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A Survay of Foundational Gentzen's Systems for Finitely-Valued Logics

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Streszczenie

The Gentzen system for n-valued logical calculi discussed here is based on the notion of a sequent. However, this notion can be defined in at least three different ways. The first defines a sequent as a finite sequence of formulas (Kirin 1985, Saloni 1972, Orłowska 1985), the second defines it as an ordered n-tuple of finite sequences or sets of formulas (Rousseau 1967, Takahasi 1967, Borowik 1984). The third way consists in defining a sequent as an ordered pair of finite sets or sequences of formulas (Fitting 1991). The assumed definition determines then the form of the rules for eliminating or introducing propositional connectives in a given sequent, and thus also the whole formalization of the system.

1. Notational Remarks

We shall now introduce two systems, which define a sequent as an ordered n-tuple of finite sets or sequences of formulas. These very similar systems differ however in the way they are defined. The system of Moto-o Takahashi is closely related to the original Gentzen system and, in fact, for $n = 2$ it coincides with the original system. The system of G. Rousseau, based on a different approach, is presented in a more concise way, without any considerations on its consequence operation. We shall start with a presentation of this latter system.

We shall assume for the sake of simplicity that $E_n = \{0, 1, \ldots, n-1\}, E_n^* = \{r, r+1, \ldots, n-1\}$. Languages of order zero have finitely many k-ary connectives (with $k \geq 0$), while first-order languages are assumed to have also finitely many quantifiers.

Let S be a language correspondingly defined over a zero-order alphabet. As usual, we shall denote formulas by $\alpha, \beta, \gamma, \ldots$ with indices, if necessary, and propositional variables by $p, q, r, \ldots$, also with indices when needed. Finite (possibly empty) sequences of formulas will be denoted by capital Greek letters $\Gamma, \Delta, \ldots$ and again indices may be added if required.

2. Many-Valued Propositional Calculi

A *sequent* is an ordered n-tuple

$$(a) \qquad \Gamma_0 \vdash \Gamma_1 \vdash \ldots \vdash \Gamma_{n-1}$$

of finite sequences of formulas. In particular, some of them may be empty. Sequents will be denoted by $\Sigma, \Pi, \ldots$ with indices, if necessary. If Γ and Δ are sequences of formulas (finite, according to our convention), then $\Gamma\Delta$ represents their usual concatenation.

Now, let Σ be a sequent as in (a) and let

$$\Pi = \Delta_0 \vdash \Delta_1 \vdash \ldots \vdash \Delta_{n-1}.$$

Then $\Sigma\Pi$ is said to be a *composition* or *concatenation* of the sequents Σ and Π and

$$\Sigma\Pi = \Gamma_0\Delta_0 \vdash \Gamma_1\Delta_1 \vdash \ldots \vdash \Gamma_{n-1}\Delta_{n-1}.$$

Let Σ be a sequent of the form (a) and take $K \subseteq E_n$. If

$$\Gamma_i = \begin{cases} \Gamma & \text{if } i \in K \\ \emptyset & \text{if } i \notin K \end{cases}$$

then the sequent Σ will be denoted by $\Gamma \vdash_K$ and, in particular, by $\Gamma \vdash_m$, when $K = \{m\}$. A sequent Σ is *atomic* if all its formulas are atomic.

We say that a valuation

$$(b) \qquad v : V \longrightarrow E_n$$

satisfies a sequent $\Sigma = \Gamma_0 \vdash \Gamma_1 \vdash \ldots \vdash \Gamma_{n-1}$ if and only if there exists a $k \in E_n$ such that $k \in h_v(\Gamma_k)$, where h_v is an extension of v to a homomorphism of S into E_n. Let $Sat(\Sigma)$ represent the set of all valuations satisfying Σ. It is easily verified that

$$(c) \qquad v\Sigma\Pi = v\Sigma \cup v\Pi.$$

A set of sequents $\Im$ is *simultaneously satisfiable* if and only if

$$(d) \qquad \bigcap_{\Sigma \in \Im} v\Sigma \neq \emptyset.$$

We say that a sequent Σ is *valid* or *tautological* and we write $\models \Sigma$ if and only if $v\Sigma = E_n^V$, i.e., when it is satisfied by every valuation

$$v : V \longrightarrow E_n.$$

Lemma 1.

Let $s_w : E_n^m \longrightarrow E_n$ be a function corresponding to a connective δ_w and let $k \in E_n$. Then there exist subsets $K_j^i \subseteq E_n$ with $i = 1, \ldots, l$, $j = 1, \ldots, m$ and $l \leq n^{m-1}$ such that $s_w(x_1, \ldots, x_m) = k$ if and only if for each i with $1 \leq i \leq l$,

$$x_1 \in K_1^i \vee \ldots \vee x_m \in K_m^i.$$

Proof. Each subset $X \subseteq E_n^m$ can be represented as a union of at most n^{m-1} cartesian products of subsets of E_n. Indeed, if $m > 1$, then X is the union of all the products

$$\{k_1\} \times \ldots \times \{k_{m-1}\} \times \{k | (k_1, \ldots, k_{m-1}, k) \in X\}$$

such that $(k_1, \ldots, k_{m-1}) \in E_n^{m-1}$.

Applying this to the set $E_n^m - s_w^{-1}(\{k\})$ we readily obtain the conclusion of the lemma. The bound n^{m-1} is the most appropriate one here, since the set $\{(k_1, \ldots, k_m) | k_1 + \ldots + k_m = 0 \pmod{n}\}$ has n^{m-1} elements, but contains no product with more than one element. □

Let $\alpha_1, \ldots, \alpha_m$ be arbitrary formulas. Then for each i such that $i = 1, 2, \ldots, l$ we denote by $\Sigma_i(\alpha_1, \ldots, \alpha_m)$ a sequent of the form

$$(e) \qquad \alpha_1 \vdash_{K_1^i} \cdots \alpha_m \vdash_{K_m^i}.$$

It can be easily deduced from lemma 3.1 that

$$(f) \qquad Sat(\delta_w), (\alpha_1, \ldots, \alpha_m) \vdash_k = \bigcap_{i=1}^{l} Sat(\Sigma_i)(\alpha_1, \ldots, \alpha_m).$$

If Σ_1 and Σ_2 are arbitrary sequents, then (f) clearly implies

$$(g) \qquad Sat(\Sigma_1), \delta_w(\alpha_1, \ldots, \alpha_m) \vdash_k \Sigma_2 = \bigcap_{i=1}^{l} Sat(\Sigma_1)\Sigma_i(\alpha_1, \ldots, \alpha_m)\Sigma_2.$$

Moreover, the sequent $\Sigma_1 \delta_w(\alpha_1, \ldots, \alpha_m) \vdash_k \Sigma_2$ is tautological if and only if for each i with $1 \leq i \leq l$ the sequents $\Sigma_1 \Sigma_i(\alpha_1, \ldots, \alpha_m)\Sigma_2$ are tautological, i.e.

$$(h) \qquad \models \Sigma_1 \delta_w(\alpha_1, \ldots, \alpha_m) \vdash_k \Sigma_2 \longleftrightarrow \forall_{1 \leq i \leq l} \models \Sigma_1 \Sigma_i(\alpha_1, \ldots, \alpha_m)\Sigma_2.$$

The considerations above and the notation assumed here make it possible to define a rule (δ_w, k) for introducing the connective δ_w for different logical values k. The rule has the following general form:

$$(\delta_w, k) \qquad \frac{\Sigma_1\Pi_1(\alpha_1,\ldots,\alpha_m)\Sigma_2;\ldots;\Sigma_1\Pi_l(\alpha_1,\ldots,\alpha_m)\Sigma_2}{\Sigma_1\delta_w(\alpha_1,\ldots,\alpha_m)\vdash_k \Sigma_2}$$

A sequent Σ is said to be an *immediate consequence* of the sequents $\Pi_1,\ldots,\Pi_l$ with respect to the rule (δ_w, k) if and only if Σ is of the form

$$\Sigma_1\delta_w(\alpha_1,\ldots,\alpha_m)\vdash_k \Sigma_2$$

and each of the sequents $\Pi_1,\ldots,\Pi_l$ is of the form $\Sigma_1\Pi_i(\alpha_1,\ldots,\alpha_m)\Sigma_2$ for $i = 1,\ldots,l$. An atomic sequent $\Sigma = \Gamma_0 \vdash \ldots \vdash \Gamma_{n-1}$ will be called *initial* if and only if all the Γ_m, for $0 \leq m \leq n-1$, have at least one common element.

The set ξ of *deducible* sequents is the least set containing all the initial sequents and closed with respect to each of the rules (δ_w, k) for every propositional connective δ_w and every $k = 0, 1, \ldots, n-1$.

Theorem 1.
Let Σ be a sequent. Then it is deducible if and only if it is tautological.

Proof. We shall prove first that every deducible sequent is tautological. Clearly every initial sequent is tautological. By (h), all direct consequences of tautological sequents are tautological, too. Thus every deducible sequent is tautological.

Assume now Σ to be tautological and let m_Σ be the maximum degree of formulas occurring in the sequent Σ. Moreover, let n_Σ be the number of all formulas of degree m_Σ which occur in Σ. If $m_\Sigma = 0$, then Σ is atomic. Every atomic and tautological sequent is initial and hence deducible.

Let then $m_\Sigma > 0$. We make an inductive assumption that all tautological sequents Π such that

$$(i) \qquad m_\Pi < m_\Sigma$$

or

$$(j) \qquad m_\Pi = m_\Sigma \text{ and } n_\Pi < n_\Sigma$$

are deducible. The number m_Σ being positive, the sequent Σ can be represented in the following form:

$$(k) \qquad \Sigma_1\delta_w(\alpha_1,\ldots,\alpha_m)\vdash_k \Sigma_2,$$

where the formula $\delta_w(\alpha_1, \ldots, \alpha_m)$ has degree m_Σ. By (h), the sequents $\Sigma_1\Pi_i(\alpha_1, \ldots, \alpha_m)\Sigma_2$ are tautological for $i = 1, \ldots, l$ and for each of them (i) or (j) holds. Therefore by the inductive assumption each of them is deducible, which implies that Σ is also deducible as a direct consequence of sequents deducible by the rule (δ_w, k), which completes the proof. □

Corollary 1. (elimination theorem)
If the sequents $\Sigma\alpha \vdash_{K_1}$ and $\alpha, \Pi \vdash_{K_2}$ are both deducible and $K_1 \cap K_2 = \emptyset$, then the sequent $\Sigma\Pi$ is deducible, too.

Theorem 2. (compactness theorem)
A set of sequents $\Im$ is simultaneously satisfiable if and only if each of its finite subsets $\Xi \subseteq \Im$ is simultaneously satisfiable.

Proof. We introduce a discrete topology on the set E_n and a product topology on E_n^V. Since $(E_n, 2^{E_n})$ is a compact Hausdorff space, also the product space E_n^V is compact Hausdorff by Tichonov's theorem. We shall prove that for any sequent Σ the set $v\Sigma$ is closed in E_n^V.

Let V_Σ be the set of all propositional variables which occur in the formulas of the sequent Σ. The set V_Σ being finite, $v\Sigma$ reduced to valuations of variables in V_Σ is also finite. Take it to be the set $\{v_1, \ldots, v_s\}$. Then

$$v\Sigma = \bigcup_{j=1}^{s} \bigcap_{p \in V_\Sigma} pr_p^{-1}(v_j(p)),$$

which implies that $v\Sigma$ is both closed an open in the product space E_n^V. This proves the theorem. □

3. Multivalued predicate calculus

In the predicate calculus presented below the multivalued quantifiers will be interpreted as functions from the power set of E_n into E_n. We shall slightly modify our notational conventions. After G. Rousseau (Rousseau 1967), we shall make a distinction between free and bounded individual variables. This distinction is not important from a formal point of view, though in several cases it will make our reasoning easier. The set of individual variables will be denoted by V, just like the set of propositional variables. We shall use $y_1, y_2, \ldots$ to represent the free variables, and $x_1, x_2, \ldots$ to denote the bound ones. The other letters of the alphabet will preserve their standard meaning. The set of all terms T and the set S of all well-formed formulas are defined in the standard way, as well as the notions of an interpretation of the language S in a non-empty set M, of satisfiability and of tautology. We shall only recall here the interpretation of quantifier formulas. If i is an interpretation, we define an interpretation i_y^d for a free variable y and an element $d \in M$ as follows:

$$(a) \qquad i_y^d(a) = \begin{cases} d & \text{if } a = y \\ i(a) & \text{otherwise} \end{cases}$$

With i_y^d defined we can now introduce the interpretation of formulas with a quantifier Q. And thus

$$(b) \qquad i(Qx\alpha(y/x)) = q(\{i_y^d(\alpha(y)) | d \in M\}).$$

Finally, we note down another two useful identities. If i is an interpretation, y - a free variable, t and t_1 are terms, and α is a formula, then

$$(c) \qquad i(t_1(y/t)) = i(t_1(y/i(t))),$$

and

$$(d) \qquad i(\alpha(y/t)) = i(\alpha(y/i(t))).$$

The notion of a sequent carries over from the propositional case, with the obvious requirement that formulas occurring in the finite sequences Γ_i, Δ_j be formulas of the predicate calculus.

Let Σ be an arbitrary sequent. We denote by T_Σ the set of all terms built of free individual variables $y_1, y_2, \ldots$ and constants and function symbols occurring in the formulas of the sequent Σ. The set T_Σ is clearly denumerable. An interpretation i in this set is called *canonical* if and only if for any term $t \in T_\Sigma$,

$$i(t) = t.$$

Lemma 2.

Let q_l be an interpretation function associated with a quantifier Q_l and assume $k \in E_n$. Then there exist $p, q \in N$ and sets $L_j^i \subseteq E_n$ with $i = 1, \ldots, p$ and $j = 0, \ldots, q$ such that for every set $L \subseteq E_n$,

$$(e)\ q_l(L) = k \iff \bigvee_{i=1}^{p} (\exists x_i \in L)(\forall y_1, \ldots y_q \in L)(x_i \in L_0^i \vee y_1 \in L_1^i \vee \ldots$$

$$\ldots \vee y_q \in L_q^i).$$

Proof. Let f be an n-ary function on E_n with values in the same set such that

$$f(x_0, \ldots, x_{n-1}) = q_l(\{k | x_k = 1\}).$$

Then obviously

$$q_l(L) = f(\chi_L(0), \ldots, \chi_L(n-1)) \text{ for } L \subseteq E_n,$$

where χ_L is the characteristic function of L. Thus by lemma 3.1 we have that for appropriate $K_j^i \subseteq E_n$,

$$q_l(L) = k \iff \bigwedge_{i=1}^{p} (\chi_K(0) \in K_0^i \vee \ldots \vee \chi_K(n-1) \in K_{n-1}^i)).$$

We can assume without loss of generality that each K_j^i is a subset of the set $\{0, 1\}$. Each expression of the form

$$\chi_K(j) \in K_j^i$$

can be replaced by an equivalent expression of the form

$$Qz_j^i \in K \ (z_j^i \in L_j^i)$$

where Q is either $\exists$ or $\forall$, depending on the set K_j^i. This yields a proof of (e). □

For any i with $1 \leq i \leq p$ we define a sequent

$$\Sigma_i(\alpha(t_i); y_1, \ldots, y_q) = \alpha(t_i) \vdash_{E_0^i} \alpha(y_1) \vdash_{E_1^i} \ldots \alpha(y_q) \vdash_{E_q^i} .$$

Lemma 3.

(i) If a sequent $\Sigma_1 \Pi(\alpha(t_i); y_1, \ldots, y_q)\Sigma_2$ is tautological for $i = 1, \ldots, p$, then the sequent $\Sigma_1 Q_l x \alpha(x) \vdash_k \Sigma_2$ is tautological too, provided the variables $y_1, \ldots, y_q$ are pairwise different and do not occur neither in the sequent $\Sigma_1 Q_l x \alpha(x) \vdash_k \Sigma_2$ nor in the terms $t_1, \ldots, t_p$.

(ii) Let Σ be an arbitrary sequent and take i to be a canonical interpretation in the set T_Σ. If the interpretation i satisfies a sequent $Q_l x \alpha(x) \vdash_k$, then there exist terms $t_1, \ldots, t_p \in T_\Sigma$ such that $i \in \bigcap_{j=1}^{p} i\Sigma_j(\alpha(t_j); y_1, \ldots, y_q)$ for any free variables $y_1, \ldots, y_q$, where $i\Sigma$ denotes the set of all interpretations which satisfy Σ.

Proof. Immediate by the definition of a canonical interpretation and by the conditions $(c), (d)$ and (e). □

The rules for introducing the connectives δ_j and the quantifiers Q_w into the sequences of a sequent Σ will be presented as schemes of the following form (Rousseau 1967):

$$(\delta_j, k) \quad \frac{\Sigma\Pi_1(\alpha_1,\ldots,\alpha_m)\delta_j(\alpha_1,\ldots,\alpha_m)\vdash_k;\ldots;\Sigma\Pi_l(\alpha_1,\ldots,\alpha_m)\delta_j(\alpha_1,\ldots,\alpha_m)\vdash_k}{\delta_j(\alpha_1,\ldots,\alpha_m)\vdash_k\Sigma}$$

$$(Q_w, k) \quad \frac{\Sigma\Pi_1(\alpha(t_1);y_1,\ldots,y_q)Q_w x\alpha(x)\vdash_k;\ldots;\Sigma\Pi_p(\alpha(t_p);y_1,\ldots,y_q)Q_w x\alpha(x)\vdash_k}{Q_w x\alpha(x)\vdash_k\Sigma}$$

We assume here that the variables $y_1,\ldots,y_q$ are pairwise different and do not occur neither in the sequent $Q_w x\alpha(x) \vdash_k \Sigma$ nor in the terms $t_1,\ldots,t_p$.

A sequent

$$(f) \qquad \Sigma = \Gamma_0 \vdash \ldots \vdash \Gamma_{n-1}$$

is basic or fundamental if and only if there is at least one element common to all the Γ_j's for $j = 0,\ldots,n-1$.

The set P of deducible sequents is the least set containing all the basic sequents, and closed with respect to the rules (δ_j, k) and (Q_w, l), where $k, l = 0,\ldots,n-1$, for all connectives δ_j and all quantifiers Q_w.

Let $\mathbf{n}$ denote a finite sequence of nonnegative integers. We shall denote by $\mathbf{n},\mathbf{j}$ the usual concatenation of the sequence $\mathbf{n}$ and a one-element sequence j. Thus for $\mathbf{n}' = \mathbf{nk}$ we have $\mathbf{n}',\mathbf{j} = \mathbf{nk},\mathbf{j}$. The empty sequence will be denoted either by $\mathbf{0}$ or by $\emptyset$, the empty set symbol. The inequality $\mathbf{n} \leq \mathbf{m}$ represents the fact that the sequence $\mathbf{n}$ is an initial segment of the sequence $\mathbf{m}$ (finite or not).

A nonempty set D of sequences is a tree if and only if

(i) if $\mathbf{n} \leq \mathbf{m}$ and $\mathbf{m} \in \mathbf{D}$, then $\mathbf{n} \in \mathbf{D}$.

(ii) for each $\mathbf{n} \in \mathbf{D}$ the set $\{\mathbf{j}|\mathbf{n},\mathbf{j} \in \mathbf{D}\}$ is finite.

If a tree D is infinite, then there exists an infinite sequence $\mathbf{m}$ such that for each $\mathbf{n}$, if $\mathbf{n} \leq \mathbf{m}$, then $\mathbf{n} \in \mathbf{D}$.

Let j be a natural number. We denote by $\mathbf{m}(j)$ the least i such that the sequence $(\mathbf{m}(0),\ldots,\mathbf{m}(j-1), i)$ is an initial segment of infinitely many elements of the tree D.

Let Σ be an arbitrary sequent. To each sequence $\mathbf{n}$ we assign inductively a sequent $\Sigma_{\mathbf{n}}$ in the following way:

(g) $\Sigma_{\mathbf{0}} = \Sigma$

(h) $\Sigma_{\mathbf{n,i}}$ is defined if and only if $\Sigma_{\mathbf{n}}$ is not basic, and in that case it is defined as follows.

Assume the length of $\mathbf{n}$ is $qn + m$, $m \in E_n$ (where n is the cardinality of the set E_n). Then

(i) if $\Sigma_\mathbf{n}$ is of the form

$$\delta_j(\alpha_1, \ldots, \alpha_r) \vdash_k \Pi,$$

then $\Sigma_{\mathbf{n},i}$ is the sequent

$$\Pi\Pi_i(\alpha_1, \ldots, \alpha_r)\delta_j(\alpha_1, \ldots, \alpha_r) \vdash_k \text{ for } i = 1, \ldots, l.$$

(ii) if $\Sigma_\mathbf{n}$ is of the form

$$Q_w x\alpha(x) \vdash_k \Pi,$$

then $\Sigma_{\mathbf{n},i}$ is the sequent

$$\Pi\Sigma_i(\alpha(t_i); y_1, \ldots, y_q)Q_w x\alpha(x) \vdash_k$$

for $i = 1, \ldots, p$, where $(t_1, \ldots, t_p)$ is the l-th element of $(T_\Sigma)^p$ under some fixed enumeration. Here l is the number of proper initial segments $\mathbf{m} \leq \mathbf{n}$ the length of which is congruent to k (modulo the cardinality of the set E_n) and such that $\Sigma_\mathbf{m}$ is of the form $Q_w x\alpha(x) \vdash_k \Pi$. As usual, we assume that the free variables $y_1, \ldots, y_q$ are pairwise different and do not occur in $\Sigma_\mathbf{n}$ nor in the terms $t_1, \ldots, t_p$.

(iii) If the sequent $\Sigma_{\vec{n}}$ has none of the forms described in (i) and (ii), let $\Sigma_{\mathbf{n},} = \Sigma_\mathbf{n}$.

As can be easily observed, the domain of the assignment $\mathbf{n} \longmapsto \Sigma_\mathbf{n}$ coincides with the tree D.

Theorem 3 (on the completeness of the multisequential predicate calculus)
A sequent Σ is tautological if and only if it is deducible.

Proof. Clearly, every fundamental sequent is deducible. The rules (δ_j, k) and (Q_w, l) preserve the property of being tautological, so every deducible sequent is tautological.Assume now that a sequent Σ is not deducible. Then $\Sigma_\mathbf{n}$ is defined for infinitely many $\mathbf{n}$, which implies the existence of an infinite sequence $\mathbf{m}$ such that $\Sigma_\mathbf{n}$ is defined for each $\mathbf{n} \leq \mathbf{m}$.

Let $l \in E_n$. Denote by S_l the set of all formulas which occur at the l-th position in some sequent $\Sigma_{\vec{n}}$ for $\vec{n} \leq \vec{m}$. Let ι be a canonical interpretation in the set of terms T_Σ. We shall prove that for any nonatomic formula α

(k) if $\alpha \in S_{\iota(\alpha)}$, then $\beta_1 \in S_{\iota(\beta)}$ for some formula β the degree of which is less than the degree of α.

Assume that $\iota(\alpha) = k$, $\alpha \in S_k$.

(i) If $\alpha = \delta_j(\alpha_1, \ldots, \alpha_r)$, then $s \in i\Sigma_j(\alpha_1, \ldots, \alpha_r)$ for $j = 1, \ldots, l$, and for some $\mathbf{n} \leq \mathbf{m}$, the sequent $\Sigma_\mathbf{n}$ is of the form $\delta_j(\alpha_1, \ldots, \alpha_r) \vdash_k \Sigma$. For $t = 1, \ldots, l$ we choose a sequent

$$\Sigma_{\mathbf{n},t} = \Sigma\Sigma_t(\alpha_1, \ldots, \alpha_r)\delta_j(\alpha_1, \ldots, \alpha_r) \vdash_k$$

so that $\mathbf{n}, i \leq \mathbf{m}$. There exist β and k_1 such that $s(\beta) = k_1$ and β occurs in the sequent $\Sigma_s(\alpha_1, \ldots, \alpha_r)$ on the k_1-st position. Consequently, $\beta \in S_{\iota(\beta)}$ and the degree of complexity of β is lower than that of α.

(*ii*) If $\alpha = Q_w x\alpha(x)$, then by lemma 3(ii) there exist terms $t_1, \ldots, t_p \in T_\Sigma$ such that for any $y_1, \ldots, y_q$, $\iota \in i\Sigma_i(\alpha(t_i); y_1, \ldots, y_q)$ for $i = 1, \ldots, l$ and, moreover, for some $\mathbf{n}$ with $\mathbf{n} \leq \mathbf{m}$, $\Sigma_\mathbf{n}$ is of the form $Q_w x\alpha(x) \vdash_k \Sigma$, while $\Sigma_{\mathbf{n},\mathbf{t}} = \Sigma\Sigma_t(\alpha(t_i); y_1, \ldots, y_q)Q_w x\alpha(x) \vdash_k$ for $t - 1, \ldots, p$ and for adequate $y_1, \ldots, y_q$. The number i is chosen so that $\mathbf{n}, s \leq \mathbf{m}$. Hence there exist β and k_1 such that $\iota(\beta) = k_1$ and β occurs in the sequent $\Sigma_s(\alpha(t_i); y_1, \ldots, y_q)$ on the k_1 position. Consequently, $\beta \in S_{s(\beta)}$ and the degree of β is less than the degree od α.

Now let ι be a canonical interpretation in the set T_Σ such that for each predicate p, if $t_1, \ldots, t_r \in T_\Sigma$, then $\iota(p)(t_1, \ldots, t_r)$ is the least k with the property $p(t_1, \ldots, t_r) \notin S_k$. Such an interpretation clearly exists. If a sequent Σ is tautological, then for some α occurring in Σ we have $\alpha \in S_{\iota(\alpha)}$. Therefore by ($k$) there exists an elementary formula β such that $\beta \in S_{i(\beta)}$, contrary to the assumption on the interpretation ι. Hence the sequent Σ cannot be tautological. □

Theorem 4.

A set $\Im$ of sequents is simultaneously satisfiable if and only if every finite subset of $\Im$ is simultaneously satisfiable.

PROOF. If a set $\Im$ is simultaneously satisfiable, then evidently each of its finite subsets is simultaneously saatisfiable, too. Assume therefore that each finite subset of $\Im$ is simultaneously satisfiable. Without loss of generality we can assume that among the truth functions on the set E_n^m we have functions $f_1, \ldots, f_u$ such that

$$f_j(x_0, \ldots, x_{n-1}) = q_j\{x_0, \ldots, x_{n-1}\} \text{ for } j = 1, \ldots, u.$$

Consider an infinite sequence of alphabets

$$A_0, A_1, \ldots$$

such that

(i) $A_0 = A$

(ii) for any $n \in N$, the primitive symbols of A_{n+1} are the symbols of A_n plus some additional constants $e_{k,y,\alpha}$, which are in a one-to-one correspondence with all triples (k, y, α), where $k \in E_n$, y is a free variable not occuring in α, which in turn is a formula over the alphabet A_n, but not over A_{n-1}.

Let

$$A^* = \bigcup_{n \in N} A_n.$$

We denote by T^* and S^* the set of all terms and the set of all formulas, correspondingly, over the alphabet A^*. We shall often write $\alpha(e_k)$ instead of $\alpha(e_{k,y,\alpha})$.

It can be proved inductively that if R is a finite set of formulas over A_{n-1}, then every interpretation of the alphabet A in a set M can be extended to an interpretation ι_n of the alphabet A_n in M in such a way that for any formula $\alpha(y)$, where y is a free individual variable,

$$\{(l_n)^d_y(\alpha(y)) | d \in M\} = \{\iota_n(\alpha(e_0)), \ldots, \iota_n(\alpha(e_{n-1}))\}.$$

Thus if R is a finite subset of S^*, every interpretation ι of the alphabet A in M extends to an intepretation ι^* of A^* in M in such a way that for each $\alpha(y) \in R$,

$$\{\iota^{*d}_{n,y}(\alpha(y)) | d \in M\} = \{\iota^*_n(\alpha(e_0), \ldots, \iota^*_n(\alpha(e_{n-1}))\}.$$

Let $\Im^* \subseteq S^*$ be a set of sequents consisting of all the sequents in $\Im$ and of all sequents of the form

1. $\alpha(t) \vdash_K (\alpha(e_0), \ldots, \alpha(e_{n-1})) \vdash_{K_1}$
2. $Q_w\alpha(x) \vdash_K (f_w(\alpha(e_0), \ldots, \alpha(e_{n-1}))) \vdash_{K_1}$

for $w = 1, \ldots, n$, $K \subseteq E_n$, $K_1 = E_n - K$, $t \in T^*$, and $\alpha(y) \in S^*$, where the variable y is free. Obviously each finite subset $R \subseteq \Im^*$ is simultaneously satisfiable.

Let ι be a canonical interpretation of the language S^* in the set of terms T^* such that for every predicate p,

$$\iota(p)(t_1, \ldots, t_r) = h_v(p(t_1, \ldots, t_r)).$$

We shall prove inductively that

3. $\iota(\alpha) = h_v(\alpha)$ for each formula $\alpha \in S^*$.

It is sufficient to prove that

$$h_v(Q_w x\alpha(x)) = q_w\{h_v(\alpha(t))|t \in T^*\}.$$

Obviously $\{h_v(\alpha(e_0)), \dots, h_v(\alpha(e_{n-1}))\} \subseteq \{h_v(\alpha(T))|t \in T^*\}$. On the other hand, since each sequent is an element of $\Im^*$, we have

$$\{h_v(\alpha(T))|t \in T^*\} \subseteq \{h_v(\alpha(e_0)), \dots, h_v(\alpha(e_{n-1}))\}.$$

Thus

$$q_w\{h_v(\alpha(t))|t \in T^*\} = q_w\{h_v(\alpha(e_0)), \dots, h_v(\alpha(e_{n-1}))\} =$$
$$= f_w(h_v(\alpha(e_0)), \dots, h_v(\alpha(e_{n-1}))) = h_v(f_w(\alpha(e_0), \dots, \alpha(e_{n-1}))).$$

The interpretation ι satisfies every formula in $\Im^*$, so its reduction to the alphabet A satisfies every formula in $\Im$, which completes the proof of the theorem. □ Recall that $E_n = \{0, \dots, n-1\}$ and that the set $E_r^* = \{r, r+1, \dots, n-1\}$, where $0 < r \leq n-1$, is the set of distinguished values. Let

$$\Gamma \| \Delta$$

represent the sequent $\Gamma \vdash_{E_n - E_r^*} \Delta \vdash_{E_r^*}$.

An interpretation ι satisfies a formula α, which shall be denoted by $\iota \in \iota^*(\alpha)$, if and only if $\iota(\alpha) \in E_r^*$. Clearly,

$$\iota^*(\beta) = \iota(\|\beta)$$

$$\iota^*(\gamma) = \iota(\gamma\|).$$

A formula α is a tautology if and only if each interpretation is in the set $\iota^*(\alpha)$. We say that α is deducible if and only if the sequent $\|\alpha$ is deducible.

A set of formulas $\Im$ is simultaneously satisfiable if and only if there is no finite subset $\Gamma \subseteq \Im$ such that the sequent $\Gamma\|$ is deducible.

Theorem 5.

A formula α is a tautology if and only if it is deducible. Moreover, a set of formulas $\Im$ is simultaneously satisfiable if and only if it is consistent.

Proof. Immediate by theorems 3 and 4. □

4. Gentzen-Takahashi Systems for Finitely-Valued Logics

We shall now introduce a Gentzen-type formalization of finitely-valued logics. It is very close to the original Gentzen's formalization of classical logic, and in fact it may be considered as an intermediate link between formalizations of the Rousseau-Takahashi type and those of Kirin. We start with a review of notions to be used in this part of the paper.

A *sequent* in an n-valued first-order predicate calculus is an ordered n-tuple

$$(a) \qquad \Gamma_0 \vdash \Gamma_1 \vdash \ldots \vdash \Gamma_{n-1}$$

of finite sets of formulas. In particular, some of the sets Γ_i for $i = 0, \ldots, n-1$ may be empty. Sequents will be usually denoted by capital Greek letters $\Sigma, \Pi, \ldots$ with indices, if necessary. For the sake of simplicity we assume some notational conventions as follows. If Γ and Δ are sets of formulas, we denote by $\Gamma\Delta$ its set-theoretic union, i.e.

$$\Gamma\Delta = \Gamma \cup \Delta.$$

Similarly, if Γ is a set of formulas and α is a formula, then $\Gamma\alpha$ or Γ, α represent the union $\Gamma \cup \{\alpha\}$. Assume now

$$\Sigma = \Gamma_0 \vdash \Gamma_1 \vdash \ldots \vdash \Gamma_{n-1},$$

$$\Pi = \Delta_0 \vdash \Delta_1 \vdash \ldots \vdash \Delta_{n-1}.$$

Then $\Sigma\Pi$ represents a sequent called the *composition* of the sequents Σ and Π, defined as follows:

$$\Sigma\Pi = \Gamma_0\Delta_0 \vdash \Gamma_1\Delta_1 \vdash \ldots \vdash \Gamma_{n-1}\Delta_{n-1}.$$

We say that the sequent Σ is a *subsequent* of Π, or that Σ is contained in Π — and we write $\Sigma \subseteq \Pi$ — if and only if for each $i \in E_n$,

$$\Gamma_i \subseteq \Delta_i.$$

Let $D \subseteq E_n$. If $\Sigma = \Gamma_0 \vdash \Gamma_1 \vdash \ldots \vdash \Gamma_{n-1}$ and

$$\Gamma_i = \begin{cases} \Gamma & \text{if } i \in D \\ \emptyset & \text{otherwise} \end{cases}$$

then the sequent Σ will be denoted by

$$\vdash_D \Gamma.$$

In particular, for $D = \{j\}$ the sequent will be denoted by $\vdash_j \Gamma$. If additionally $\Gamma = \{\alpha\}$, then we write $\vdash_j \alpha$.

A sequent $\Sigma = \Gamma_0 \vdash \Gamma_1 \vdash \ldots \vdash \Gamma_{n-1}$ is said to be *atomic* if and only if all its formulas are atomic. It is *satisfiable* if and only if there exist a valuation

$$v : S \longrightarrow E_n$$

and a $j \in E_n$ such that $j \in v(\Gamma_j)$. We say that a sequent is *tautological* if and only if for each valuation v there exist a $j \in E_n$ and a formula $\alpha \in \Gamma_j$ such that $v(\alpha) = j$.

A formula α is a *tautology* if and only if the sequent

$$\vdash_{E_n^*} \alpha$$

is tautological.

Sequents of the form $\alpha \vdash \alpha \vdash \ldots \vdash \alpha$ for $\alpha \in S$ will be called *axiomatic* or just *axioms.*

We shall now introduce rules for connecting formulas occurring in sets of sequents by propositional connectives or, speaking in other terms, for introducing a connective into a set of formulas in a sequent. Also rules for adding quantifiers to formulas in sequents will be presented.

Recall that $E_n = \{0, 1, \ldots, n-1\}$ and define

$$E_j^+ = \{i \in E_n | i \geq j\},$$

$$E_j^- = \{i \in E_n | i \leq j\}.$$

Moreover, let $s\#(j_1, \ldots, j_k; j)$ represent the statement that the function $s : E_n^m \longrightarrow E_n$ minimally satisfies the system $(j_1, \ldots, j_k; j)$.

Now, let $\Sigma = \Gamma_0 \vdash \Gamma_1 \vdash \ldots \vdash \Gamma_{n-1}$ and take $\sigma \in \mathbf{A}$ to be an arbitrary m-ary propositional connective; let $\exists$ and $\forall$ be quantifiers and assume the function $s : E_n^m \longrightarrow E_n$ is an interpretation of the connective σ. Then the rules for introducing connectives and quantifiers into the set Γ_j for $j \in E_n$ and a sequent Σ are represented by the following schemes:

$$(\sigma j) \quad \frac{\{\Sigma \vdash_{j_1} \alpha_{j_1} \vdash_{j_2} \alpha_{j_2} \ldots \vdash_{j_k} \alpha_{j_k} | 1 \leq j_i \leq m,\ s\#(j_1, \ldots, j_k; j)\}}{\Sigma \vdash_j \sigma(\alpha_1, \ldots, \alpha_m)}$$

$$(\exists j) \quad \frac{\Sigma \vdash_{E_j^-} \alpha(z); \Sigma \vdash_j \alpha(y)}{\Sigma \vdash_j \exists_x \alpha(x)}$$

$$(\forall j) \quad \frac{\Sigma \vdash_{E_j^+} \alpha(z); \Sigma \vdash_j \alpha(y)}{\Sigma \vdash_j \forall_x \alpha(x)}$$

where the variable z does not occur in the conclusions of the rules $(\exists j)$ and $(\forall j)$.

The rule of extension:

(ex) $\frac{\Sigma}{\Pi}$ provided $\Sigma \subseteq \Pi$,

The rule of cut:

(ct) $\frac{\Sigma \vdash_j \alpha; \Pi \vdash_k \alpha}{\Sigma\Pi}$, if $j \neq k$.

A sequent Σ is *deducible* if and only if there exists a proof tree

$$\mathcal{D} = (D, r, \Sigma)$$

such that Σ is its root, D is a set of sequents and r is a relation defined as follows: $(\Sigma, \Pi) \in r$ if and only if in the set $\{(\sigma j) | \sigma \in \mathbf{A}, j \in E_n\} \cup \{(\exists j) | j \in E_n\} \cup \{(\forall j) | j \in E_n\} \cup \{(ex), (ct)\}$ there is a rule such that the sequent Σ is one of its premises, Π is its conclusion, and every leaf of the tree is an axiom.

A formula α is *deducible* if and only if the sequent $\vdash_{E_r^*} \alpha$ is deducible. Before proving completeness and consistency of the Gentzen system we state some auxiliary lemmas without proof or with a short sketch of the proof only.

Lemma 4.

Assume $\Sigma(x)$ is an arbitrary sequent with a free individual variable x. Let $\Sigma(y)$ represent the sequent obtained from $\Sigma(x)$ by a sound substitution of the variable y for x. If $\Sigma(x)$ is deducible, then $\Sigma(y)$ is deducible, too. □

Theorem 6.

Let $D_k \subseteq E_n$ for $k = 1, \ldots, s$ and let D_k^* be increasing sequences of elements of D_k. If for each k with $1 \leq k \leq s$ the sequent $\Sigma \vdash_{D_k^*} \alpha$ is provable, then so is the sequent $\Sigma \vdash_{\bigcap_{k=1}^{s} D_k} \alpha$.

Proof. It suffices to prove the theorem for $s = 2$. Let $D_1 \cap D_2 = \{m_1, \ldots, m_t\}$. If $D_1 \cap D_2$ is empty, then the theorem holds trivially. Assume then that

$$D_1 = \{m_1, \ldots, m_t, n_1, \ldots, n_r\},$$

$$D_2 = \{m_1, \ldots, m_t, l_1, \ldots, l_v\}.$$

When $v = 0$, the theorem again holds trivially. Assume $t > 0$ and let the corresponding inductive assumptions hold. Then

$$\frac{\dfrac{\Sigma \vdash_{m_1,\ldots,m_t,n_1} \alpha; \Sigma \vdash_{m_1,\ldots,m_t,l_1,\ldots,l_v} \alpha}{\Sigma \vdash_{m_1,\ldots,m_t} \alpha \; \Sigma \vdash_{m_1,\ldots,m_t,l_1,\ldots,l_{v-1}} \alpha}}{\Sigma \vdash_{m_1,\ldots,m_t,l_1,\ldots,l_{v-1}} \alpha}$$

due to the rules of cut and extension, since $n_1 \neq l_v$. By the inductive assumption we infer that the sequent

$$\Sigma \vdash_{D_1 \cap D_2} \alpha$$

is deducible. Let now $r > 0$. An application of the rule of extension yields deducibility of the sequents

$$\Sigma \vdash_{m_1,\ldots,m_t,n_1,\ldots,n_{r-1}} \alpha \vdash_{n_r} \alpha$$

$$\Sigma \vdash_{m_1,\ldots,m_t,n_1,\ldots n_{r-1}} \alpha \vdash_{l_1,\ldots,l_v} \alpha,$$

which implies that the sequent

$$\Sigma \vdash_{m_1,\ldots,m_t,n_1,\ldots,n_{r-1}} \alpha$$

is deducible, too. Hence by the inductive aassumption the sequent

$$\Sigma \vdash_{m_1,\ldots,m_t} \alpha$$

is also deducible, which completes the proof of the theorem. □

Corollary 1.

Let C and D be disjoint subsets of the set E_n. If the sequents $\Sigma \vdash_C \alpha$ and $\Pi \vdash_D \alpha$ are deducible, then the sequent $\Sigma\Pi$ is also deducible. □

Corollary 2.

If all the sequents $\Sigma \vdash_{E_n - \{m\}}$ are deducible for $m \in E_n$, then Σ is also deducible. □

Lemma 5.

Let σ_j be an arbitrary propositional connective and assume s_j is its interpretation. If $w \neq s_j(m_1, \ldots, m_k)$, then the sequent

$$\vdash_{E_n - \{m_1\}} \alpha_1 \cdots \vdash_{E_n - \{m_k\}} \alpha_k \vdash_{E_n - \{w\}} \sigma_j(\alpha_1, \ldots, \alpha_k)$$

is also deducible.

Proof. Let $v = s_j(m_1, \ldots, m_k)$. By application of the rule of extension we deduce from the axioms the following sequents:

$$\vdash_{E_n - \{m_1\}} \alpha_1 \ldots \vdash_{E_n - \{m_k\}} \alpha_k \vdash_{m_j} \alpha_t$$

for $t \in \{m_1, \ldots, m_k\}$. Applying the rule (σj) we obtain the sequent

$$\vdash_{E_n-\{m_1\}} \alpha_1 \ldots \vdash_{E_n-\{m_k\}} \alpha_k \vdash_v \sigma_j(\alpha_1, \ldots, \alpha_k).$$

Now the rule of extension yields the conclusion of the lemma, if $v \neq w$. □

Lemma 6.

For any $m \in E_n$ the following sequents are deducible:

(a) $\vdash_{\{0,\ldots,m-1\}} \forall_x \alpha(x) \vdash_{\{m,\ldots,n-1\}} \alpha(y)$

(b) $\vdash_{\{m+1,\ldots,n-1\}} \exists_x \alpha(x) \vdash_{\{0,\ldots,m\}} \alpha(y)$

where $\alpha(y)$ is a sound substitution of an arbitrary variable y for x in the formula $\alpha(x)$.

Proof. We shall prove (b) by induction on m. Let $m = 0$. Then the sequent (b) is an axiom and is therefore deducible. Assume now that the sequent (b) is deducible for some $m \geq 0$. We shall prove that the sequent

$$(c) \qquad \vdash_{\{m+2,\ldots,n-1\}} \exists_x \alpha(x) \vdash_{\{0,\ldots,m+1\}} \alpha(y)$$

is deducible provided $m + 1 \leq n - 1$. Let z be a free variable not occurring in the formula $\exists_x \alpha(x)$ and not equal to y. Then clearly the sequent

$$\vdash_{\{m+1,\ldots,n-1\}} \exists_x \alpha(x) \vdash_{\{0,\ldots,m\}} \alpha(z)$$

is deducible by inductive assumption. Hence the sequent (c) is deducible according to the following fragment of the tree:

$$\cfrac{\cfrac{\vdash_{\{m+1,\ldots,n-1\}} \exists_x \alpha(x) \vdash_{\{0,\ldots,m\}} \alpha(z)}{\vdash_{\{m+1,\ldots,n-1\}} \exists_x \alpha(x) \vdash_{\{0,\ldots,m-1\}} \alpha(z) \vdash_{\{0,\ldots,m\}} \alpha(y); \vdash_{\{m+1,\ldots,n-1\}} \exists_x \alpha(x) \vdash_{\{0,\ldots,m-1\}} \alpha(y) \vdash_m \alpha(y)}}{\vdash_{\{m+1,\ldots,n-1\}} \exists_x \alpha(x) \vdash_{\{0,\ldots,m-1\}} \alpha(y) \vdash_m \exists_x \alpha(x)}$$

In the above deduction we have first applied the rule of extension and then the rule for the introduction of the existential quantifier $\exists$.

The proof for (a) is analogous. □

Lemma 7.

For each $k \in E_n$ and every increasing sequence D^*_k of elements in E_n (where D^*_0 is the empty sequence), if the sequent

$$(a) \qquad \Sigma \vdash_{D^*_k} \exists_x \alpha(x) \vdash_{D^*_k} \alpha(y)$$

is deducible and the variable y does not occur neither in the formula $\exists_x \alpha(x)$ nor in the sequent Σ, then the sequent

$$\Sigma \vdash_{D^*_k} \exists x \alpha(x)$$

is deducible, too. Moreover, if the sequent

$$(b) \qquad \Sigma \vdash_{D_k^*} \forall_x \alpha(x) \vdash_{D_k^*} \alpha(y)$$

is deducible with the same assumptions on y as before, then the sequent

$$\Sigma \vdash_{D_k^*} \forall_x \alpha(x)$$

is also deducible.

Proof. The proof is inductive over k and uses Lemma 4, the rule of extension and the rules for the introduction of the quantifiers $\exists$ and $\forall$. □

Let $k \in E_n$ and let $\Sigma = \Gamma_0 \vdash \Gamma_1 \vdash \ldots \vdash \Gamma_{n-1}$ be an arbitrary sequent. We shall write $\Sigma - \vdash_k \alpha$ to represent the sequent

$$\Gamma_0 \vdash \ldots \vdash \Gamma_k - \{\alpha\} \vdash \ldots \vdash \Gamma_{n-1}.$$

We define the following rule:

$$(mix) \qquad \frac{\Sigma; \Pi}{\Sigma - \vdash_k \alpha \ \Pi - \vdash_m \alpha} \text{ provided } k \neq m.$$

Lemma 8.

If a sequent Σ is deducible, then it can also be deduced using the rule (mix), but not the rule of cut (ct). □

Lemma 9. (Takahashi 1967)

If the sequents Σ and Π can be deduced without the rules (ex) and (ct), then the sequent $\Sigma - \vdash_k \alpha \ \Pi - \vdash_m \alpha$ is also deducible for $k \neq m$. □

Theorem 7. (on cut elimination)

If a sequent Σ is deducible, then it can be also deduced without using the rule of cut.

Proof. Immediate by Lemmas 8 and 9. □

Let Σ be an arbitrary sequent. If for each $m \in E_n$ the sequent

$$\Sigma \vdash_{E_n - \{m\}} \alpha \vdash_m \beta$$

is deducible, then we say that the formulas α and β are equivalent modulo Σ and we represent this fact by $\alpha \equiv \beta \, (\text{mod } \Sigma)$. If Σ is empty, then α and β are said to be equivalent and we write $\alpha \equiv \beta$.

Lemma 10.

The relations $\equiv$ and $\equiv$(mod Σ) of formula equivalence and of formula equivalence modulo Σ, respectively, are equivalence relations.

Proof. Reflexivity is trivial due to the rule of extension. To prove symmetry, assume that $\alpha \equiv \beta \pmod{\Sigma}$. Observe that for any $k \in E_n$ such that $k \neq m$, the sequent

$$\Sigma \vdash_{E_n-\{k\}} \beta \vdash_{E_n-\{m\}} \alpha$$

is deducible. Thus, by Theorem 6 and the fact that $\{k\} = \bigcap_{m \neq k} (E_n - \{m\})$, the sequent

$$\Sigma \vdash_{E_n-\{k\}} \beta \vdash_k \alpha$$

is also deducible.

Transitivity follows from the deducibility of the sequents

$$\Sigma \vdash_{E_n-\{m\}} \alpha \vdash_m \gamma \vdash_m \beta \text{ and } \Sigma \vdash_{E_n-\{m\}} \alpha \vdash_m \gamma \vdash_{E_n-\{m\}} \beta$$

and from Corollary 3.1. □

Lemma 11.

Let $\alpha \equiv \beta$. If a sequent Σ is deducible, then each sequent Π obtained from Σ by substituting the formula β for some occurrences of the formula α is also deducible.

Proof. It suffices to prove the theorem for the case when β is substituted for one occurrence of α only. Let $\Sigma = \Lambda \vdash_m \alpha$ and $\Pi = \Lambda \vdash_m \beta$ and assume Σ is deducible. Applying the rule of extension, we can deduce the sequent

$$\Lambda \vdash_m \beta \vdash_m \alpha.$$

Since $\alpha \equiv \beta$ by assumption, the sequent

$$\Lambda \vdash_m \beta \vdash_{E_n-\{m\}} \alpha$$

is also deducible, which proves the lemma due to Corollary 3.1. □

Theorem 8. (on consistency)

If a sequent Σ is deducible, then it is tautological.

Proof. Recall that a sequent is deducible if there exists a proof tree d_Σ with root Σ such that every leaf of d_Σ is an initial axiomatic sequent. Similarly, Σ is tautological if and only if for each n-valued structure $\mathfrak{M}$ and for each valuation v of the language S in E_n there exist a $j \in E_n$ and a formula $\alpha \in \Gamma_j$ such that $v(\alpha) = j$. Clearly every initial sequent is deducible (the only leaf coincides then with the root of the tree) and it is trivially tautological. Assume now that the premises of the rule (σj) are tautological sequents. Then its conclusion is tautological too, since $j = s(j_1, \ldots, j_k)$. The adequacy of the rules $(\exists j)$ and $(\forall j)$ can be easily verified, while the rule (ex) is trivially adequate. A consideration of the three cases relative to the rule of cut easily shows that tautological premises always yield a tautological conclusion. □

Corollary 3.

The empty sequent is not deducible in the n-valued Gentzen-Takahashi calculus. □

Theorem 9. (Takahashi)

If a sequent Σ is tautological, then it is deducible.

Proof. Evidently, it is sufficient to prove that if a sequent $\Sigma = \Gamma_0 \vdash \ldots \Gamma_{n-1}$ is not deducible, then there exist a structure $\mathfrak{M}$ and a valuation $v : S \longrightarrow E_n$ such that for any $j \in E_n$ and each formula $\alpha \in \Gamma_j$, $v(\alpha) \neq j$. Let us then enumerate all the formulas of the language S in a sequence

$$\beta_1, \beta_2, \beta_3, \ldots, \beta_k, \ldots$$

We define a sequence of sequents

$$\Sigma_0, \Sigma_1, \Sigma_2, \ldots$$

and a sequence of values $w_1, w_2, w_3, \ldots$ so that the following conditions are satisfied:

(*a*) the sequents Σ_k are not deducible for $k \geq 0$,

(*b*) $\Sigma_{k-1} \subseteq \Sigma_k$ for $k > 0$,

(*c*) $\vdash_{w_k} \beta_k \subseteq \Sigma_k$ for $k > 0$.

We take the sequent Σ as Σ_0. Assume that the sequents Σ_l have been inductively defined for $l < m$. Then by corollary 3.2 there exists a value $w \in E_n$ such that the sequent $\Sigma_{m-1} \vdash_{E_n - \{w\}} \beta_m$ is not deducible. Let w_m be the least of such values. If β_m is of the form $\forall_x \gamma(x)$ or $\exists_x \gamma(x)$, then let z be the first free variable (under some numeration of variables) which does not occur neither in the sequent Σ_{m-1} nor in the formula β_m. Then the sequent Σ_m is defined by the following equality:

$$\Sigma_m = \Sigma_{m-1} \vdash_{E_n - \{w_m\}} \beta_m \vdash_{E_n - \{w_m\}} \gamma(z),$$

and in every other case, i.e. when β_m is not of the form $\forall_x \gamma(x)$ or $\exists_x \gamma(x)$, the sequent Σ_m is defined as follows:

$$\Sigma_m = \Sigma_{m-1} \vdash_{E_n - \{w_m\}} \beta_m.$$

It can be easily seen that the sequence

$$\Sigma_0, \Sigma_1, \ldots$$

defined as above satisfies the conditions $(a), (b)$ and (c).

Let $\mathfrak{M}$ be an n-valued structure for the system considered here and let v be a valuation of S in $\mathfrak{M}$. It can be proved inductively (Takahashi 1967) with respect to the complexity degree of the formula β_m, that

$$(d) \qquad v(\beta_m) = w_m.$$

This follows from the definition of the sequence of sequents $\Sigma_0, \Sigma_1, \ldots$ and from the adequate choice of the formulas β_m.

Let now $\beta_m \in \Gamma_j$ in the sequent Σ. Since $\Sigma = \Sigma_0$, the conditions $(a), (b), (c)$ and (d) imply that $w_m \neq j$ for each $j \in E_n$. Thus the sequent Σ is not tautological. □

Corollary 4.

A sequent Σ is deducible if and only if Σ is strictly deducible if and only if Σ is tautological.

Proof. Immediate by Theorems 3.7, 3.8 and 3.9. □

The Gentzen-type system for an n-valued first-order logical calculus, as presented above, can be generalized in various ways. One of them implies a more extensive notion of a propositional connective, considered by Moto-o-Takahashi [Takahashi 1967]. Questions related to the consequence operation in such systems shall not be considered here.

5. A Gentzen System for a Particular Class of Finitely Valued Logics

We shall now introduce a Gentzen-type formalization for a certain class of finitely valued propositional logics, where a sequent is defined — as in classical logic - as a pair of sets of formulas. This formalization is due to M. C. Fitting [Fitting 1991], to whom it served as a basis for research on multivalued modal logics.

First we formulate the precise semantical foundations for the class of logics presented. We assume that the truth values of our logic form a finite lattice $\mathbf{L}$ with the least element 0 and the greatest element 1, which will be also called *falsity* and *truth*, and with a lattice ordering $\leq$. Naturally we assume $0 \neq 1$. The elements of the universe of $\mathbf{L}$ will become logical constants in the language of our logic. In the sequel we shall only consider finite lattices.

Let $A, B \subseteq \mathbf{L}$. We say that $A \leq B$ if and only if

$(a) \forall_{x \in A} \exists_{y \in B} (x \leq y)$ and

$(b) \forall_{y \in B} \exists_{x \in A} (x \leq y)$.

We write $A < B$ if and only if $A \leq B$ and not $B \leq A$. It can be easily verified that the relation $\leq$ is transitive in the powerset $P(\mathbf{L})$, though it is not antisymmetric. The sets $\{0\}$ and $\{1\}$ are the least and the greatest elements, correspondingly, with respect to the relation $\leq$.

Lemma 12.

Let $A, B \subseteq \mathbf{L}$. If A and B are antichains and $A \leq B$ and $B \leq A$, then $A = B$. □

Corollary 5.

The class of all antichains in the lattice $\mathbf{L}$ is partially ordered by the relation $\leq$. □

A subset $A \subseteq \mathbf{L}$ is said to be a *spanning* set if and only if A is an antichain in $\mathbf{L}$ and

$$\forall_{x \in \mathbf{L}} \exists_{y \in A} (x \leq y \vee y \leq x).$$

Thus $\{0\}$ and $\{1\}$ are spanning sets. Each maximal (with respect to inclusion) antichain is a spanning set, too. Moreover, every antichain can be extended to a spanning set.

Let $A \subseteq \mathbf{L}$ be a spanning set. We say that y is *under* A if and only if

$$\exists_{x \in A} (y \leq x)$$

and y is *over* A if and only if

$$\exists_{z \in A} (z \leq y).$$

An element y is said to be *strictly under* A if and only if

$$\exists_{x \in A} (y \leq x \ \&\ y \notin A).$$

Correspondingly, y is strictly over A if and only if

$$\exists_{z \in A} (z \leq y \ \&\ y \notin A).$$

It can be proved that for any spanning set $A \subseteq \mathbf{L}$ and an arbitrary $x \in \mathbf{L}$, either $x \in A$ or x is strictly under A or x is strictly over A.

Theorem 10.

Let A and B be spanning sets in a lattice $\mathbf{L}$, and let $A \leq B$. Assume moreover that there are elements of $\mathbf{L}$ which are both strictly over A and strictly under B. Then there exists a spanning set C such that $A < C < B$.

Proof. Let C_0 be a maximal antichain consisting of those elements of $\mathbf{L}$ which are both strictly over A and strictly under B. Clearly $C_0 \neq \emptyset$. Extend C_0 to a set C_1 by adding all those elements in A, which do not destroy the property of the set of being an antichain. Next we add to C_1 all those elements $y \in B$ which again preserve the property of being an antichain. Let C be the set so obtained. By construction C is obviously an antichain. Let $x \in \mathbf{L}$. If x is not both strictly over A and strictly under B, then x is under A or x is over B. Consider the following possible cases:

(1) Assume x is both strictly over A and strictly under B. We shall say then that x is strictly between A and B. If $x \in C_0$, then the conclusion of the theorem is trivially true. If $x \notin C_0$, then x is comparable with some element $y \in C_0$, since we have assumed that C_0 is a maximal antichain of elements strictly between A and B. Thus each element strictly between A and B is comparable with some element of C.

(2) Let now x be under A. Then, clearly, for some $y \in A$, $x \leq y$. If $y \in C_1$, then x is under C_1, which implies that x is under C. If $y \notin C_1$, then adding y to C_1 yields a set which is not an antichain. The elements of A being incomparable, there exists a $z \in C_0$ such that y and z are comparable. If $y \leq z$, then $x \leq z$ and consequently x is under C. The assumption $z \leq y$ gives rise to contradiction.

(3) Assume x is over B. Hence there exists an element $y \in B$ such that $y \leq x$. If $y \in C$, then evidently x is over C. On the other hand, if $y \notin C$, then y is comparable with some element in C. The set B being a maximal antichain, y must be comparable with some element in C_1. Two cases are possible:

   (a) The element y is comparable with some element $z \in C_0$. Then if $z \leq y$, we get $z \leq x$ and thus x is over C. Observe that $y \leq z$ is not possible.

   (b) The element y is comparable with some $z \in C_1 - C_0$. If $z \leq y$, then $z \leq x$ and x is over C. Let therefore $y \leq z$. Since $z \in A$ and $A \leq B$, for some y_1 we have $z \leq y_1$, which implies $y \leq z \leq y_1$. The set B being an antichain, we infer that $y = z = y_1$. Thus $y \leq x$ and $z \leq x$ imply that x is over C.

We shall prove now that $A < C$ and $C < B$. Evidently we have $A \leq C$ and $C \leq B$. If $C \leq A$, then $A = C$, which is not possible, since C has lements which are strictly over A. Hence $A < C$. Similarly we prove that $C < B$. □

Assume now that $B \subseteq \mathbf{L}$ is a spanning set different from the set $\{0\}$. Thus there exists a spanning set Y such that $Y < B$; one of them is the set $\{0\}$. The lattice $\mathbf{L}$ being finite, there exists a spanning set $A < B$, maximal with respect to the relation $\leq$ in the class of spanning subsets of B. We shall call it an immediate predecessor of B.

Lemma 13.

If $B \subseteq \mathbf{L}$ is a spanning set not equal to $\{0\}$, then there exists an immediate predecessor of B such that for any $x \in \mathbf{L}$, either x is under A or x is over B.

Proof. It is evident that such a set exists. Theorem 3.10 implies that if A is an immediate predecessor of B, then no element of the lattice $\mathbf{L}$ is strictly between A and B, i.e. no element of the lattice is both strictly over A and strictly under B. Consequently, each $x \in \mathbf{L}$ is under A or over B. □

As we have mentioned at the beginning of this section, we can assume without loss of generality that the elements of the lattice $\mathbf{L}$ coincide with the propositional constants of our language. Thus we have in our alphabet propositional variables, logical constants, the connectives for the disjunction ($\vee$) and for the conjunction ($\wedge$) and the implication symbol ($\Rightarrow$). The set of well-formed formulas is defined in a standard way, nevertheless it may be convenient to recall its definition here. Thus, the set $S_{\mathbf{L}}$ of formulas is the least set $\mathbf{Y}$ satisfying the following conditions:

(a) $\mathbf{V} \subseteq \mathbf{Y}$ and $|\mathbf{L}| \subseteq \mathbf{Y}$, where $|\mathbf{L}|$ is the universe of the lattice $\mathbf{L}$. Hence propositional variables and constants are formulas; they will be called *atomic formulas* or just *atoms*, for short.

(b) If $\alpha, \beta \in \mathbf{Y}$, then $\alpha \vee \beta \in \mathbf{Y}, \alpha \wedge \beta \in \mathbf{Y}$ and $\alpha \Rightarrow \beta \in \mathbf{Y}$.

A *valuation* of the language $S_{\mathbf{L}}$ in the lattice $\mathbf{L}$ is any mapping

$$v : \mathbf{V} \cup |\mathbf{L}| \longrightarrow \mathbf{L}$$

which is an identity on the set $|\mathbf{L}|$. Every such valuation extends naturally to a homomorphism

$$h^v : S_{\mathbf{L}} \longrightarrow \mathbf{L}$$

in the following way:

(a) $h^v(\alpha) = v(\alpha)$ for $\alpha \in \mathbf{V} \cup |\mathbf{L}|$,

(b) $h^v(\alpha \vee \beta) = h^v(\alpha) \vee h^v(\beta)$,

(c) $h^v(\alpha \wedge \beta) = h^v(\alpha) \wedge h^v(\beta)$,

where $\wedge, \vee$ are the lattice operations, while the implication satisfies the following conditions:

(d$_1$) $h^v(\alpha \Rightarrow \beta) = 1$ and only if and only if $h^v(\alpha) \leq h^v(\beta)$,

(d$_2$) if $h^v(\alpha) > h^v(\beta)$, then $h^v(\alpha \Rightarrow \beta) < 1$, where 1 is the unit of the lattice $\mathbf{L}$ and $x > y$ is defined as $y < x$.

Let $\alpha, \beta, \gamma, \ldots$ be arbitrary formulas. Implications of formulas will be denoted by Greek letters $\kappa, \lambda, \mu \ldots$, while sets of implications will be denoted by capital Greek letters $\Gamma, \Delta, \Lambda, \ldots$. The expression Γ, κ represents the set $\Gamma \cup \{\kappa\}$. A *sequent* is an ordered pair (Γ, Δ) of sets of implications and it will be denoted by

$$\Gamma \vdash \Delta.$$

A sequent $\Gamma \vdash \Delta$ is *tautological* if and only if for each valuation v either there exists an element $\kappa \in \Gamma$ such that $h^v(\kappa) < 1$ or there exists an element $\lambda \in \Delta$ such that $h^v(\lambda) = 1$.

We shall now introduce a system $G_{\mathbf{L}}$ consisting of axiomatic sequents and rules, with a standard definition of a proof tree.

The axiomatic sequents are the following:

(a$_1$) $\kappa \vdash \kappa$

(a$_2$) $\kappa \Rightarrow \lambda, \lambda \Rightarrow \mu \vdash \kappa \Rightarrow \mu$

(a$_3$) $\vdash \alpha \wedge \beta \Rightarrow \alpha$

(a$_4$) $\vdash \alpha \wedge \beta \Rightarrow \beta$

(a$_5$) $\alpha \Rightarrow \beta, \alpha \Rightarrow \gamma \vdash \alpha \Rightarrow \beta \wedge \gamma$

(a$_6$) $\vdash \alpha \Rightarrow \alpha \vee \beta$

(a$_7$) $\vdash \beta \Rightarrow \alpha \vee \beta$

(a$_8$) $\alpha \Rightarrow \gamma, \beta \Rightarrow \gamma \vdash \alpha \vee \beta \Rightarrow \gamma$

(a$_9$) for any $s, t \in \mathbf{L}$, if $s \leq t$, then $\vdash s \Rightarrow t$

(a$_{10}$) for any $s, t \in \mathbf{L}$, if $s > t$, then $s \Rightarrow t \vdash$

The rules:

weakening:

(os) $\dfrac{\Gamma \vdash \Delta}{\Gamma, \Lambda \vdash \Delta, \Pi}$, where Γ, Λ stands for the union $\Gamma \cup \Lambda$

cut:

(ct) $\dfrac{\Gamma \vdash \Delta, \kappa; \Gamma, \kappa \vdash \Delta}{\Gamma \vdash \Delta}$

compression:

$$(s_1)\quad \frac{\Gamma \vdash \Delta, \gamma \Rightarrow \alpha; \Gamma, \gamma \Rightarrow \beta \vdash \Delta}{\Gamma, \alpha \Rightarrow \beta \vdash \Delta}$$

$$(s_2)\quad \frac{\Gamma \vdash \Delta, \beta \Rightarrow \gamma; \Gamma, \alpha \Rightarrow \vdash \Delta}{\Gamma, \alpha \Rightarrow \beta \vdash}$$

implicational rules:

$$(ip)\quad \frac{\{\Gamma, t \Rightarrow \alpha \vdash \Delta, \{t \Rightarrow \beta | t \in |\mathbf{L}|\}}{\Gamma \vdash \Delta, \{\alpha \Rightarrow \beta\}}$$

$$(in)\quad \frac{\{\Gamma, \beta \Rightarrow t \vdash \Delta, \{\alpha \Rightarrow t | t \in |\mathbf{L}|\}}{\Gamma \vdash \Delta, \alpha \Rightarrow \beta}$$

Lemma 14.

The compression rules (s_1) and (s_2) are deducible from other rules in the system introduced above.

Proof.

$$\frac{\dfrac{\Gamma \vdash, \gamma \Rightarrow \alpha; \gamma \Rightarrow \alpha, \alpha \Rightarrow \beta \vdash \gamma \Rightarrow \beta}{\Gamma, \alpha \Rightarrow \beta \vdash \Delta, \gamma \Rightarrow \beta; \Gamma, \gamma \Rightarrow \beta \vdash \Delta}}{\Gamma, \alpha \Rightarrow \beta \vdash \Delta}$$

Deducibility of (s_2) is proved similarly. □

Lemma 15.

The sequent $\vdash \alpha \Rightarrow \alpha$ is deducible for any formula $\alpha \in S_{\mathbf{L}}$.

Proof. For each $t \in \mathbf{L}$, in view of the sequent $t \Rightarrow \alpha \vdash t \Rightarrow \alpha$ being an axiom, the rule (ip) yields $\vdash \alpha \Rightarrow \alpha$. □

The rules (s_1) and (s_2) are deducible, as stated in Lemma 3.14. On the other hand, with these two rules in the system, the axiom (a_2) is derivable.

Lemma 16.

The axiom (a_2) can be deduced from (a_1) using the rules $(os), (ct), (s_1)$ and (s_2).

Proof.

$$\frac{\dfrac{t \Rightarrow \alpha \vdash t \Rightarrow \alpha}{t \Rightarrow \alpha \vdash t \Rightarrow \gamma, t \Rightarrow \beta, t \Rightarrow \alpha} \quad \dfrac{t \Rightarrow \beta \vdash t \Rightarrow \beta}{t \Rightarrow \alpha, t \Rightarrow \beta \vdash t \Rightarrow \gamma, t \Rightarrow \beta}}{t \Rightarrow \alpha, \alpha \Rightarrow \beta \vdash t \Rightarrow \gamma, t \Rightarrow \beta}$$

It can be observed that $t \Rightarrow \alpha \vdash t \Rightarrow \alpha$ and $t \Rightarrow \beta \vdash t \Rightarrow \beta$ are axioms, and the proof involves the weakening rule and the compression rule (s_1).

$$\frac{\dfrac{t \Rightarrow \gamma \vdash t \Rightarrow \gamma}{t \Rightarrow \alpha \vdash t \Rightarrow \gamma, t \Rightarrow \gamma, t \Rightarrow \alpha} \quad \dfrac{t \Rightarrow \gamma \vdash t \Rightarrow \gamma}{t \Rightarrow \alpha, t \Rightarrow \gamma, t \Rightarrow \beta, \vdash t \Rightarrow \gamma, t \Rightarrow \alpha}}{\alpha \Rightarrow \beta, t \Rightarrow \alpha, t \Rightarrow \gamma \vdash t \Rightarrow \gamma}$$

The sequents so obtained will become premises in the following proof tree, in which we apply successively the rules (s_1) and (ip):

$$\frac{\dfrac{t \Rightarrow \alpha, \alpha \Rightarrow \beta \vdash t \Rightarrow \gamma, t \Rightarrow \beta; \alpha \Rightarrow \beta, t \Rightarrow \alpha, t \Rightarrow \gamma \vdash t \Rightarrow \gamma}{\alpha \Rightarrow \beta, \beta \Rightarrow \gamma, t \Rightarrow \alpha \vdash t \Rightarrow \gamma}}{\alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vdash \alpha \Rightarrow \gamma}$$

which completes the proof. □

Let $W = \{w_1, \ldots, w_k\}$ be the set of distinguished values. Assume $0 \notin W$. A formula $\alpha \in S_{\mathbf{L}}$ is a theorem of the system $G_{\mathbf{L}}$ if and only if the sequent $\vdash w_1 \Rightarrow \alpha, \ldots, w_k \Rightarrow \alpha$ is deducible in $G_{\mathbf{L}}$.

Lemma 17.

The sequents $\vdash \alpha \Rightarrow 1$ and $\vdash 0 \Rightarrow \alpha$ are theorems, for any $\alpha \in S_{\mathbf{L}}$.

Proof. Immediate from the axioms (a_9) and (a_{10}) and the rules (os), (ip), (in). For example:

$$\frac{\dfrac{\vdash t \Rightarrow 1}{t \Rightarrow \alpha \vdash t \Rightarrow 1}}{\vdash \alpha \Rightarrow 1}$$

□

Lemma 18. (Fitting 1991)

Let $A = \{x_1, x_2, \ldots, x_m\}$ and $B = \{y_1, \ldots, y_s\}$ be spanning sets in $\mathbf{L}$, and assume A is an immediate predecessor of B. Then, for any formula $\alpha \in S_{\mathbf{L}}$, the following sequent is deducible in $G_{\mathbf{L}}$:

$$\vdash \alpha \Rightarrow x_1, \ldots, \alpha \Rightarrow x_m, y_1 \Rightarrow \alpha, \ldots, y_s \Rightarrow \alpha.$$

Theorem 11.

The following rule

$$\frac{\{\Gamma, \alpha \Rightarrow t, t \Rightarrow \alpha \vdash \Delta | t \in \mathbf{L}\}}{\Gamma \vdash \Delta}$$

is deducible in $G_{\mathbf{L}}$.

Proof. Assume the sequent

$$\Gamma, \alpha \Rightarrow t, t \Rightarrow \alpha \vdash \Delta$$

is deducible for any $t \in \mathbf{L}$. We say that a spanning set A is complete with respect to a sequent $\Gamma \vdash \Delta$ and a formula α if and only if for each $x \in A$ the sequent $\Gamma, x \Rightarrow \alpha \vdash \Delta$ has a proof, i.e. is deducible. Obviously the set $\{1\}$, where 1 is the unit of the lattice $\mathbf{L}$, is complete with respect to $\Gamma \vdash \Delta$ and α, since due to lemma 3.17, our assumption and the rule of cut we have

$$\frac{\Gamma, \alpha \Rightarrow 1, 1 \Rightarrow \alpha \vdash \Delta; \vdash \alpha \Rightarrow 1}{\Gamma, 1 \Rightarrow \alpha \vdash \Delta}.$$

Let now $B = \{y_1, \ldots, y_k\}$ be a complete spanning set, and assume $A = \{x_1, \ldots, x_m\}$ is an immediate spanning predecessor of B. The sequent

$$\vdash \alpha \Rightarrow x_1, \ldots, \alpha \Rightarrow x_m, y_1 \Rightarrow \alpha, \ldots, y_k \Rightarrow \alpha$$

is obviously deducible by Lemma 3.18. The set B being complete with respect to $\Gamma \vdash \Delta$ and α, the sequents $\Gamma, y_i \Rightarrow \alpha \vdash \Delta$ are deducible for each $i = 1, \ldots, k$. Applying k times the rule of cut (ct), we deduce the sequent

$$\Gamma \vdash \Delta, \alpha \Rightarrow x_1, \ldots, \alpha \Rightarrow x_m.$$

Applying our assumption, the rule (ct) and the fact that A is an antichain, we get successively

$$\frac{\Gamma, \alpha \Rightarrow x_1, x_1 \Rightarrow \alpha \vdash \Delta; \Gamma \vdash \Delta, \alpha \Rightarrow x_1, \ldots, \alpha \Rightarrow x_m}{\Gamma x_1 \Rightarrow \alpha \vdash \Delta, \alpha \Rightarrow x_2, \ldots, \alpha \Rightarrow x_m}$$

$$\frac{x_1 \Rightarrow \alpha, \alpha \Rightarrow x_2 \vdash}{x_1 \Rightarrow x_2; x_1 \Rightarrow x_2 \vdash}$$

Thus by an application of the rule of cut we arrive at

$$\frac{\Gamma, x_1 \Rightarrow \alpha \vdash \Delta, \alpha \Rightarrow x_2, \ldots, \alpha \Rightarrow x_m; x_1 \Rightarrow \alpha, \alpha \Rightarrow x_2 \vdash}{\Gamma, x_1 \Rightarrow \alpha \vdash \Delta, \alpha \Rightarrow x_3, \ldots, \alpha \Rightarrow x_m}.$$

Iterating this procedure we eliminate $x_3, x_4, \ldots, x_m$, which proves the deducibility of the sequent

$$\Gamma, x_1 \Rightarrow \alpha \vdash \Delta$$

for every choice of the element x_1.

Similarly we prove deducibility of each sequent

$$\Gamma, x_i \Rightarrow \alpha \vdash \Delta$$

for $i = 1, \ldots, m$, which implies the completeness of A with respect to $\Gamma \vdash \Delta$ and α.

Take now the set $\{1\}$. Choose C_1 to be its immediate spanning predecessor and let C_2 be an immediate spanning predecessor of C_1. Continuing this choice of predecessors we get a decreasing sequence of spanning sets. This sequence is obviously finite and its very last element is the set $\{0\}$, where 0 - as usual - represents the zero of the lattice $\mathbf{L}$. It follows from theorem 3.11 that each element of this sequence is complete with respect to $\Gamma \vdash \Delta$ and α. In particular, the set $\{0\}$ is complete, which proves the deducibility of $\Gamma, 0 \Rightarrow \alpha \vdash \Delta$.

Hence

$$\frac{\Gamma, 0 \Rightarrow \alpha \vdash \Delta; \vdash 0 \Rightarrow \alpha}{\Gamma \vdash \Delta},$$

which proves the deducibility of the sequent $\Gamma \vdash \Delta$. □

Now, let κ be an implication and X - a set of implications. The set X is κ-inconsistent if and only if there exists a finite set $\Gamma \subseteq X$ such that the sequent $\Gamma \vdash \kappa$ has a proof in the system $G_{\mathbf{L}}$. A set which is not κ-inconsistent will be called κ-consistent. A κ-consistent set X can be extended in a natural way to a maximal κ-consistent set.

Lemma 19.

If X is a maximal κ-consistent set and for some $\Gamma \subseteq X$ the sequent $\Gamma \vdash \lambda$ is deducible, then $\lambda \in X$.

Proof. Immediate by an application of the rule of cut. □

Theorem 12.

If X is a maximal κ-consistent set, then for each formula $\alpha \in S_{\mathbf{L}}$ there exists exactly one $x \in \mathbf{L}$ such that $x \Rightarrow \alpha$ and $\alpha \Rightarrow x$ are in X.

Proof. Observe that for $x \neq y$ the formulas $x \Rightarrow \alpha, \alpha \Rightarrow x, y \Rightarrow \alpha$ and $\alpha \Rightarrow y$ cannot all be in X. Indeed, if this were so, then assuming, for instance that $x \not\leq y$, we get by an application of the rule of cut

$$\frac{x \Rightarrow \alpha, \alpha \Rightarrow}{x \vdash x \Rightarrow y; x \Rightarrow y \vdash} x \Rightarrow \alpha, \alpha \Rightarrow y \vdash$$

Applying now the rule of weakening we infer that X is κ-inconsistent. Assume then that there is no x such that both $x \Rightarrow \alpha$ and $\alpha \Rightarrow x$ are in X. Then, due to the finiteness of the lattice $\mathbf{L}$ there exists a finite set $\Gamma \subseteq X$ such that the sequent $\Gamma, \alpha \Rightarrow x, x \Rightarrow \alpha \vdash \kappa$ is deducible in $G_{\mathbf{L}}$, for any $x \in \mathbf{L}$. Theorem 3.11 implies that $\Gamma \vdash \kappa$ is deducible, which contradicts the κ-consistency of the set X. □

Theorem 13. (Fitting 1991)

If an implication κ is a tautology, then it is deducible in $G_{\mathbf{L}}$.

Proof. Supppose κ is not deducible. Evidently the empty set $\emptyset$ is κ-consistent, so extend it to a maximal κ-consistent set X. Define a valuation

$$v : \mathbf{V} \cup |\mathbf{L}| \longrightarrow \mathbf{L}$$

as follows:

$$v(\alpha) = x \text{ if and only if } \alpha \Rightarrow x, x \Rightarrow \alpha \in X.$$

Due to theorem 3.12 this valuation is well defined, so we can extend it in a standard way to a homomorphism

$$h^v : S_{\mathbf{L}} \longrightarrow \mathbf{L}.$$

It is easy to see that, for each $\alpha \in S_{\mathbf{L}}$, the value $h^v(\alpha)$ is an element $y \in \mathbf{L}$ such that $y \Rightarrow \alpha$ and $\alpha \Rightarrow y$ are in X. Let $\kappa = \alpha \Rightarrow \beta$, and assume $h^v(\alpha) = x$ and $h^v(\beta) = y$. If $x \leq y$, then the set X is κ-inconsistent, which follows from the scheme below by an application of the rule of cut:

$$\frac{\alpha \Rightarrow x, y \Rightarrow \beta, x \Rightarrow y \vdash \alpha \Rightarrow \beta \vdash x \Rightarrow y}{\alpha \Rightarrow x, y \Rightarrow \beta \vdash \alpha \Rightarrow \beta}.$$

Thus it is not true that $x \leq y$, so $h^v(\kappa) = h^v(\alpha \Rightarrow \beta) \neq 1$, which completes the proof of the theorem. □

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