Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XXII (2017)
23–29
DOI http://dx.doi.org/10.16926/m.2017.22.02

GEOGEBRA AND PROBLEMS OF TANGENT CIRCLES

MARTIN BILLICH

Abstract

There are many mathematical programmes for teaching and learning geometry from middle school to the university level. Dynamic software package Geogebra can help students to explore and understand more concepts in geometry on their own. In this paper we are concentrating in one of the most interesting and less known two-dimensional transformations like inversion with respect to a circle supported by GeoGebra. The use of this transformation through GeoGebra makes possible a number of elegant solutions to classical construction problems in geometry. Contribution of this paper is presentation of some problems which require the construction of the circle tangent to given circles.

1. Introduction

By computers with dynamic geometry software we are able to solve problems which are difficult to solve by classical approach. GeoGebra is a multiplatform mathematical software that is simple to use and which is successfully applied at all levels of education in geometry, algebra and calculus. The basic idea of GeoGebra's interface is to provide two presentations of each mathematical object in its algebra and graphics windows. If we change an object in one of these windows, its presentation in the other one will be immediately updated. The main advantages of using GeoGebra are:

- GeoGebra offers easy-to-use interface, multilingual menus, commands and help.
- The use of GeoGebra in the classroom does not require any prior training for students. Students can learn how to use the main features directly during problem-solving activities and exercises.
- In GeoGebra environment students can manipulate variables easily by simply dragging free objects around the plane of drawing, or by using sliders and, in fact, students can see how the dependent objects will be affected.

• *Martin Billich* — e-mail: martin.billich@ku.sk
Catholic University in Ružomberok.

- The use of GeoGebra allows students to explore a wider range of function types, and provides students to make the connections between symbolic and visual representations.
- The algebra input allows the user to generate new objects or to modify those already existing, by the command line.
- The worksheet files can easily be exported or published as web pages.

One of the transformations in GeoGebra environment is the so-called *inversion* (or "*reflection about circle*"). This map was first introduced by J. Steiner about 1830. The inversion have many properties in common with line reflection. An important property of line reflection is that it preserves basic geometric classifications, straight lines go into straight lines and circles go into circles. Although the inversion preserves the class of lines and circles but can transform a line into a circle and a circle into a line. This and other exceptional properties of inversion serve as a foundation for its effectiveness in solving various problems in geometry.

In next section we discuss the main characteristics of inversion as well as some properties to be used throughout the paper. The use of inversion allows us to develop a unified method of solution for many problems in elementary geometry, especially those concerning construction of circles tangent to one or several circles and, supported by GeoGebra.

2. The Inversion

In plane, suppose ω is a circle with center O and radius R. We call the transformation that sends an arbitrary point P distinct from O into point P' lying on the ray $\overrightarrow{OP}$ such that $|OP| \cdot |OP'| = R^2$ the *inversion relative to circle* ω. The inversion relative ω will be also called the inversion with center O and degree R^2 and ω will be called the circle of inversion.

In this paper we shall fix $\mathcal{I}$ as the inversion relative to circle of radius R with center O. First, we will establish the following basic properties of inversion $\mathcal{I}$.

- Inversion is a transformation from $\mathbb{R}^2 \setminus \{O\}$ onto itself.
- Inversion is a bijection of order 2, that is if P' is the image of P, then P is the image of P'.
- Points on the circle of inversion stay fixed.
- Points inside [outside] of the circle of inversion are moved outside [inside].
- A line l going through the center of inversion is invariant under the inversion.
- Let $P \neq O$ be a point inside circle ω of inversion. To determine the position of P' on the ray $\overrightarrow{OP}$, draw a chord through P perpendicular

to OP meeting ω at T and S. Then the tangents to ω at T and S meet at the inverse point P'.

- If P is a point outside ω, then the two tangents from P to ω determine the chord TS and, the intersection between TS and the ray $\overrightarrow{OP}$ gives the inverse point P'.
- Suppose the points P and Q are different from each other and from the point O and, that the points O, P and Q are noncollinear. If $P' = \mathcal{I}(P)$ and $Q' = \mathcal{I}(Q)$, then the triangles OPQ and $OQ'P'$ are similar. That is, $\angle POQ \cong \angle Q'OP'$.

The following properties regarding the inversion of lines and circles hold (for details, see [2, 3, 6]):

- The inverse of any line l not through the center O of inversion, is a circle through O (minus the point O itself), and the diameter through O of this circle is perpendicular to l.
- The inverse of any circle through the center O of inversion (with O omitted) is a line perpendicular to the diameter through O, that is a line parallel to the tangent at O to the circle.
- The inverse of a circle not passing through the center O of inversion is a circle not passing through O. In particular, every circle orthogonal to the circle of inversion is its own inverse.
- Under the inversion the angle between lines [circles] is equal to the angle between their images.
- The angle between a circle and a line is equal to the angle between the images of these figures under the inversion.

3. Problems of tangent circles

In this section we will illustrate the type of problems that can be solved by method of inversion relative to circle in the GeoGebra environment. It is obvious that all of the construction problems of this section could be solved by students in the classroom with the help of a ruler and a compass too.

Now consider two problems (see also [1, 2]) which require the construction of circles tangent to several circles.

Problem 1. Construct all circles which are tangent to two given circles c_1, c_2 and pass through a given point O, lying outside c_1 and c_2.

Suppose c is one of the desired circles. Let $\mathcal{I}$ be an inversion with center O. Then $\mathcal{I}$ maps the circles c_1, c_2 into the circles c_1', c_2' and the circle c into a common tangent c'. It is obvious that the solution to the problem are circles which are the images of the common tangents of the circles c_1', c_2' under the inversion $\mathcal{I}$. Since in generally there are four of these tangents (in figure 1 they are shown by dotted lines), the problem has four solutions.

All the steps above may be demonstrated in the environment of software GeoGebra.

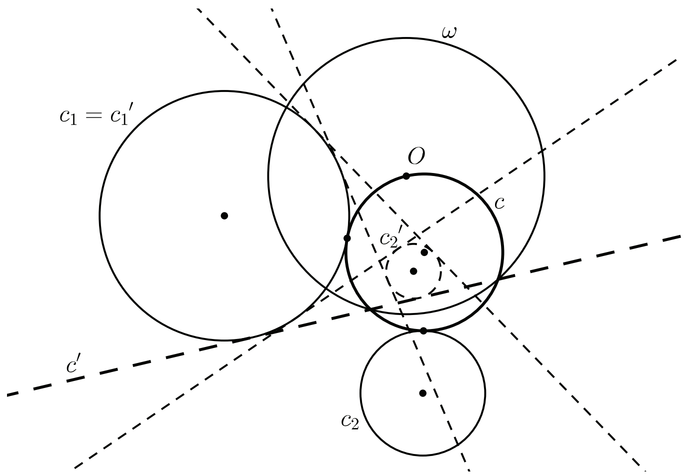

Figure 1

In our case a circle c_1 is orthogonal to the circle of inversion ω and therefore c_1 is invariant under this inversion ($c_1 = c_1'$).

Problem 2. (*Problem of Apollonius*). Construct all circles which are tangent to three given circles c_1, c_2 and c_3 of radii r_1, r_2 and r_3 around the points O_1, O_2 and O_3, respectively.

The three-circle case generally has eight solutions. Number of solution circles depend on the arrangement of the given circles. We shall present solutions to this problem for two different position of the circles c_1 and c_2:

A. The circles c_1 and c_2 are tangent (externally) at a single point P.
Solution. Suppose the circle c of radius r and centered at O is one of the desired circles. Let $\mathcal{I}$ be an inversion with center P and radius R such that the circle of inversion intersect the circles c_1 and c_2 in pairs of distinct points or/and the circle c_3 is orthogonal to c. The inversion $\mathcal{I}$ maps the circles c_1 and c_2 into a pair of parallel lines c_1' and c_2', and the circle c_3 into a circle c_3'. In the case of orthogonality of circles c_3 and c, the circle c_3 is invariant under this inversion ($c_3 = c_3'$). The circle c is taken by the inversion $\mathcal{I}$ into a circle c', which is tangent to circle c_3' and to both the parallel lines c_1' and c_2'. In this way, the solution of our problem has been reduced to a simpler construction problem: to construct all circles tangent to a given pair of parallel lines and to a given circle. Since in generally there are four of these tangent circles (in figure 2 they are shown by dotted lines), the problem has four solutions. As shown in figure 2, all the previsious steps

may be demonstrated in the GeoGebra environment. With the help of this software, by dragging the center O_3 of given circle c_3 around the plane of drawing, we can see how the solution circles will be changed.

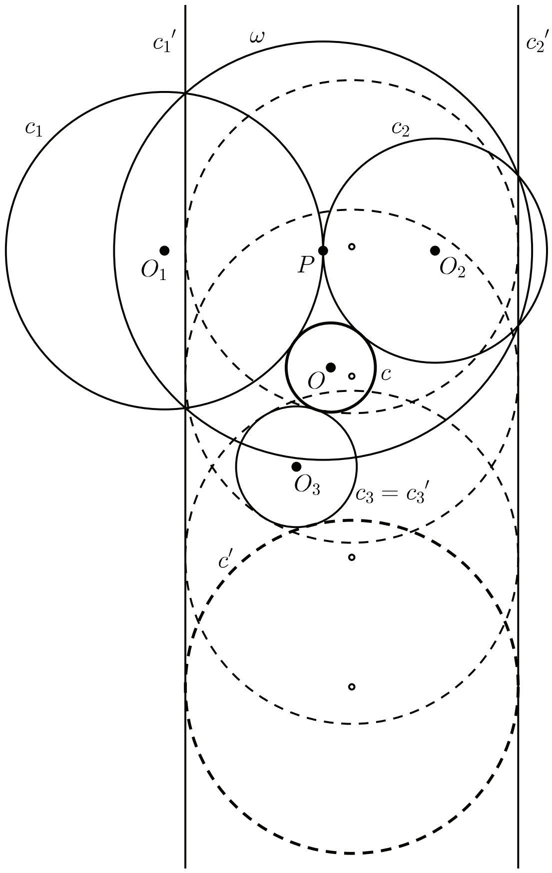

Figure 2

B. The circles c_1 and c_2 have no points in common (and c_3 is a circle outside c_1 and c_2).

Solution. Suppose the circle c of radius r and centered at O is one of the desired circles (figure 3). We connect the segment O_1O_2 from the centers of the circles c_1 and c_2 and draw circles of radii $r_1 + d$, $r_2 + d$, $r_3 + d$ with centers O_1, O_2, O_3, respectively, where

$$d = \frac{|O_1O_2| - (r_1 + r_2)}{2}.$$

We denote the constructed circles by $\overline{c}_1$, $\overline{c}_2$, $\overline{c}_3$ respectively (in figure 3 they are shown by dotted lines). Let $\overline{c}$ be the circle concentric with a circle c of radius $\overline{r} = r - d$. It is obvious that if we can construct the circle $\overline{c}$, we can easily construct the circle c. It is easy to see that $\overline{c}$ is tangent to the circles $\overline{c}_1$, $\overline{c}_2$, $\overline{c}_3$. The circles $\overline{c}_1$, $\overline{c}_2$ are constructed so that they are tangent to one another at some point P. This reduces it to a case **A**.

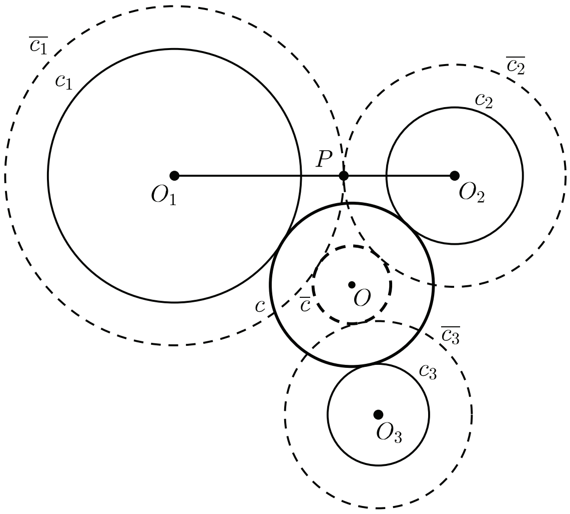

Figure 3

We leave the finalization of solution of this problem to the reader and suggest that the reader verify that the pair of the circles c_1 and c_3 or c_2 and c_3 could be used in place of the pair c_1 and c_2 in the above construction. Using the software GeoGebra it could be obtained all eight solution circles (see figure 4, [4]).

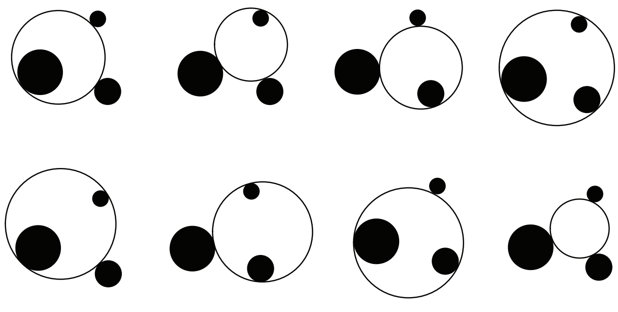

Figure 4

4. Conclusion

In this paper, some construction problems in plane geometry which require the construction of the tangent circles, by the inversion relative to circle and with the help of software GeoGebra, are introduced. The main strategy of inversion is to transform a given tangency problem into another problem of tangent circles that is simpler to solve. Subsequently, the solutions to the original problem are found from the solutions of the transformed problem by undoing the transformation. The usefulness of inversion can be increased significantly by resizing of given circles (see also [1, 5]). One usually inverts in a circle centered at a point where given circles intersect, since these will all be sent to lines. Quite often, only the center of inversion matters, and not the radius of the circle. Inversion, supported by dynamic geometry software like GeoGebra, is useful for solving various problems in elementary geometry.

References

[1] I. Ya. Bakel'man, *Inversions*, Chicago, IL: University of Chicago Press, 1974.

[2] M. Billich, *Circle inversion and problems on tangent circles*, In: Proceedings of International Congress IMEM 2009, Catholic University in Ružomberok, 2009, 492-496.

[3] H. S. M. Coxeter, *Introduction to geometry*, John Wiley & Sons., 1961.

[4] D. Gisch; J. M. Ribando, *Apollonius' Problem: A Study of Solutions and Their Connections*, American Journal of Undergraduate Research 3, 2004, 15–25.

[5] Z. Sklenáriková, *K metódam riešenia Apolloniovej úlohy*, In: Matematika v proměnách veku III, Edícia Dějiny matematiky, Praha, 2004, 45-55.

[6] Z. Sklenáriková, J. Čižmár, *Elementárna geometria euklidovskej roviny*, FMFI UK Bratislava, 2002.

Received: July 2017

Martin Billich
Catholic University in Ružomberok,
Department of Mathematics, Faculty of Education,
Place A. Hlinka 56/1, 034 01 Ružomberok, Slovakia
E-mail address: martin.billich@ku.sk