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Some Propositional Calculus Oppositional with Respect to the Intuitionistic Propositional Calculus

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Abstract. A fragmentary system of the classical propositional calculus, in which the law $CNN\alpha\alpha$ is valid instead of the law $C\alpha NN\alpha$, is presented.

In this article we present a fragmentary system of the classical propositional calculus, where the law $CNN\alpha\alpha$ is valid instead of the law $C\alpha NN\alpha$ which is typical to the intuitionistic sentential calculus*.

The both formulas, which are called the laws of the double negation, are the laws of the classical propositional calculus. In preparation of this paper we have used the results from [1].

In B. Mates' opinion [2], in the stoical system the following primitive rules of inferention, which are called "nonprovable", are valid:

1. The first rule (nonprovable inference) produces the successor from a conditional statement and its predecessor.

*In this work we use nonbrackets symbols of J. Łukasiewicz. The symbol C means the functor of implication ($C\alpha\beta = \alpha \to \beta$), whereas the symbol N means the functor of negation ($N\alpha = \neg\alpha$) .

2. The second rule produces a statement contradictory to the predecessor from a conditional statement and a statement contradictory to its successor.

3. The third rule having the negation of conjuction and one of its terms produces a statement contradictory to another term.

4. The fourth rule having the disjunction and one of its terms produces a statement contradictory to another term.

5. The fifth rule having the disjunction and a statement contradictory to one of its terms produces another term.

In the above "nonprovables" there appears the concept of "the statement contradictory to some statement", which is more general than the idea of "the negation of some statement". If the symbol $\neg\alpha$ means the statememt contradictory to the statement α, then the second "nonprovable" can be written symbolically as:

$$C\alpha\beta, \neg\beta \mid \neg\alpha. \tag{tr}$$

The scheme (tr) is the scheme for the following rules of transposition (contraposition):

$$C\alpha\beta, N\beta \mid N\alpha, \tag{tr1}$$

$$C\alpha N\beta, \beta \mid N\alpha, \tag{tr2}$$

$$CN\alpha\beta, N\beta \mid \alpha, \tag{tr3}$$

$$CN\alpha N\beta, \beta \mid \alpha. \tag{tr4}$$

The first "nonprovable" can be presented in the form of the following scheme:

$$C\alpha\beta, \alpha \mid \beta. \tag{mp}$$

We restrict our contemplation to the implication-negation language and ignore some considerations about the third, fourth and fifth "nonprovables". Using the consequence function defined on the implication-negation language S, we can formulate the rules (mp), (tr1)-(tr4) as:

$$Cxy, x \in Cn(X) \Rightarrow y \in Cn(X), \tag{RD}$$

(TR1) $$Cxy, Ny \in Cn(X) \Rightarrow Nx \in Cn(X),$$

(TR2) $$CxNy, y \in Cn(X) \Rightarrow Nx \in Cn(X),$$

(TR3) $$CNxy, Ny \in Cn(X) \Rightarrow x \in Cn(X),$$

(TR4) $$CNxNy, y \in Cn(X) \Rightarrow x \in Cn(X),$$

for every $x, y \in S$ and every $X \subseteq S$.

We also add the classical deduction theorem [4] to the above formulas as:

(DT) $$y \in Cn(X \cup \{x\}) \Rightarrow Cxy \in Cn(X),$$

for every $x, y \in S$ and every $X \subseteq S$.

As the expression (RD) is equivalent to the expression

(RD′) $$Cxy \in Cn(X) \Rightarrow y \in Cn(X \cup \{x\}),$$

therefore the expressions (RD′) and (DT) can be written in the form of the following equivalence:

(CH) $$y \in Cn(X \cup \{x\}) \Leftrightarrow Cxy \in Cn(X),$$

where $x, y \in S$, $X \subseteq S$.

The expression (CH) generates in one-to-one manner the positive implication sentential calculus of Hilbert [3,4]. This calculus can be based on the following set of the axioms:

$$A = \{CC\alpha\beta CC\beta\gamma C\alpha\gamma, C\alpha C\beta\alpha, CC\alpha C\alpha\beta C\alpha\beta\}.$$

Moreover, let us consider the following theorem about the indirect deduction [1]:

(IDT-I) $$z, Nz \in Cn(X \cup \{y\}) \Rightarrow Ny \in Cn(X \cup \{Nz\}),$$

(IDT-II) $$z, Nz \in Cn(X \cup \{y\}) \Rightarrow Ny \in Cn(X),$$

(IDT-III) $$z, Nz \in Cn(X \cup \{Ny\}) \Rightarrow y \in Cn(X \cup \{Nz\}),$$

(IDT-IV) $$z, Nz \in Cn(X \cup \{y\}) \Rightarrow y \in Cn(X),$$

and the following expression, which is typical for intuicionistic implication-negation sentential calculus INT:

(IC) $$[z, Nz \in Cn(X) \vee (z, Nz \in Cn(X \cup \{y\}) \wedge Ny \notin Cn(X))] \Rightarrow$$
$$\Rightarrow [v \in Cn(X) \wedge [z, Nz \in Cn(X \cup \{y\}) \Rightarrow Ny \in Cn(X)]]$$

for every $x, y, z, v \in S$ and $X \subseteq S$.

In the work [1] it was shown that:

Theorem 1. Under the expression (CH), the transpositions (TR1)-(TR4) are equivalent to indirect deduction theorems (IDT-I)–(IDT-IV), respectively.

It should be noted that generally it is easier to prove some properties of sentential calculus using the indirect deduction theorem rather than by other means.

Theorem 2. There are the following relations of resulting between the expressions (IC), (IDT-I–(IDT-IV):

The rules (tr1)-(tr4) characterizing the second "nonprovable" correspond to the following laws of propositional calculus, respectively:

(a1) $$CC\alpha\beta CN\beta N\alpha,$$

(a2) $$CC\alpha N\beta C\beta N\alpha,$$

(a3) $$CCN\alpha\beta CN\beta\alpha,$$

(a4) $$CCN\alpha N\beta C\beta\alpha.$$

We denote by $ST.I - ST.IV$ the propositional calculi based on the following sets of axioms, respectively:

$$A_1 = A \cup \{a1\}, A_2 = A \cup \{a2\}, A_3 = A \cup \{a3\}, A_4 = A \cup \{a4\},$$

where A is a set of the axioms of implication sentential calculus of Hilbert.

We denote sets of theorems of the abovementioned calculi by $T_1 - T_4$, respectively. T is a set of theorems of Hilbert's sentential calculus, whereas T_{INT} is a set of theorems of intuicionistic implication-negation sentential calculus.

It can be proved [1] that the relations of inclusion are valid for the sets T, $T_1 - T_4$ (figure 2).

$$T \subset T_1 \subset T_2 \cap T_3 \subset T_2 \subset T_{INT} \subset T_4, T_3 \subset T_4.$$

In the work [1] it was proved that:

Theorem 3. The expressions (CH), (TR-j) generate in one-to-one manner the sentential calculi $ST.j$ $(j = 1, 2, 3, 4)$.

From the above theorem and the theorem 2 it follows that the sets of theorems T_{INT} and T_3 cross over and are included in the set of theorems T_4, i.e. in the set of theorems of the classical implication-negation propositional calculus.

The sets $T_2 - T_4$, T_{INT} can be obtained in the following way: *

$$T_2 = Der(T_1 \cup \{CC\alpha N\alpha N\alpha\}),$$

$$T_3 = Der(T_1 \cup \{CNN\alpha\alpha\}),$$

$$T_{INT} = Der(T_2 \cup \{C\alpha CN\alpha\beta\}),$$

$$T_4 = Der(T_2 \cup \{CNN\alpha\alpha\}) = Der(T_3 \cup \{CCN\alpha\alpha\alpha\}).$$

On the basis of the indirect deduction theorem (or by another means) we can prove that the following sets of theorems include the corresponding expressions:

$T_{INT} - T_3$: $C\alpha NN\alpha$, $CNNN\alpha N\alpha$, $CC\alpha N\alpha N\alpha$, $CCNN\alpha N\alpha N\alpha$, $CCNN\alpha N\beta C\beta N\alpha$, $CCN\alpha N\beta C\beta NN\alpha$.

$T_3 - T_{INT}$: $CNN\alpha\alpha$, $CCN\alpha\alpha CN\beta\alpha$, $CCN\alpha\beta CN\beta\alpha$.

$T_3 \cap T_{INT}$: $C\alpha N\alpha\beta$, $CCN\alpha\beta CN\alpha CN\beta\alpha$, $CC\alpha N\beta C\beta N\alpha$, $CC\alpha\beta CN\beta N\alpha$, $CC\alpha\beta C\alpha CN\beta N\alpha$, $CCN\alpha\beta CN\alpha CN\beta\alpha$, $CC\alpha\beta CC\alpha N\alpha CN\beta N\alpha$.

$T_4 - (T_3 \cup T_{INT})$: $CCN\alpha\alpha\alpha$, $CCN\alpha N\beta C\beta\alpha$.

It is obvious that the system INT includes the theorem $CNNN\alpha N\alpha$, which is a weak version of the theorem $CNN\alpha\alpha$ belonging to the system $ST.3$. The both systems include the law of overflow $C\alpha CN\alpha\beta$. The system INT has valuable elaborations and has deep philosophical and semantic substantiation, whereas the system $ST.3$, which is

*The symbol $Der(X)$ means a set of formulas belonging to X or a set of formulas which are obtained on the basis of X using the rule of detachment.

crossing with it, distinguishes itself among stoical systems as an oppositional one with respect to the system INT. The systems $ST.1$ and $ST.2$ are included in the system INT. The system $ST.3$ has not yet any valuable elaboration and proper semantic interpretation.

References

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