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Investigation of routes on various grids

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Problems in the curriculum have existed since ancient times, if not earlier. For example, an Egyptian mathematical manuscript (the Ahmes Papyrus) consisting of a collection of problems, was written about 1650 B.C. This has not changed in nearly 4000 years.

The central idea of problems has been a common to school mathematics historically and to school mathematics today. Traditionally, when we have spoken about problems we have meant word problems and especially so-called story problems. From our contemporary view, these are only one particular kind of problem. These are problems with the help of which we present pupils with a pseudo – real world. Through this type of problem we present students with a situation or a task and in the form of a question with a goal that the student must achieve. We can say that these are „goal – aimed" problems, or in contemporary terminology closed problems. We require that the student choose some previously learned algorithm (if he/she knows more than one), and with this perform a calculation. They are often placed at the end of a set of algorithmic exercises and with help of them we practise what we have just learned. In many countries today, as in our own, these problems are of central importance, and it is by solving such problems that we can assess the student. Here is a typical example of a story problem when we must calculate with fractions:

Story problem:
On a trip at the end of school year there were 25 pupils. $\frac{3}{5}$ of them are girls. How many girls and how many boys participated in the trip?

A great change occurred in the concept of the problem when we began to consider the relationship between the student and a problem. Reys et al. (1984) argued that „a problem involves a situation in which a person wants something and does not know immediately what to do to get it". In this explanation of a problem, it is very important that the pupil wants to achieve the goal. If for example, we give to our student the task of making a present for his/her mother for Mothers' Day, it is problem for him/her only if they want to do it (in this new approach).

In recent times, we have focused much more on the method by which a problem is solved than on the solution to that problem. We know that in history there were many particular methods of solving problems, for example in the Ahmes papyrus, there is the so-called *Method of false position* (see Kopka (1993)). However, the methods by which problems were solved were not of central importance. Today, this position has changed and methods are now given greater importance and are stressed more than the actual goal of the problem. The main reason for this new position is that one method can often be used to solve many different problems (but we do not advocate unique methods). The relative importance of the goal has also changed with the introduction of the notion of open – ended problems. A problem which is ópen-endedleaves the goal(s) open for the student to decide what it should be. This might mean two students having different goals for the same problem.

Frobisher (1994) suggests that a problem is a situation that has interest and appeal to a child, who therefore wishes to explore the situation more fully in order to gain an understanding of it. Goals arise naturally during the exploration and are determined not by the setter of the problem but by the child. We would like to say that Frobisher's definition of a notion problem is much closer to what we call today *investigation* in school mathematics than to the „task-goal" definition of a problem.

There are many suggestions to explain what investigation is, what is common to a problem and investigation, and what the differences are. Mathematical educators do not agree on these concepts. For example, HMI (1985) stated „in broad terms it is useful to think of problem-solving as being a convergent activity where pupils have to reach a solution to a defined problem, whereas investigative work should be seen as a more divergent activity". This definition is probably interesting from a theoretical point of view, but it is not very useful from practical point of view. At this time we could say

- a problem must be such that the person working on it believes there is a solution
- an investigation looks for strategies which might lead to a generalization.

Let me give you one example of investigations. The base of this example is taken from Kirkby (1986).

Example (investigations of routes on a various grids).
We can develop an investigation based on the study of routes on a square grid. Consider a square grid on which a path can be traced along the horizontal and vertical lines.

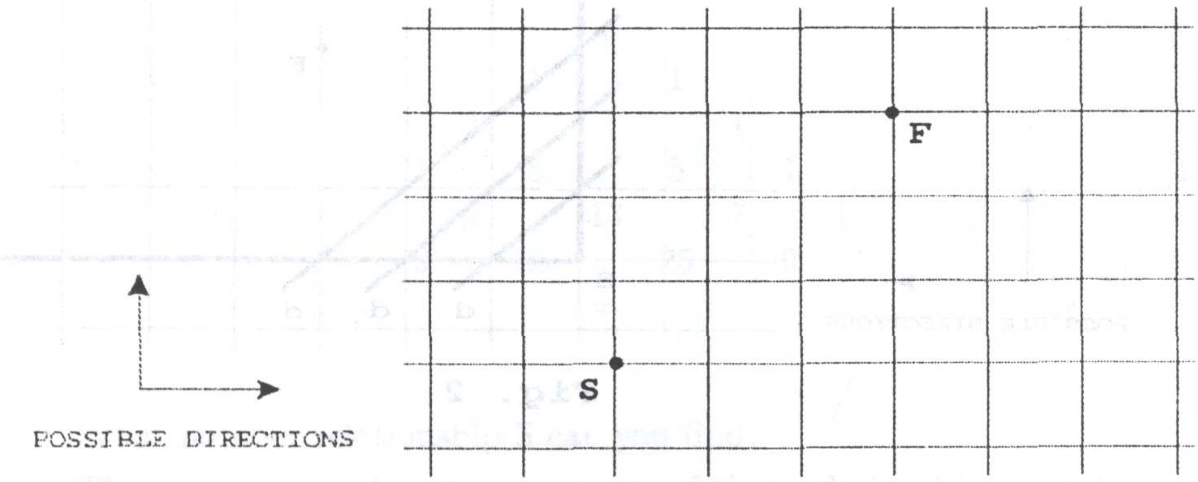

Fig. 1

How many different routes are there from a given point S to given point F? See Fig. 1.
Investigate methods of recording these routes.
How can you be sure that you have found all the routes? (See remark 1, second way)
Move F to a different position – F'. How many routes now?
Investigate the number of different routes for other positions of F.
This produces a pattern known as Pascal's Triangle. Investigate the triangle.

```
                1
            1       1
        1       2       1
    1       3       3       1
1       4       6       4       1
1   5       10      10      5   1
```

Remark 1. The first way of solution: The numbers on the grid can be obtained, step by step, by writing down the numbers at the points along the diagonals $d_0, d_1, d_2, \dots$ marked in Fig.2. As you have just seen, the emerging pattern is the famous Pascal's Triangle.

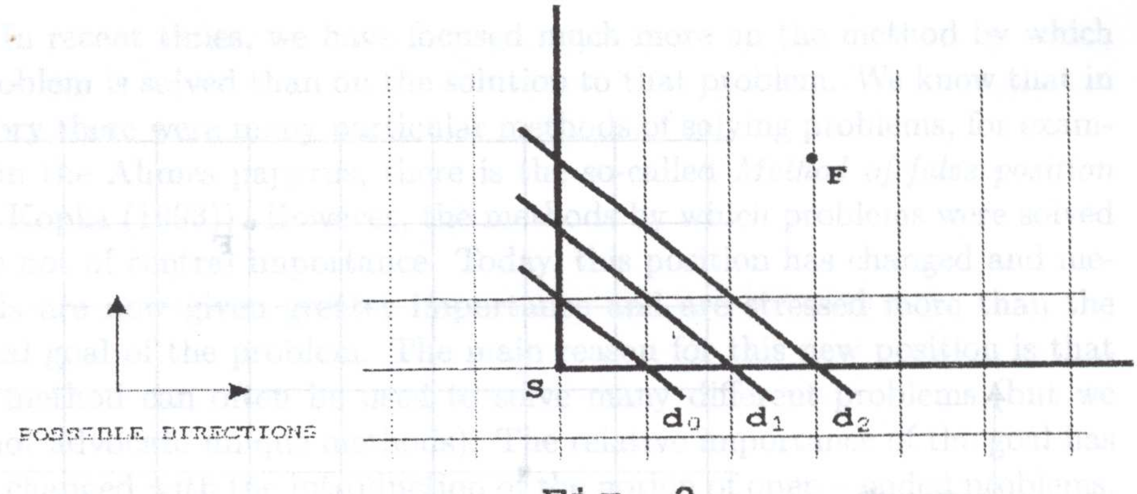

Fig. 2

The second way of solution: The number at each point can be calculated directly as follows: label the horizontal rows and the vertical columns of the grid by $0, 1, 2, \ldots$. Call a step on the grid a move from one point to a neighbouring point. Each routes from S to F in the m-th row and the nth column consists of $m+n$ steps, n of which must be in the horizontal and m in the vertical direction. It is left to us to decide which n of the $m+n$ steps should be horizontal. Therefore: The number of different routes leading from S to F is equal to the number of choice of n out of $m+n$ steps. It follows that the number in the m-th row and the n-th column is $\frac{(m+n)!}{m!n!}$. But we have taken the first way of solution and we are going to use it in the following part, too.We can continue.

Investigate routes on a different grid. See Fig. 3. How many different routes are there from S to F?

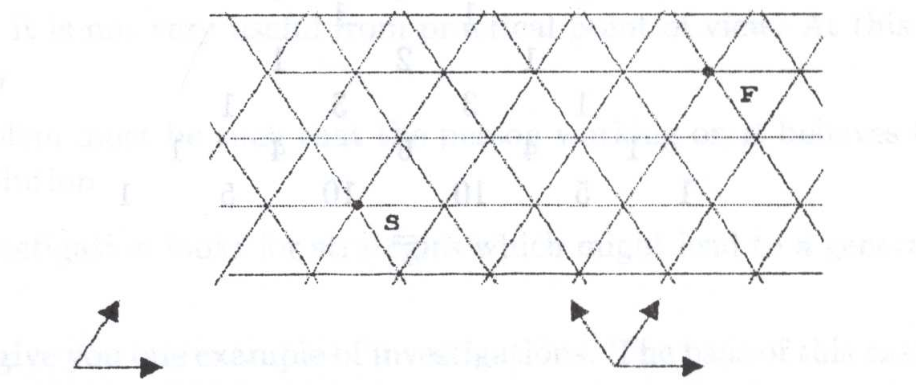

Fig. 3

Investigate routes for other positions of F. In case a) we'll get Pascal's triangle and in case b) the following number triangle:

1

1 1

1 3 1

1 5 5 1 (1)

1 7 13 7 1

1 9 25 25 9 1

=

What number relationship S can you find?

There is a very interesting example of that relationship: Numbers of two parallel diagonals $3-5, 5-13, 7-25, \ldots$ are the smallest and the largest numbers of any Pythagorean triple. (A triple $[x, y, z]$, where $x, y, z \in N$, is pythagorean one if $x^2 + y^2 = z^2$.) We can prove it later by induction.

Remark 2. Next question would be to find an expression for the number of routes leading from S to the point F in the m-th row and the n-th column.

Suppose that we take r steps diagonally upwards; than the number of horizontal steps taken must be $n - r$, and the number of vertical steps $m - r$. The r diagonal, $n - r$ horizontal and $m - r$ vertical steps can be taken in any order. Therefor, the number of those routes that involve r diagonal steps is equal to the number of permutations of the total of $r + (n - r) + (m - r) = n + m - r$ steps, of which r are of one kind, $n - r$ of a second kind and $m - r$ of a third kind. This number equals to

$$\frac{(n + m - r)!}{r!(n - r)!(m - r)!}$$

But we can take any number of diagonal steps ranging from 0 to the smaller one of the numbers m and n.Thus the total number of routes leading from S to F is given by the sum

$$\sum \frac{(n + m - r)!}{r!(n - r)!(m - r)!} \qquad (2)$$

But we do not use this way of solution.

Suppose we extended the idea to considering routes on a three dimensional grid, e.g. a cubic grid. See Fig. 4.

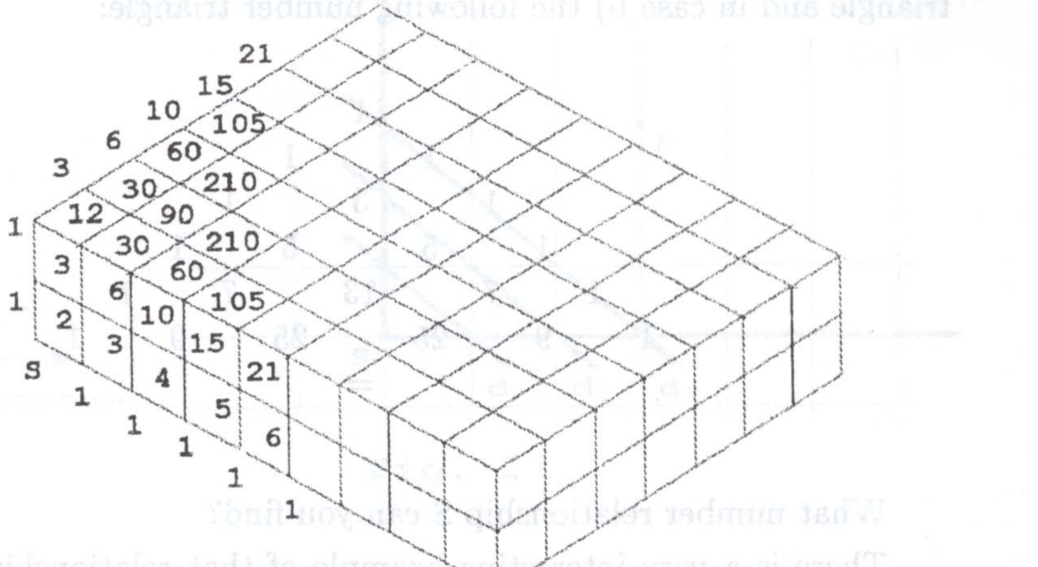

Fig. 4

The bottom layer of the cubic grid will produce the Pascal's triangle. The triangles for second and third layer are given bellow.

Investigate the triangles. Are there any relationships between Pascal's triangle and second and third triangle? (The second is obtained by multiplying the rows of PT by $1, 2, 3, 4, \ldots$ and the third one by multiplying the rows of PT by $3, 6, 10, 15, \ldots$)

Find the patterns of routes to the fourth layer.

1
2 2
3 6 3
4 12 12 4
5 20 30 20 5
6 30 60 60 30 6

=

1
3 3
6 12 6
10 30 30 10
15 60 90 60 15
21 105 210 210 105 21

=

Remark 3.(different formulations of problems)

We could formulate our „basic" problem in a little different way.

A „chessboard" is bounded from the below and from the left only (Fig. 5). A rook is placed on the square S in the lower left corner, and can move horizontally or vertically. For each square on Fig. 5 investigate the number of shortest path the rook can take from F to that square, and write this number in the square.

As we know this produces a pattern known as Pascal's Triangle

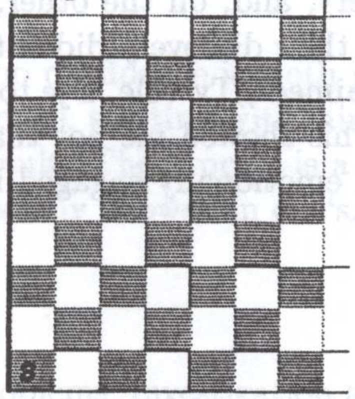

Fig. 5

The next problem can be formulated in the following way: Instead of rook put a king on the square S. The king is allowed to move in three directions only: from left to right, vertically upwards and diagonally upwards to the right. For each squares in Fig. 5 investigate the number of paths leading the kings from S to that square, and write the number in the square.

As we know number of pattern can be constructed step by step, proceeding along the diagonals $d_0, d_1, d_2, \dots$ (See Fig. 2) . This produces number triangle (1).

Next task is to find an expression for the number of paths leading from S to the square F in the m-th row and the n-th column. Suppose that the king takes r steps diagonally upwards; than the total number of path leading from S to F is given by the sum (2).

A three-dimensional chessboard is constructed from congruent cubes arranged in layers. The board is limited by three planes only: from below, from the behind and from the left (see Fig. 6). A rook is placed at the corner cube O. Investigate the number of shortest routes leading from O to any cell (that is cube) of the board. Examine the number pattern obtained.

Let us speak only about the second way of solution. Call a step on the board a path from the centre of one cell to the centre of the neighbouring cell. (Two cells are neighbours if they share a face.)

Any path from O to $P(i, j, k)$ consists of $i + j + k$ steps, of which i

have to be to the right, j to the front and k to the top. Thus the number of different paths leading from O to P is the number of permutations of $i+j+k$ steps, of which i are of one kind, j of a second kind k and of a third kind. This number equals to $\frac{(i+j+k)!}{i!j!k!}$.

In attempting investigations with second year students in the gymnasium (age = 16 years), I found that, on the one hand, they experienced great delight in discovery, and, on the other, great disappointment when later they realized that their discovery did not work out. Once, to give one example a student exclaimed „Ty vole, a je to v pytli!" (a colloquial Czech barnyard vulgarity). This showed me not that the student was rude, but that he was deeply and emotionally engaged in the work.

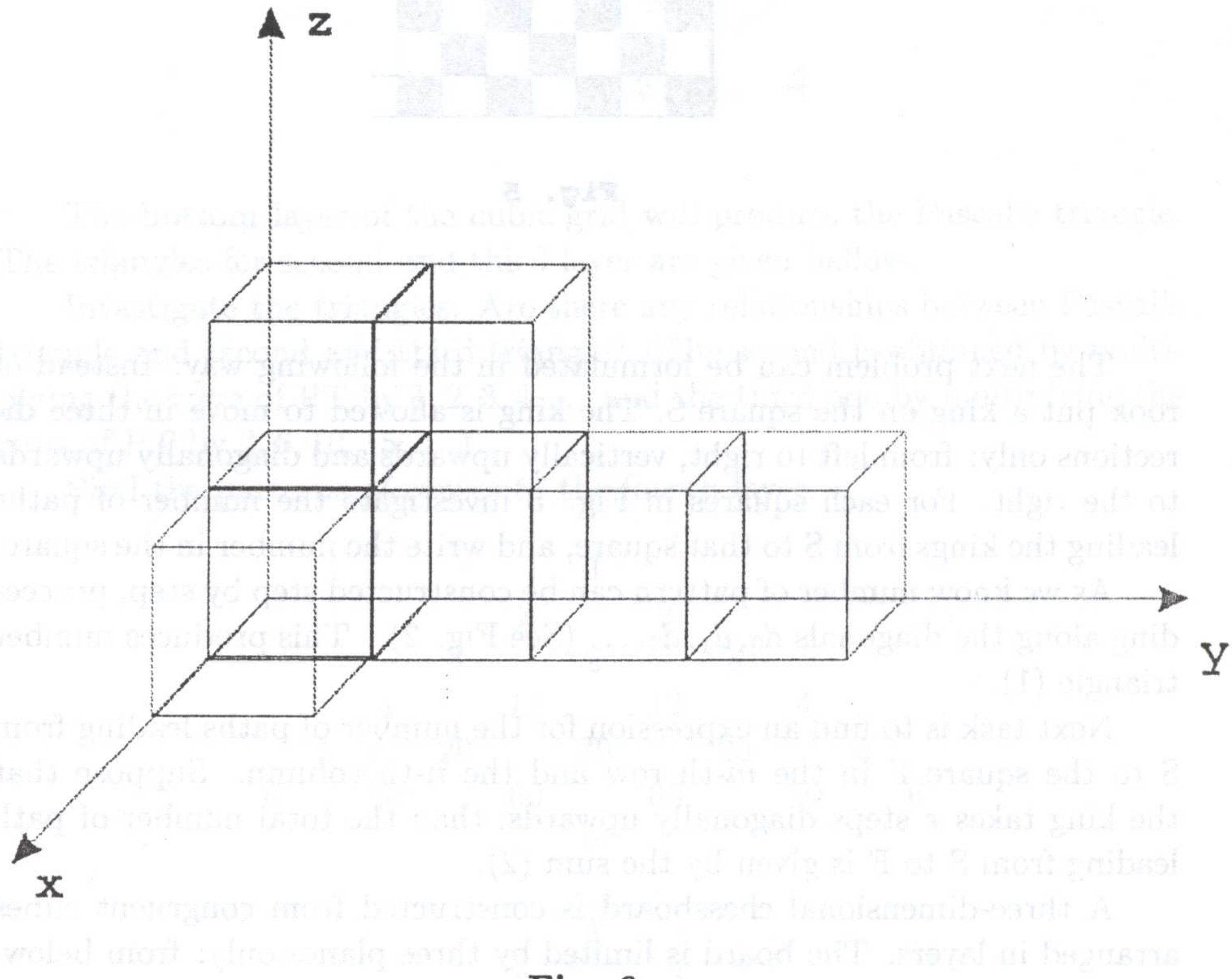

Fig. 6

One point which teachers must be sensitive to is that if students are given situations where they cannot make discoveries, they will lose interest very quickly.

When students work on problems by the method of generating problems or when they investigate, it is better if they work in small groups (e.g. two, three or four). Working in small groups gives students the opportunity to express their views, to discuss or disagree with each other, and to develop their arguments. It is the beginning of making mathematical proofs. Sometimes students can work individually on their own particular investigation or sometimes the whole class can work on the same investigation.

We say that students are free when they investigate and after it solve problems. But a teacher must lead them. This leading must be very subtle, very indirect and very gentle. Students must sometimes believe that they have done it alone.

Children can only develop confidence in both problem solving and investigations by constant practice, starting with fairly simple problems and progressing to the more difficult. The process is a much more flexible approach to thinking than is usually allowed in one's.

References

[1] Frobisher, L. (1994) Problems, Investigation and Investigative approach. In Issues in Teaching of Mathematics, Cassell London

[2] HMI (1985) Mathematics from 5 to 16, London HMSO

[3] Kirkby, D.(1986) Investigation Bank Book 5, Sheffield, Dickins & Son

[4] Kopka, J. (1993) Problem Solving - Method of Generated Problems, In Selected Topics from Maths Education 2, University of Oslo

[5] Kopka, J. (1994) Isolated and Non-isolated Problems in School Mathematics, In Selected Topics from Maths Education 3, University of Oslo

[6] Kopka, J. (1996) New Approaches to Problems and Investigations in School Mathematics, In Selected Topics from Maths Education 4, University of Oslo

[7] Reys et al (1985) Helping Children Learn Mathematics, Englewood Cliffs, NJ:Prentice-Hall

[8] Solvang, R.(1994) Thoughts about the free phase in connection to the use of generator problems, In Selected Topics from Maths Education 3, University of Oslo