here, the first three elements are atoms, and the last three are not atoms.*

Proposition 9. *If* X *is a relevant set for an algebra* $\mathbf{A}$ *and every congruence* $\theta \in PCon(X)$ *contains a minimal element from* $PCon(X)$, *then* $\mathbf{ConA}$ *is atomic with the set of atoms consisting of all the minimal elements from* $PCon(X)$.

Proof. By Proposition 5 every nontrivial congruence θ contains a congruence from $PCon(X)$ and by assumption it contains a minimal element from $PCon(X)$. Finally, by Proposition 8 it contains an atom. $\square$

Corollary 1. *If an algebra* $\mathbf{A}$ *has a finite relevant set, then* $\mathbf{ConA}$ *is an atomic lattice with a finite number of atoms.*

Propositions 7, 8 and the last corollary yield that if X is a finite relevant set for an algebra $\mathbf{A}$, then the number of atoms in $\mathbf{ConA}$ is not greater than the number of two-element subsets of X, i.e. $\binom{|X|}{2}$. We describe the lower bound on the number of atoms in Theorem 2.

Proposition 10. *If* **ConA** *is atomic with* $At = \{\theta(a_i, b_i) : i \in I\}$*, then* $\bigcup\{a_i, b_i\}$ *is a relevant set for* **A**.

Proof. If **ConA** is atomic, then every nontrivial congruence contains an atom. Thus it contains a pair of generators (a_i, b_i) of this atom. □

Corollary 2. *If* **ConA** *is atomic, then there is a relevant set for* **A** *of cardinality at most* $2|At|$.

The above facts lead to the following conclusion:

Corollary 3. *Let* **A** *be a nontrivial algebra. Then* **ConA** *is atomic with a finite set of atoms iff there exists a finite relevant set for* **A**.

In the next examples we show algebras with no finite relevant sets, and we also answer the natural question if atomicity of the congruence lattice is equivalent to minimality of a relevant set. The answer is 'no'.

Example 5. *Let* $\mathbf{C} = (Z, s)$ *be the set of integers with a unary successor operation. Then* $Con\mathbf{C}$ *has no atoms since every principal congruence is of the form* $\theta(m, n), m < n$*, and then* $\theta(n, n + (n - m)) \subset \theta(m, n)$*, so every principal congruence properly includes other principal congruence. From this we conclude that there is no finite relevant set for* **C**. *However, notice that there are different infinite relevant sets, for example, all the integers* Z *and all the even integers* $2Z$.

Example 6. *Let* $\mathbf{N} = (N, s)$ *be the set of natural numbers with a unary successor operation. Then, as in the above example,* $Con\mathbf{N}$ *has no atom and there is no finite relevant set for* **N**.

The algebra **C** in the previous example has an atomless congruence lattice. In the next example we have an algebra for which the congruence lattice has exactly one atom.

Example 7. *Let* $\mathbf{A} = \mathbf{C}\dot{\cup}\mathbf{C_2}$ *be the disjoint sum of algebras as defined above. Then* **ConA** *is not atomic, but has one atom* $\theta(0_2, 1_2)$*, where* $0_2, 1_2$ *are elements from* $\mathbf{C_2}$. *There is no finite relevant set for* **A** *but every relevant set contains elements* $0_2, 1_2$.

The following example shows that there exists a minimal relevant set but the congruence lattice is not atomic. Hence the existence of minimal relevant set does not yield atomicity of congruence lattice. Examples 9, 10 show algebras with atomic congruence lattice and minimal relevant sets.

Example 8. *Let* $\mathbf{A} = (N\dot{\cup}N', s)$ *be the disjoint sum of two copies of naturals with a unary operation* s *such that* s *is the ordinary successor operation on the first copy* N*, i.e.* $s(n) = n + 1$*, and for the second copy* N',

$s(n') = n+1$, where $n \in N$ and $n' \in N'$. Then **ConA** *is not atomic, but has infinitely many atoms $\theta(n,n')$. The minimal relevant set for* **A** *is equal to $N\dot{\cup}N'$ and obviously $|N\dot{\cup}N'| = |At|$.*

Example 9. *Let $A = \{\left(\dot{\bigcup}\{-\mathbf{n}\}\right)\dot{\cup}\{0\} \colon n \in N \setminus \{0\}\}$ be the disjoint sum of sets $-\mathbf{n} = \{-n_n, (-n+1)_n, \ldots, -1_n\}$ and zero, and let $\mathbf{A} = (A; s)$. The unary operation s is defined as $s(0) = 0$, $s(a_i) = (a+1)_i$ if $a_i < -1_i$ and $s(a_i) = 0$ in opposite case. Then principal congruences are of the form of $\theta(a_n, b_m)$, atoms are of the form of $\theta(-1_n, 0)$ for $n \in N \setminus \{0\}$ and $\theta(-1_n, -1_m)$ for $n \neq m$, $n, m \in N \setminus \{0\}$. The minimal relevant set for* **A** *is equal to $\{0\} \cup \{-1_n \colon n \in N \setminus \{0\}\}$.*

In the next example we have an uncountable algebra with a countable set of atoms and a countable minimal relevant set. Moreover, we construct a descending chain of relevant sets such that the meet is not relevant.

Example 10. *Let $\mathbf{A} = \prod \mathbf{Z_2}$ be a countable power of the two element group. The elements of* **A** *are binary countable sequences and for any $\underline{a} \in A$ we see that $\underline{a}+\underline{a} = \underline{0}$ where $\underline{0} = (0,0,\ldots,0)$. Every principal congruence is an atom since every congruence class is two-element. Hence there are uncountably many atoms. Moreover, for any $\underline{a}, \underline{b} \in A$ we see that $\theta(\underline{a}, \underline{b}) = \theta(\underline{0}, \underline{a}+\underline{b})$ and $\theta(\underline{0}, \underline{a}) = \theta(\underline{b}, \underline{a}+\underline{b})$.*

We show now that there exists a set X which is a minimal relevant set for **A**. *It consists of $\underline{0}$ and all the sequences with 1 on the first coordinate. The cardinality of X is equal to the cardinality of the set of atoms according to Theorem 2. Let us prove that X is a minimal relevant set. We show that X is relevant and for any $\underline{a} \in \mathbf{A}$ the set $X_{\underline{a}} = X \setminus \{\underline{a}\}$ is not relevant. To prove the first assertion let us take any principal congruence $\theta(\underline{0}, \underline{a})$. If $a_1 = 1$ then, obviously, $\underline{0}, \underline{a} \in X$. If $a_1 = 0$, then take $\underline{a'} = (1, a_2, a_3, \ldots)$ and notice that $\theta(\underline{0}, \underline{a}) = \theta(\underline{a'}, \underline{a}+\underline{a'})$ and $\underline{a'}, \underline{a}+\underline{a'} \in X$. So, X is relevant.*

Let now $\underline{a} \in X$ and take $X_{\underline{a}}$. If $\underline{a} = \underline{0}$, then $\theta(\underline{0}, \underline{1})$ cannot be generated by any pair of elements from $X_{\underline{0}}$. For any $\underline{b} \in X_{\underline{0}}$ it holds that $\underline{0}+\underline{b} \in X_{\underline{0}}$ and $\underline{1}+\underline{b} \notin X_{\underline{0}}$. Analogously, if $\underline{a} \neq \underline{0}$, then $\theta(\underline{0}, \underline{a})$ cannot be generated by any pair of elements from $X_{\underline{a}}$.

Notice also that there exists an infinite descending chain of relevant sets, the intersection of which is not relevant. To construct it let C consist of all the sequences with a finite number of $1's$ and let $X_n = C \cup B_n$, where $B_n = \{\underline{b} \colon b_i = 0 \, for \, i < n, \, b_n = 1\}$. It is easy to check that every X_n is relevant and $X_1 \supset X_2 \supset X_3 \supset \ldots$, $|X_n| > |N|$, $|\bigcap X_n| = |C| = |N|$.

Notice that examples in Example 3 show that relevant sets for a given algebra can be of different cardinality. It follows from the fact that to obtain a relevant set we can choose any pair of generators of every atom.

Proposition 10 shows the relationship between atomicity of the congruence lattice and existence of a relevant set of cardinality limited by the cardinality of the set of atoms. The next theorem shows that the existence of a minimal relevant set forces the existence of atoms with limited cardinality. It would be desirable to show that the existence of a minimal relevant set forces atomicity of the congruence lattice but we have not been able to do this.

Theorem 2. *Let* $\mathbf{A}$ *be a nontrivial algebra and let* At *denote the set of atoms in its congruence lattice. If there exists a minimal relevant set* X *for* $\mathbf{A}$ *then*

(1) *if* X *is finite, then* $\mathbf{ConA}$ *is atomic with a finite number of atoms and* $|At| \geq \frac{|X|}{2}$,
(2) *if* X *is infinite, then* $|At| = |X|$.

Proof. The finite case follows from Corollaries 1, 2, 3. We give here a common proof for both finite and infinite cases. Let X be a minimal relevant set for $\mathbf{A}$ and $a \in X$. Then $X_a = X \setminus \{a\}$ is not relevant, so there is a congruence $\varphi \neq \nabla_{\mathbf{A}}$ separating X_a and not separating X. Hence there is $b \in X_a$ such that $\theta(a,b) \subseteq \varphi$. Moreover, $\theta(a,b)$ is minimal in $PCon(A)$, for if $c \neq d$, $c,d \in X_a$, then if $\theta(c,d) \subset \theta(a,b) \subseteq \varphi$, then φ does not separate X_a.

Finally, for every $a \in X$ there exists an atom $\theta(a,b), b \in X_a$, so we have at least $\frac{|X|}{2}$ atoms if $|X|$ is finite and even, and $\frac{|X|+1}{2}$ atoms if $|X|$ is finite and odd. When X is infinite we have $|X|$ atoms. □

Corollary 4. *If* $\mathbf{ConA}$ *is not atomic and the set of atoms is finite, then a minimal relevant set for* $\mathbf{A}$ *does not exist.*

Proof. If a minimal relevant set exists and is finite, then $\mathbf{ConA}$ is atomic, and if this set is infinite, then $\mathbf{ConA}$ has infinitely many atoms. □

4. η-MAXIMAL ALGEBRAS

In this section we consider minimality of relevant sets under cardinality.

Definition 3. *Let* $\mathbf{A}$ *be a nontrivial algebra. A congruence* $\theta \in Con\mathbf{A} \setminus \nabla_{\mathbf{A}}$ *is* η*-maximal for* $\mathbf{A}$ *iff* $\eta = min\{|X| : X \subseteq A, X \neq \emptyset,\ \theta\ is\ X{-}maximal\ for\ \mathbf{A}\}$. *We say that* $\mathbf{A}$ *is* η*-maximal if* Δ_A *is* η*-maximal for* $\mathbf{A}$.

Proposition 11. *A nontrivial algebra* $\mathbf{A}$ *is* 1*-maximal iff* $\mathbf{A}$ *is simple.*

Proof. If $\mathbf{A}$ is 1-maximal, then $\Delta_{\mathbf{A}}$ is $\{a\}$-maximal for some $a \in A$, so by Proposition 1 we get that $\mathbf{A}/\Delta_{\mathbf{A}} \cong \mathbf{A}$ is simple. If $\mathbf{A}$ is simple with $|A| \geq 2$, then $\Delta_{\mathbf{A}}$ is X-maximal for every nonempty $X \subseteq A$, in particular for $|X|=1$. □

Proposition 12. *A nontrivial algebra* $\mathbf{A}$ *is 2-maximal iff it is subdirectly irreducible but not simple.*

Proof. If $\mathbf{A}$ is 2-maximal, then there exists a two-element relevant set $\{a, b\}$. By Proposition 2 $\mathbf{A}$ is subdirectly irreducible and it is not simple for it is not 1-maximal.

Assume that $\mathbf{A}$ is not simple and η-maximal. Then $|\eta| \geq 2$. Moreover, there exists a two-element relevant set for $\mathbf{A}$ because $\mathbf{A}$ is subdirectly irreducible. Hence $\eta = 2$. □

The next theorem is analogous to Proposition 6.

Theorem 3. *A congruence* θ *is* η*-maximal for* $\mathbf{A}$ *iff* $\mathbf{A}/\theta$ *is* η*-maximal.*

Proof. $\mathbf{A}/\theta$ is simple for $\eta = 1$ by Proposition 1 so it is 1-maximal by Proposition 11. $\mathbf{A}/\theta$ is subdirectly irreducible for $\eta = 2$ so, by Proposition 12, $\mathbf{A}/\theta$ is 2-maximal.

Assume now that θ is η-maximal, $\eta > 2$ and $\mathbf{A}/\theta$ is ξ-maximal. Then there exists a relevant set X of cardinality η. By Proposition 6 $\mathbf{A}/\theta$ is X/θ-maximal, but θ separates X, so $|X/\theta| = |X| = \eta$. Hence $\xi \leq \eta$.

On the other side, if $\mathbf{A}/\theta$ is ξ-maximal, then there is a set Y/θ relevant for $\mathbf{A}/\theta$ and $|Y/\theta| = \xi$. Thus $|Y/\theta| = |Y'| = \xi$, where Y' is the set of selected elements (one from each congruence class): $y' \in [y]_\theta$, $y \in Y$. Thus $\eta \leq \xi$ and, in consequence, $\eta = \xi$. □

Example 11. *Based on the previous examples notice that* $\mathbf{C_n}$ *is* $m+1$*-maximal, where* m *is a number of prime factors of* n. $\mathbf{Z_2} \times \mathbf{Z_2}$ *is* 3*-maximal. Algebras from Examples 5, 7, 8, 9 are* $|N|$*-maximal, wherein the last one is uncountable.*

Let us consider now finite maximality, to emphasize this finiteness we use symbol n instead of η. The following fact is a consequence of Proposition 10 and Theorem 2.

Proposition 13. *Let* $\mathbf{A}$ *be a nontrivial* n*-maximal algebra. Then* $Con\mathbf{A}$ *is atomic with a finite set of atoms and the following inequalities hold:*
$\left[\frac{n-1}{2}\right] + 1 \leq k \leq \binom{n}{2}$, *where* $k = |At|$.

Proof. Let X be a relevant set for $\mathbf{A}$ and $|X| = n$. Then X is a minimal relevant set, so by Proposition 2 we get the lower bound. The upper bound follows from Proposition 8, since $At \subseteq PCon\,(X)$. □

The next examples show that we cannot get better bounds.

Example 12. (1) *If* $\mathbf{A}$ *is a disjoint sum of* n *trivial mono-unary algebras, then* $\mathbf{A}$ *is* n*-maximal and every pair of two different elements from* A *generates a new atom. Hence* $k = \binom{n}{2}$.

(2) $\mathbf{Z_2} \times \mathbf{Z_2}$ *is* 3*-maximal and has* 3 *atoms. Hence* $\binom{n}{2} = \binom{3}{2} = 3 = k$.
(3) $\mathbf{C_{15}}$ *is* 3*-maximal and has* 2 *atoms, so* $n = 3, k = 2$. *Hence* $\left[\frac{n-1}{2}\right] + 1 = \left[\frac{3-1}{2}\right] + 1 = 2 = k$.
(4) *Any non-simple subdirectly irreducible algebra is* 2*-maximal and has one atom. Hence* $\left[\frac{2-1}{2}\right] + 1 = 1$.
(5) $\mathbf{C_{30}}$ *is* 4*-maximal and has* 3 *atoms, so* $n = 4, k = 3$. *Hence* $\left[\frac{n-1}{2}\right] + 1 = \left[\frac{4-1}{2}\right] + 1 = 2 < k$ *and* $\binom{n}{2} = \binom{4}{2} = 6 > k$.

Corollary 5. *Under assumptions from the last proposition we get that* $\lceil \frac{1+\sqrt{1+8k}}{2} \rceil \leq n \leq 2k$, *where* $\lceil\,\rceil$ *denotes rounding up to the nearest integer.*

Proof. It is enough to show the left inequality. Notice that $k \leq \binom{n}{2} = \frac{n(n-1)}{2}$ and hence $n^2 - n - 2k \geq 0$. Solving the last inequality we get that $n \geq \frac{1+\sqrt{1+8k}}{2}$. □

Theorem 4. *If* **ConA** *is atomic with* $|At| = \kappa \geq 2$, *then* **A** *is a subdirect product of* κ *subdirectly irreducible algebras. Moreover, this decomposition is proper, i.e. there is no isomorphic copy of* **A** *in this product.*

Proof. Let $\kappa \geq 2$ and $At = \{\theta(a_i, b_i), i \in I\}$ and for every $i \in I$ let θ_{a_i,b_i} be a maximal congruence separating $\{a_i, b_i\}$. Then as in Birkhoff's Theorem $\mathbf{A}/\theta_{a_i,b_i}$ is subdirectly irreducible and $\bigcap\{\theta_{a_i,b_i} : i = 1, \ldots, k\} = \Delta_A$. Hence **A** is a subdirect product of subdirectly irreducible algebras $\mathbf{A}/\theta_{a_i,b_i}$. Moreover, **ConA** is atomic, so $\theta_{a_i,b_i} \neq \Delta_A$ and hence this decomposition is proper. □

For a finite numbers of atoms $\kappa = k$ this theorem gives a subdirect decomposition into $k = |At|$ factors. By Proposition 13 $k \leq \binom{n}{2}$, so we have at most $\binom{n}{2}$ factors for any n-maximal algebra. The next theorem limits the number of factors to $n - 1$.

Theorem 5. *If* **A** *is* n*-maximal with* $n \geq 2$, *then* **A** *is a subdirect product of at most* $n - 1$ *subdirectly irreducible algebras.*

Proof. If $n = 2$, then **A** is subdirectly irreducible and $2 - 1 = 1$ so the assertion is true. Assume that the assertion is true for some fixed $n \geq 2$. Let **A** be $n+1$-maximal. Then there is a minimal relevant set $X \subseteq A$ such that $|X| = n + 1$. As in the proof of Theorem 10 for any $a \in X$ we take $X_a = X \setminus \{a\}$. Then there exists a congruence $\varphi \neq \Delta_A$ such that φ separates X_a. Let Φ be a maximal congruence with this property, i.e. Φ is X_a-maximal. Then there exists $b \in X_a$ such that $(a, b) \in \Phi$ and $\theta(a, b)$ is an atom in $Con\mathbf{A}$. Then $\Phi \cap \theta_{a,b} = \Delta_A$ and hence **A** is a subdirect product of $\mathbf{A}/\Phi$ and $\mathbf{A}/\theta_{a,b}$ wherein the last algebra is subdirectly irreducible. Since

Φ is X_a-maximal and $|X_a| = n$, we get by inductive assumption that $\mathbf{A}/\Phi$ is m-maximal for some $m \leq n$. By induction $\mathbf{A}/\Phi$ is a subdirect product of at most $n-1$ subdirectly irreducible algebras. Finally, $\mathbf{A}$ is a subdirect product of at most $n-1+1=n$ subdirectly irreducible algebras. $\square$

Notice that if κ from Theorem 4 is infinite, then $\mathbf{A}$ is κ-maximal, so we can formulate a generalization of the last theorem:

Corollary 6. *If $Con\mathbf{A}$ is atomic and $\mathbf{A}$ is κ-maximal, then $\mathbf{A}$ is a subdirect product of κ subdirectly irreducible algebras.*

References

[1] Birkhoff, G.: On the structure of abstract algebras, Proc. Cambridge Philos. Soc. **31**, 433-454, (1935). DOI: 10.1017/S0305004100013463
[2] Burris, S.; Sankappanavar, H. P. A Course in Universal Algebra, Springer-Verlag, (1981)
[3] Grätzer, G.: The order of principal congruences of a bounded lattice. Algebra Universalis. **70(1)**, 95–105 (2013). DOI:10.1007/s00012-013-0242-3
[4] Grätzer, G.: General Lattice Theory. Second edition. Birkhauser Verlag, Basel (1998).
[5] Grätzer, G. , Wehrung F. eds.: Lattice Theory: Special Topics and Applications, Vol. 1. Birkhauser Verlag, Basel (2014)
[6] Staruch Bożena: Extensions of Partial Structures and Their Application to Modelling of Multiagent Systems. In: Monitoring, Security, and Rescue Techniques in Multiagent Systems. Part 1. Advances in Soft Computing Vol. 28., South-Holland, Rotterdam, 293–304 (2005)
[7] Staruch Bożena, Staruch Bogdan: First Order Theories for Partial Models, Studia Logica, **80(1)**, 105–120 (2005). DOI: 10.1007/s11225-005-6779-1
[8] Staruch Bożena, Staruch Bogdan: Maximal congruences for partial structures. Paper presented at the 50th Summer School on General Algebra and Ordered Sets, Matej Bel University in Banska Bystrica, Novy Smokovec, 2–7 September (2012)

Received: October 2016

Bogdan Staruch
University of Warmia and Mazury, Olsztyn, Poland,
Department of Mathematics and Computer Science,
ul. Słoneczna 54, 10-710 Olsztyn, Poland
E-mail address: `bstar@uwm.edu.pl`

Bożena Staruch
University of Warmia and Mazury,
Department of Mathematics and Computer Science,
ul. Słoneczna 54, 10-710 Olsztyn, Poland
E-mail address: `bostar@matman.uwm.edu.pl`