

**FRACTIONAL HEAT CONDUCTION IN AN INFINITE  
ROD WITH HEAT ABSORPTION PROPORTIONAL  
TO TEMPERATURE**

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ABSTRACT

The one-dimensional time-fractional heat conduction equation with heat absorption (heat release) proportional to temperature is considered. The Caputo time-fractional derivative is utilized. The fundamental solutions to the Cauchy and source problems are obtained using the Laplace transform with respect to time and the exponential Fourier transform with respect to the spatial coordinate. The numerical results are illustrated graphically.

1. INTRODUCTION

The classical heat conduction is based on the Fourier law which constitutes the linear dependence between the heat flux and the temperature gradient. As a result, the temperature  $T$  satisfies the standard parabolic equation

$$(1) \quad \frac{\partial T}{\partial t} = a \Delta T,$$

where  $t$  is time,  $\Delta$  denotes the Laplace operator,  $a$  is the heat diffusivity coefficient.

If volume heat absorption proportional to temperature occurs in the medium, then instead of (1) we obtain [10]

$$(2) \quad \frac{\partial T}{\partial t} = a \Delta T - bT.$$

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The cases  $b > 0$  and  $b < 0$  correspond to absorption and release of heat, respectively.

Time-nonlocal generalizations of the Fourier law were studied by many authors (see, for example, [5], [8], [14], [15], [18] and references therein). The time-nonlocal dependence between the heat flux and the temperature gradient with the “long-tail” power kernel [11], [12], [13] can be interpreted in terms of fractional integrals and derivatives and leads to the time-fractional heat conduction equation

$$(3) \quad \frac{\partial^\alpha T}{\partial t^\alpha} = a \Delta T, \quad 0 < \alpha \leq 2.$$

The time-fractional counterpart of equation (2) has the following form

$$(4) \quad \frac{\partial^\alpha T}{\partial t^\alpha} = a \Delta T - bT, \quad 0 < \alpha \leq 2.$$

Here  $\frac{\partial^\alpha f}{\partial t^\alpha}$  is the Caputo fractional derivative [3], [6], [9]:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, \quad n-1 < \alpha < n,$$

where  $\Gamma(x)$  is the gamma function,  $n$  is an integer.

## 2. FUNDAMENTAL SOLUTION TO THE CAUCHY PROBLEM

Consider the time-fractional heat conduction equation (4) in an infinite rod:

$$(5) \quad \frac{\partial^\alpha T}{\partial t^\alpha} = a \frac{\partial^2 T}{\partial x^2} - bT, \quad -\infty < x < \infty, \quad 0 < t < \infty, \quad 0 < \alpha \leq 2,$$

under the initial conditions

$$(6) \quad t = 0: \quad T = p_0 \delta(x), \quad 0 < \alpha \leq 2,$$

$$(7) \quad t = 0: \quad \frac{\partial T}{\partial t} = 0, \quad 1 < \alpha \leq 2,$$

where  $\delta(x)$  is the Dirac delta function. The constant coefficient  $p_0$  is introduced to obtain the nondimensional quantities used in numerical calculations (see Eq. (14)).

The zero condition at infinity

$$(8) \quad \lim_{x \rightarrow \pm\infty} T(x, t) = 0$$

is also assumed.

To solve the problem (5)–(8), the integral transform technique will be employed. Recall that the Laplace transform with respect to time  $t$  is defined as [2]

$$\begin{aligned}\mathcal{L}\{f(t)\} &= f^*(s) = \int_0^\infty f(t) e^{-st} dt, \\ \mathcal{L}^{-1}\{f^*(s)\} &= f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) e^{st} ds, \quad t > 0,\end{aligned}$$

where the asterisk denotes the transform,  $s$  is the Laplace transform variable,  $c$  is the positive fixed number such that all the singularities of  $f^*(s)$  lie to the left of the vertical line known as the Bromwich path of integration. The Caputo fractional derivative has the following Laplace transform rule [3], [6], [9]:

$$\mathcal{L}\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad n-1 < \alpha < n.$$

The exponential Fourier transform with respect to the spatial coordinate  $x$  is used in the domain  $-\infty < x < \infty$  and has the form [17]:

$$\begin{aligned}\mathcal{F}\{f(x)\} &= \tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{ix\xi} dx, \\ \mathcal{F}^{-1}\{\tilde{f}(\xi)\} &= f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \tilde{f}(\xi) e^{-ix\xi} d\xi, \\ \mathcal{F}\left\{\frac{d^2 f(x)}{dx^2}\right\} &= -\xi^2 \tilde{f}(\xi).\end{aligned}$$

Applying the integral transforms to (5)–(8), we get

$$(9) \quad \tilde{T}^* = \frac{p_0}{\sqrt{2\pi}} \frac{s^{\alpha-1}}{s^\alpha + a\xi^2 + b}.$$

To invert the Laplace transform, the following equation [3], [6], [9]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha + c}\right\} = E_\alpha(-ct^\alpha)$$

is used, where

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \quad z \in C,$$

is the Mittag-Leffler function in one parameter  $\alpha$ .

Hence, the solution is expressed as

$$(10) \quad T(x, t) = \frac{p_0}{\pi} \int_0^\infty E_\alpha[-(a\xi^2 + b)t^\alpha] \cos(x\xi) d\xi.$$

Consider several particular cases of the solution (10). The classical heat conduction corresponds to the value  $\alpha = 1$ . In this instance the Mittag-Leffler function reduces to the exponential function

$$E_1 [-(a\xi^2 + b)t] = \exp [-(a\xi^2 + b)t].$$

The integral in Eq. (10) is evaluated as [16]

$$(11) \quad \int_0^\infty e^{-ax^2} \cos(bx) dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \exp\left(-\frac{b^2}{4a}\right).$$

Therefore,

$$(12) \quad T(x, t) = \frac{p_0}{2\sqrt{\pi at}} \exp\left(-\frac{x^2}{4at} - bt\right).$$

The solution (12) is presented in [1], [10].

Another particular case is obtained for  $\alpha = 1/2$ . Using the following expression for the Mittag-Leffler function [14]

$$E_{1/2}(-x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2 - 2ux} du,$$

changing the order of integration and taking into account (11), we get

$$(13) \quad T(x, t) = \frac{p_0}{\pi\sqrt{2at^{1/4}}} \int_0^\infty \frac{1}{\sqrt{u}} \exp\left(-u^2 - \frac{x^2}{8at^{1/2}u} - 2bt^{1/2}u\right) du.$$

For  $b = 0$  the solution (13) coincides with the corresponding solution to the time-fractional diffusion-wave equation [14].

The results of numerical calculations are shown in Figs. 1–3. In calculations we have used the nondimensional quantities

$$(14) \quad \bar{T} = \frac{\sqrt{at^{\alpha/2}}}{p_0} T, \quad \bar{x} = \frac{x}{\sqrt{at^{\alpha/2}}}, \quad \bar{b} = bt^\alpha.$$

### 3. FUNDAMENTAL SOLUTION TO THE SOURCE PROBLEM

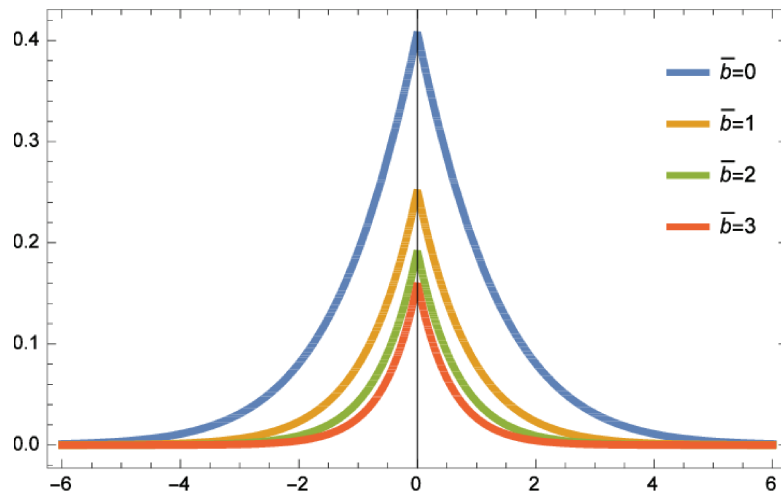
Next, we consider the time-fractional heat conduction equation with the source term

$$(15) \quad \frac{\partial^\alpha T}{\partial t^\alpha} = a \frac{\partial^2 T}{\partial x^2} - bT + w_0 \delta(x) \delta(t), \quad -\infty < x < \infty, \quad 0 < \alpha \leq 2,$$

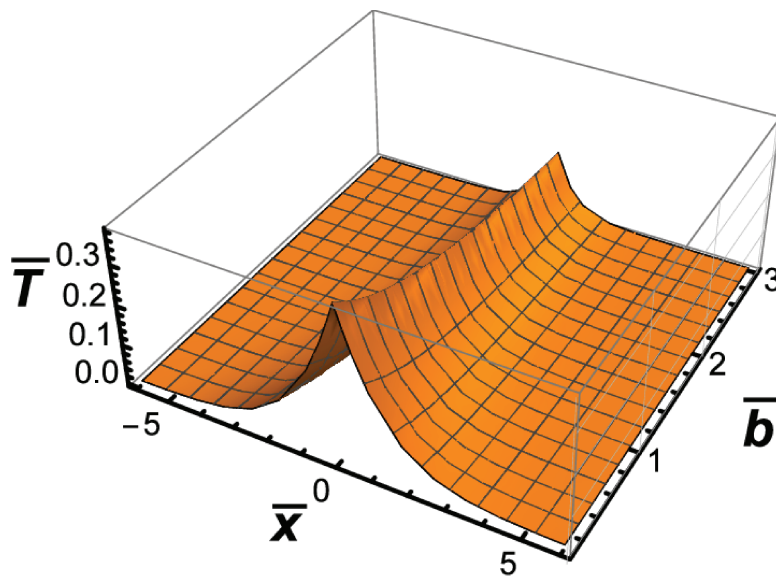
under zero initial conditions

$$(16) \quad t = 0: \quad T = 0, \quad 0 < \alpha \leq 2,$$

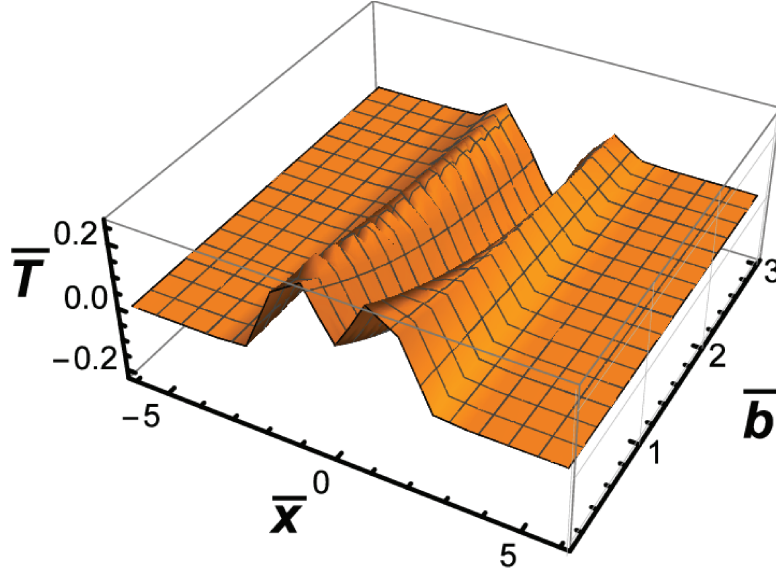
$$(17) \quad t = 0: \quad \frac{\partial T}{\partial t} = 0, \quad 1 < \alpha \leq 2.$$



**Figure 1** Solution to the Cauchy problem for  $\alpha = 0.5$  and various values of  $\bar{b}$ .



**Figure 2** Solution to the Cauchy problem for  $\alpha = 0.5$ .



**Figure 3** Solution to the Cauchy problem for  $\alpha = 1.5$ .

The constant coefficient  $w_0$  is introduced to get the nondimensional quantity used in numerical calculations (see Eq. (24)).

The integral transform technique gives

$$(18) \quad \tilde{T}^* = \frac{w_0}{\sqrt{2\pi}} \frac{1}{s^\alpha + a\xi^2 + b}.$$

By virtue of the fact that [3], [6], [9]

$$\mathcal{L}^{-1} \left\{ \frac{s^{\alpha-\beta}}{s^\alpha + c} \right\} = t^{\beta-1} E_{\alpha,\beta}(-ct^\alpha),$$

where  $E_{\alpha,\beta}(z)$  is the Mittag-Leffler function in two parameters  $\alpha$  and  $\beta$

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \quad \beta > 0, \quad z \in C,$$

the fundamental solution to the source problems is expressed as

$$(19) \quad T(x,t) = \frac{w_0 t^{\alpha-1}}{\pi} \int_0^\infty E_{\alpha,\alpha}[-(a\xi^2 + b)t^\alpha] \cos(x\xi) d\xi.$$

In the case of the classical heat conduction equation ( $\alpha = 1$ ), the solutions (10) and (19) coincide and yield (12).

For heat conduction with  $\alpha = 1/2$ , we have [14]

$$E_{1/2, 1/2}(-x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2 - 2ux} u \, du$$

and

$$(20) \quad T(x, t) = \frac{w_0}{\pi \sqrt{2at^{3/4}}} \int_0^\infty \sqrt{u} \exp\left(-u^2 - \frac{x^2}{8aut^{1/2}} - 2bt^{1/2}u\right) du.$$

When the coefficient  $b = 0$ , the solution (20) coincides with the corresponding solution to the time-fractional diffusion-wave equation presented in the book [14].

For the wave equation corresponding to  $\alpha = 2$ , the Mittag-Leffler function is expressed as

$$(21) \quad E_{2,2}[-(a\xi^2 + b)t^2] = \frac{\sin\left(t\sqrt{a\xi^2 + b}\right)}{t\sqrt{a\xi^2 + b}}.$$

Taking into account the following integral [16]

$$(22) \quad \int_0^\infty \frac{\sin\left(c\sqrt{x^2 + \gamma^2}\right)}{\sqrt{x^2 + \gamma^2}} \cos(\beta x) \, dx = \begin{cases} \frac{\pi}{2} J_0\left(\gamma\sqrt{c^2 - \beta^2}\right), & 0 < \beta < c, \\ 0, & 0 < c < \beta, \end{cases}$$

where  $J_0$  is the Bessel function, from the general expression (19) we get the corresponding solution

$$(23) \quad T(x, t) = \begin{cases} \frac{w_0}{2\sqrt{a}} J_0\left[\sqrt{b(at^2 - x^2)/a}\right], & 0 < |x| < \sqrt{at}, \\ 0, & \sqrt{at} < |x| < \infty. \end{cases}$$

Dependence of the nondimensional fundamental solution to the source problem for the time-fractional heat conduction equation

$$(24) \quad \bar{T} = \frac{\sqrt{at}^{\alpha/2-1}}{w_0} T$$

on the similarity variable  $\bar{x}$  is presented in Figs. 4–7 for different values of the order of fractional derivative  $\alpha$  and the parameter  $\bar{b}$ . Recall that other nondimensional quantities are the same as in Eq. (14).

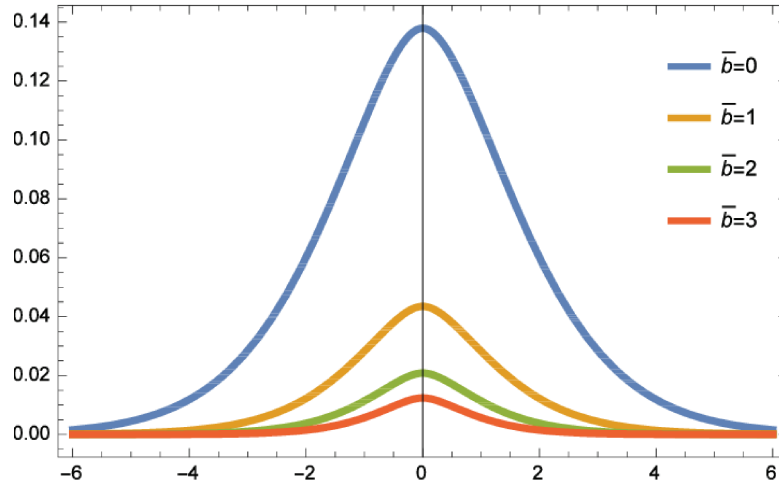


Figure 4 Solution to the source problem for  $\alpha = 0.5$  and various values of  $\bar{b}$ .

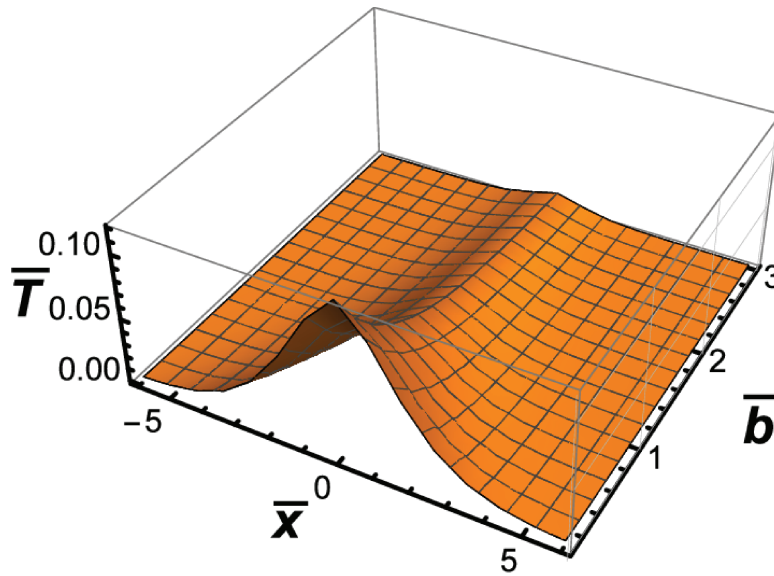
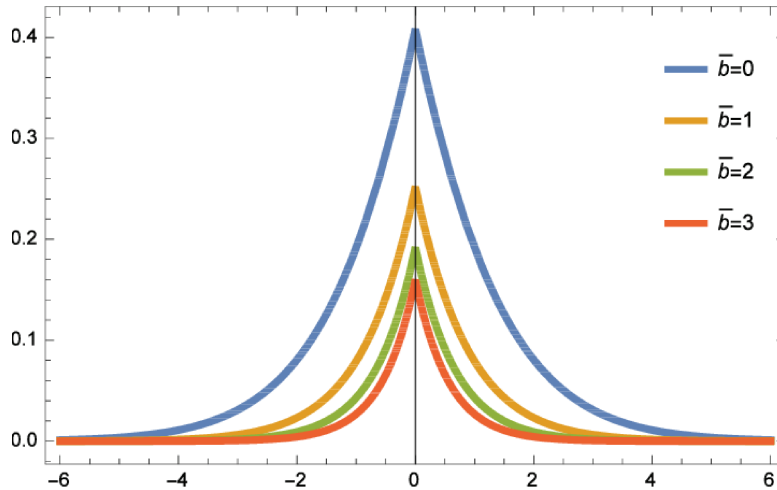


Figure 5 Solution to the source problem for  $\alpha = 0.5$ .





**Figure 6** Solution to the source problem for  $\alpha = 1.5$ .

#### 4. CONCLUSIONS

We have studied the time-fractional heat conduction equation with heat absorption proportional to temperature in the case of one Cartesian spatial coordinate. The fundamental solutions to the Cauchy and source problems have been obtained. Several particular cases have been considered, specifically the solutions to the classical heat conduction equation ( $\alpha = 1$ ) and to the wave equation ( $\alpha = 2$ ).

It should be emphasized that the time-fractional heat conduction equation with the Caputo fractional derivative of order  $0 < \alpha \leq 2$  for  $0 < \alpha \leq 1$  interpolates between the so-called localized heat conduction ( $\alpha = 0$ ) and the standard heat conduction ( $\alpha = 1$ ), whereas for  $1 < \alpha \leq 2$  interpolates between the classical heat conduction ( $\alpha = 1$ ) and the theory of heat conduction described by the wave equation ( $\alpha = 2$ ). The wave equation for temperature is obtained as a consequence of time-nonlocal generalization of the Fourier law with constant memory kernel and no fading of memory; such theory was developed by Nigmatullin [7] and Green and Naghdi [4]. It is evident from the solution (23) that for the wave equation there appear two wave fronts at  $x = \pm\sqrt{at}$ . The results of numerical calculations have been presented graphically.

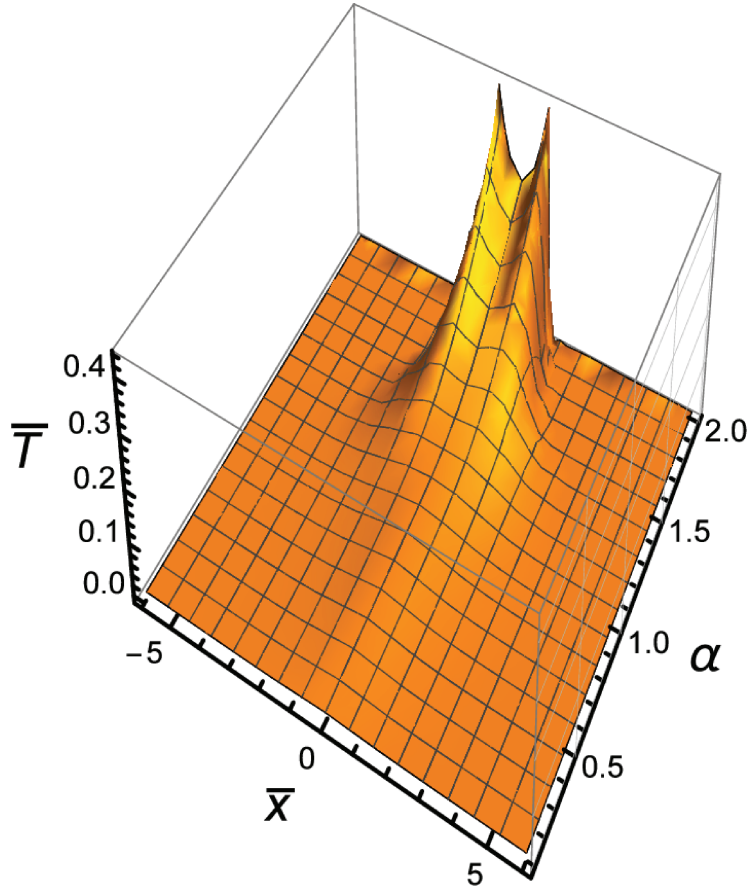


Figure 7 Solution to the source problem for  $\bar{b} = 2$ .

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