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About Two Different Ways of Characterization of Łukasiewicz's Three-valued Modal Propositional Calculus

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In this article we consider two systems of Łukasiewicz's three-valued modal propositional calcules. One of them is the system based on such primary terms as the disjunction (A), negation (N) and necessity (L), whereas the second is based on such primary terms as the implication (C), negation (N) and definitively improved by modal necessity terms. The both systems are definitively equivalent.

The $A - N - L$ system
Developing Łukasiewicz's ideas, Słupecki [6] explains the intuitive bases of three-valued logic under assumption that propositions describe events (present, future or past) and it is possible to assign to these events the logical value (1/2). Based on an analysis of the logical values of compound propositions J. Słupecki concludes that the tables of logical values of disjunction, conjunction and negation should have the following form, respectively:

(I)

A	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

K	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

	N
0	1
1/2	1/2
1	0

or $Axy = max(x,y)$, $Kxy = min(x,y)$, $Nx = 1 - x$, where $x, y \in \{0, 1/2, 1\}$.
The above tables coincide with those proposed by J. Lukasiewicz.

J. Słupecki noticed that the set of reasonable formulas incorporating, apart from propositional variables, only the functors A, K, N and being the tautology in the matrix

$$M = (\{0, 1/2, 1\}, \{1\}, \{A, K, N\})$$

is empty.

Łukasiewicz's implication C having the table of logical values

(II)

C	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

$$(Cxy = min(1, 1 - x + y), \text{ where } x, y \in \{0, 1/2, 1\}),$$

is not definable by the functors A, K, N, because $C1/2\,1/2 = 1$, whereas any reasonable formula $\alpha(x,y)$ written in terms of A, K, N has the logical value 1/2 for $x - y = 1/2$.

When the system with primary terms A, N or K, N is improved by modal terms then we obtain the system definitively equivalent to the system with the primary terms C, N, i.e. any term of one system is definable in another system. There are the following tables of logical values for modal functors of necessity L and possibility M

(III)

p	Lp	Mp
0	0	0
1/2	0	1
1	1	1

The functors L and M do not take the logical value 1/2. They are mutually definable using the negation

(IV)
$$M\alpha = NLN\alpha,$$

$$L\alpha = NMN\alpha.$$

We can define the implication functor using the terms A, K, N, M as follows

(V) $$C\alpha\beta = AAN\alpha\beta MKN\alpha\beta.$$

Because of complicated and unclear structure of this definition, it is worthy of attention to construct the calculus with the primary terms to be among the following triples:

$$(A, N, L), (A, N, M), (K, N, L), (K, N, M).$$

In the paper [7] the propositional calculus was constructed with the primary terms (A, N, L), the primary rule of substitution, and the rule of detachment, according to the scheme

(VI) $$ANL\alpha\beta, \alpha/\beta.$$

The formula $ANL\alpha\beta$ does not define the formula $C\alpha\beta$ obviously. Therefore, we use the symbol F instead of ANL. The functor F differs from the functor C only for $p = \frac{1}{2}$ and $q = 0$. It has the following table of logical values*

(VII)

Fpq	0	1/2	1
0	1	1	1
1/2	1	1	1
1	0	1/2	1

In the $A-N-L$ axiomatic system we use the equivalence functor E, which can be defined using the functors K and F as follows:

(VIII) $$E\alpha\beta = NCE_1\alpha\beta NE_1\alpha\beta,$$

where

$$K\alpha\beta = NAN\alpha N\beta,$$

$$E_1\alpha\beta = KKF\alpha\beta F\beta\alpha KFN\alpha N\beta FN\beta N\alpha$$

and the functor C is defined by (V). The axioms of the $A-N-L$ system are (see [7]):

*Let us notice that the functor F is definable by the use of the functor C as follows: $F\alpha\beta = C\alpha C\alpha\beta$.

$A1.$ $FFFpqrFFrpFsp$,*

$A2.$ $FFpqFApqq$,

$A3.$ $FpApq$,

$A4.$ $FpAqp$,

$A5.$ $FpNNp$,

$A6.$ $FNNpp$,

$A7.$ $FNApqNp$,

$A8.$ $FNApqNq$,

$A9.$ $FNpFNqNApq$,

$A10.$ $FLpp$,

$A11.$ $NLNApNp$,

$A12.$ $ALpNLp$.

It was proved in [7] that the system based on the above-mentioned axioms is complete with respect to the matrix

$$M = (\{0, 1/2, 1\}, \{1\}, \{A, N, L\}), \quad \text{(IX)}$$

where the functors A, N, L are defined by tables (I), (III).

The $A - N - L$ system.
It should be noted that Wajsberg's axiomatics [8] for implication-negation propositional calculus was among the first versions of axiomatics for the three-valued propositional calculus of Łukasiewicz. M. Wajsberg has accepted the following axioms:

$a1.$ $CqCpq$,

$a2.$ $CCpqCCqrCpr$,

$a3.$ $CCCpNppp$,

*The axiom $A1$ is equivalent to the known Łukasiewicz formula, which is the only axiom of two-valued implication propositional calculus.

$a4$. $CCNqNpCpq$.

The rules of inference are the rule of substitution and the rule of detachment for implication C. M. Wajsbergnotices, that the axiom $a3$ could be replaced by the axiom $a3'$:

$a3'$. $CCCpCpqpp$.

It is possible to introduce the functors of disjunction (A), conjunction (K) and equivalence (E) for the system under consideration by definitions:

$$\begin{aligned} A\alpha\beta &= CC\beta\alpha\alpha, \\ K\alpha\beta &= NAN\alpha N\beta, \\ E_1\alpha\beta &= KC\alpha\beta C\beta\alpha, \\ E\alpha\beta &= NCE_1\alpha\beta NE_1\alpha\beta. \end{aligned}$$

The functor E has the following table of logical values:

(X)

E	0	1/2	1
0	1	0	0
1/2	0	1	0
1	0	0	1

M. Wajsberg [8] presented a proof of independence of adopted axioms and a proof of completeness of axiomatic system with respect to the matrix*

(XI) $$M^* = (\{0, 1/2, 1\}, \{1\}, \{C, N\}),$$

where $Cxy = min(1, 1 - x + y)$, $Nx = 1 - x$, $x, y \in \{0, 1/2, 1\}$.

In order to receive the three-valued modal propositional calculus of Łukasiewicz it is sufficient to add to a language the modal term of necessity (L) setting the definition (see [5]):

$$L\alpha = NC\alpha N\alpha.$$

A table of values for this functor is the same as the table (III). The functor of possibility (M) can be defined as follows (see [5]):

$$M\alpha = CN\alpha\alpha.$$

*M. Wajsberg used the logic value 2 instead of 1/2. The corresponding matrix is isomorphic to the matrix M^*.

Concluding remarks

Owing to the possibility of defining the modal functors L and M using the functors C and N, some doubts are cast on modal character of the $C - N - L$ calculus. The following critical comments concerning the three-valued logic can be found in [4, p. 365]: "Originally philosophical hopes were pining on this logic, for example, additional definition possibilities (compared to the classical logic) by creation the starting point for defining the possibility and necessity terms. It should be emphasized that the use of such possibilities in the three-valued logic of Łukasiewicz has serious consequences ... it causes rejection of one of the most obvious and useful rules – the rule of cancellation for implication. This results in the lack of the deduction theorem for this logic...". Let us note that J. Łukasiewicz has constructed some four-valued modal propositional calculus with modal functor of necessity and possibility [3] (see also[1]). Modal functors were already used in ancient Greece by stoics in propositional calculus and by Aristotle in syllogistics, however research in propositional calculus (and other systems) with modal functors developed after publishing C.J.Lewis' works. This has led many people to believe that C.J.Lewis is the originator of the modal logic systems (see [2]).

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