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Category approach to $\mathcal{R}\,\mathcal{L}_4$– sets

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In this paper I'm going to investigate the subcategory of the category of sets valued by some Heyting algebra.

The notion of a Heyting algebra valued set was introduced by Scott [1] in 1972 in his work an the intuitionistic set theory. The categories of Heyting algebra valued sets were investigated and described by D. Higgs in [3] and [4].

We recall that the notion of a *Heyting algebra* is equivalent to the notion of a pseudo - Boolean algebra and the notion of complete lattice in which the following equality holds:

$$a \wedge \bigvee_{t\in T} a_t = \bigvee_{t\in T} (a \wedge a_t)$$

(the condition is refferred to as infinite distributivity).

Let $\mathcal{A} = (A, \leq)$ be a complete Heyting algebra.
By the *set valued by the algebra* $\mathcal{A}$, shortly: an $\mathcal{A}$ – set, we will mean any pair (U, δ) such that U is a set and $\delta : U \times U \to A$ is a mapping satisfying the following conditions:

$$(a_1)\ \forall x, y \in U \quad \delta(x,y) = \delta(y,x),$$

$$(a_2)\ \forall x, y, z \in U \quad \delta(x,y) \wedge \delta(y,z) \leq \delta(x,z).$$

The intuitive sense of the definition of the $\mathcal{A}$– set is this: for any two given elements x, y of the set U, the element $\delta(x,y)$ of the Heyting algebra $\mathcal{A}$ defines the extent with respect to which the element x is equal to y.
Let $P(U,\delta)$ denotes, for an arbitrary $\mathcal{A}$ – set $\mathcal{U} = (U, \delta)$, the set of all mappings $\alpha : U \to A$ satisfying the following conditions:

$$(b_1)\ \forall x \in U \quad \alpha(x) \leq \delta(x,x),$$

$$(b_2)\ \forall x, y \in U \quad \alpha(x) \wedge \delta(x,y) \leq \alpha(y).$$

We define the subset $S(U,\delta)$ of $P(U,\delta)$ in the following way:

$$(1) \qquad S(U,\delta) = \{\alpha \in P(U,\delta) : \forall x, y \in U \quad \alpha(x) \wedge \alpha(y) \leq \delta(x,y)\}.$$

The elements of the set $S(U,\delta)$ are called *singletons*.
For each $\mathcal{A}$ – set $\mathcal{U} = (U,\delta)$, we define the mapping $\Gamma_\delta : S(U,\delta) \times S(U,\delta) \to A$ as follows:

$$(2) \qquad \Gamma_\delta(\alpha,\beta) = \bigvee\{\alpha(x) \wedge \beta(x) : x \in U\}.$$

Then the pair $(S(U,\delta),\Gamma_\delta)$ is also an $\mathcal{A}$ – set.
Let us denote by $\mathcal{L}_4$ the chain with the underlying set $L_4 = \{0,1,2,3\}$.
A $\mathcal{RL}_4$ *– set*[1] is any $\mathcal{L}_4$ – set $\mathcal{U} = (U,\delta)$ which satisfies the following conditions:

$(r_1)\ \forall x \in U \quad 1 \le \delta(x,x)$,

$(r_2)\ \forall x,y \in U \quad [2 \le \delta(x,y) \Rightarrow x = y]$,

$(r_3)\ \forall x,y \in U \quad [\delta(x,y) = 1 \Rightarrow \delta(x,x) = \delta(y,y)]$,

$(r_4)\ \forall x \in U \quad [\delta(x,x) = 2 \Rightarrow \exists y \in U \quad \delta(x,y) = 1]$.

To each complete Heyting algebra $\mathcal{A}$ the category of all $\mathcal{A}$ – sets is assigned. $\mathcal{A}$ – **Set** denotes the category which objects are all $\mathcal{A}$ – sets and let the morphisms from one object $\mathcal{U} = (U,\delta)$ to another $\mathcal{W} = (W,\sigma)$ be all the triples $(\mathcal{U}, f, \mathcal{W})$, where f is $\mathcal{A}$ – *function*, i. e. f is a function from $U \times W$ to A satisfying the following conditions:

$(m_1)\ \forall x,x' \in U \quad \forall y \in W \quad f(x,y) \wedge \delta(x,x') \le f(x',y)$,

$(m_2)\ \forall x \in U \quad \forall y,y' \in W \quad f(x,y) \wedge \sigma(y,y') \le f(x,y')$,

$(m_3)\ \forall x \in U \quad \forall y,y' \in W \quad f(x,y) \wedge f(x,y') \le \sigma(y,y')$,

$(m_4)\ \forall x \in U \quad \bigvee\{f(x,y) : y \in W\} = \delta(x,x)$.

If $(\mathcal{U}, f, \mathcal{W})$ and $(\mathcal{W}, f', \mathcal{V})$ are morphisms from $\mathcal{U} = (U,\delta)$ to $\mathcal{W} = (W,\delta)$ and from $\mathcal{W}$ to $\mathcal{V} = (V,\gamma)$, respectively, then the triple $(\mathcal{U}, f' \circ f, \mathcal{V})$ is the composition of these morphisms, where

$$(f' \circ f)(x,z) = \bigvee\{f(x,y) \wedge f'(y,z) : y \in W\},$$

for all $(x,z) \in U \times V$.
By the identity morphism we shall mean any triple of the form $(\mathcal{U}, f, \mathcal{U})$, where $\mathcal{U} = (U,\delta)$ is any $\mathcal{A}$-set.

Each $\mathcal{A}$–function $f : U \times W \to A$ can be treated as a „characteristic function" of a „subset" of the set $U \times W$. For each pair (x,y) belonging to $U \times W$, $f(x,y)$ is interpreted as the element of the algebra A which defines the degree of relatedeness of the element y to x through f.

The following theorems are true for the category $\mathcal{A}$ – **Set**:

[1] In [5] A. Obtułowicz gives the representation of Pawlak's rough sets by means of $\mathcal{RL}_4$–sets.

T.1. A morphism $(\mathcal{U}, f, \mathcal{W})$ from an object $\mathcal{U} = (U, \delta)$ to an object $\mathcal{W} = (W, \sigma)$ is a monomorphism iff

$$f(x,y) \wedge f(x',y) \leq \delta(x,x'),$$

for all $x, x' \in U$ and $y \in W$.

T.2. A morphism $(\mathcal{U}, f, \mathcal{W})$ is an epimorphism iff

$$\bigvee\{f(x,y) : x \in U\} = \sigma(y,y),$$

for every $y \in W$.

T.3. If $(\mathcal{U}, f, \mathcal{W})$ is both monomorphism and an epimorphism, then it is an isomorphism.
The proofs of these results can be found in Higgs [4]. The following corollary readily follows from theorems **T.1. - T.3.**:
Corollary.
For every $\mathcal{A}$ – set $\mathcal{U} = (U, \delta)$, the triple $(\mathcal{U}, f, \mathcal{S})$, where $\mathcal{S} = (S(U,\delta), \Gamma_\delta)$ and $f : U \times S(U,\delta) \to A$ is defined as:

$$(3) \qquad f(x,\beta) = \delta(x,x) \wedge \Gamma_\delta(\alpha_x, \beta),$$

for all $x \in U$, $\beta \in S(U,\delta)$ and where $\alpha_x(y) = \delta(x,y)$ for all $y \in U$, is an isomorphism in the category $\mathcal{A}$- **Set.**

Let $\mathcal{RL}_4$ – **Set** denote the category which objects are all $\mathcal{RL}_4$ – sets and let the morphisms from an object $\mathcal{U}$ to an object $\mathcal{W}$ be all triples of the form $(\mathcal{U}, f, \mathcal{W})$, where f is an $\mathcal{L}_4$ – function.

The composition of morphisms is defined accordingly to the equality (3). The identity morphism in the category $\mathcal{RL}_4$ – **Set** is any triple of the form $(\mathcal{U}, \delta, \mathcal{U})$, where $\mathcal{U} = (U, \delta)$ is any $\mathcal{RL}_4$ – set.

The category $\mathcal{RL}_4$ – **Set** is the full subcategory of the category $\mathcal{L}_4$ – **Set**. Moreover, this category has products and a terminal object. The product $\mathcal{U} \times \mathcal{W}$ of two object $\mathcal{U} = (U, \delta)$, $\mathcal{W} = (W, \sigma)$ is defined as $\mathcal{U} \times \mathcal{W} = (U \times W, \xi)$, where

$$\xi((x,y),(x',y')) = \delta(x,x') \wedge \sigma(y,y'),$$

for all $x, x' \in U$ and all $y, y' \in W$.
A terminal object in the category $\mathcal{RL}_4$ – **Set** is any pair (U, τ) such that U is a one - element set, i.e. $U = \{x\}$ and $\tau(x,x) = 3$.

The singletons defined by (1) are useful in proving certain facts connected to the notion of isomorphic closedness of the subcategory relative to its supercategory (cf. [2]).

It applies to the subcategory $\mathcal{RL}_4$ – **Set** of the category $\mathcal{L}_4$ – **Set**. We shall recall the definition of isomorphic closedness of a subcategory. The subcategory $\mathcal{B}$ of a category $\mathcal{C}$ is *isomorphically closed* if any $\mathcal{C}$–object (i.e. an object of the category $\mathcal{C}$) which is isomorphic with a $\mathcal{B}$ – object is a $\mathcal{B}$ – object, too.

Theorem.

The $\mathcal{RL}_4$ – **Set** category is not an isomorphically closed subcategory of the $\mathcal{L}_4$ – **Set** category.

Proof. Let $\mathcal{U} = (U, \delta)$ be an $\mathcal{RL}_4$ – set such that $\delta(x_o, x_o) = 3$ for some $x_o \in U$. Let $S_o(U, \delta)$ be the set of all singletons for $\mathcal{U}$. We define the functions $\alpha_{x_o} : U \to L_4$, $\beta_{x_o} : U \to L_4$ in the following way

$$\alpha_{x_o}(x) = \delta(x_o, x),$$

$$\beta_{x_o}(x) = \begin{cases} 2 & \text{if } x = x_o, \\ \delta(x_o, x) & \text{otherwise.} \end{cases}$$

Clearly both α_{x_o} and β_{x_o} belong to the set $S_o(U, \delta)$. By Corollary, the $\mathcal{RL}_4$ – set $\mathcal{U}$, which is clearly an $\mathcal{L}_4$ – set, too, is isomorphic (in the $\mathcal{L}_4$ – **Set** category) to the $\mathcal{L}_4$ – set $\mathcal{Z}_\delta = (S_o(U, \delta), \Gamma_\delta)$, where Γ_δ is given by the formula (2). However, $\Gamma_\delta(\alpha_{x_o}, \beta_{x_o}) = 2$ and $\alpha_{x_o} \neq \beta_{x_o}$, so the function Γ_δ does not satisfy the condition (r_2) of definition of $\mathcal{RL}_4$ – sets. This means that $\mathcal{Z}_\delta$ is not an $\mathcal{RL}_4$ – set.

The proof is complete.

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