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Generalized Integral Means Preserving Convexity of Higher Orders

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Abstract
Some generalizations of the classical integral means are characterized. Such type characterizations are obtained by requiring that the corresponding integral operator preserves given function classes.

Consider an operator

$$f \longmapsto F(f),$$

given by the formula:

$$F(f)(x) := \begin{cases} 0 & \text{for} \quad x = 0 \\ \frac{n}{x^n} \int\limits_0^x t^{n-1} f(t)dt & \text{for} \quad x \neq 0. \end{cases}$$

This operator transforms the class $K(b)$, $b > 0$, all real functions, convex on $[0, b]$ vanishing at zero, into itself. In particular, for $n := 1$ it reduces itself to the usual integral mean. More generally, given a suitable function φ (instead of " $x \longmapsto x^n$ ") one may consider an integral operator of the form

$$F_\varphi(f)(x) := \begin{cases} 0 & \text{for} \quad x = 0 \\ \frac{1}{\varphi(x)} \int\limits_0^x \varphi'(t) f(t)dt & \text{for} \quad x \neq 0. \end{cases} \tag{1}$$

This operator was first considered by Gh.Toader in [6] He has proved that the inclusion $F_\varphi(K(b)) \subset K(b)$ forces φ to be proportional to a power function. More exactly:

Theorem (Gh. Toader [6]). *If the operator F_φ given by* (1) *on the class $K(b)$ preserves the convexity, then there exist a real constant $k > 0$ and an $a > 0$ such that $\varphi(t) = kt^a$, $t \in [0,b]$. Conversely, if $\varphi(t) = kt^a$, $t \in [0,b]$, $a > 0$, $k \neq 0$, then the operator F_φ given by* (1) *transforms the class $K(b)$ into itself.*

With an accuracy of denotations this operator was testing by C. Mocanu (1982) in his publication [4] J.B. Lacković in his doctoral dissertation [3] was dealing with

$$F^\varphi(f)(x) := \begin{cases} 0 & \text{for} \quad x = 0 \\ \frac{1}{\int\limits_0^x \varphi(t)dt} \int\limits_0^x \varphi(t)f(t)dt & \text{for} \quad x \in (0,b], \end{cases}$$

for the clas $C(b)$, $b > 0$, all real functions, continuous on $[0,b]$ and vanishing at zero, where φ is a given function from $C(b)$, positive on $(0,b]$; plainly, we have

$$F^\varphi = F_{(x \longmapsto \int\limits_0^x \varphi(t)dt)}.$$

In what follows we shall replace the derivative φ' of the function φ in Toader's operator by another given function. More precisely, we shall replace φ' by an arbitrary positive continuous function. So, we shall consider a pair (φ, ψ) of continuous functions enjoying the following properties: $\varphi, \psi : [0,b] \longrightarrow \mathbb{R}, \varphi(0) = 0, \varphi(x) \neq 0$ for $x \in (0,b]$. In this way we define an integral mean $F_{\varphi,\psi}$ on the space $C(b)$ of all continuous real functions on $[0,b]$, vanishing at zero with the aid of the formula

$$(2) \qquad F_{\varphi,\psi}(f)(x) := \begin{cases} 0 & \text{for} \quad x = 0 \\ \frac{1}{\varphi(x)} \int\limits_0^x \psi(t)f(t)dt & \text{for} \quad x \in (0,b] \end{cases}$$

for $f \in C(b)$.

Clearly, we have:

$$F_\varphi = F_{\varphi,\varphi'}.$$

We will prove a result similar to that due to Gh. Toader in [7] for the integral mean $F_{\varphi,\psi}$ just introduced.

In what follows, we shall show that under some additional assumptions upon the given functions the demand that the corresponding operator $F_{\varphi,\psi}$ transforms the class $K_n(b)$, $b>0$ of all (regular) real convex functions of n-th order on the interval $[0,b)$, i.e. C^{n+1}-functions with nonnegative $(n+1)$-st derivative, into itself, determines the analytic forms of φ and ψ.

We shall first show that the integral operator $F_{\varphi,\psi}$ given by (2) can be represented as the sum of some special operators.

Theorem 1. *Let $\varphi,\psi:[0,b]\longrightarrow\mathbb{R}$ be continuous functions enjoying the following properties: $\varphi(0)=0$, $\varphi(x)>0$ and $\psi(x)>0$ for $x\in(0,b]$ and let φ be (right-hand side) differentiable at zero. If the operator $F_{\varphi,\psi}:C(b)\longrightarrow\mathbb{R}^{[0,b]}$ given by (2) transforms the class $K_n(b)$ into itself, then φ is a C^1-function on the interval $(0,b]$ and there exist a polynomial w of order at most n on the interval $[0,b]$ such that*

$$F_{\varphi,\psi}(f)(x)=\frac{1}{\varphi(x)}\int_0^x\frac{w'(t)}{t}\varphi(t)f(t)dt+\frac{1}{\varphi(x)}\int_0^x\frac{w(t)}{t}\varphi'(t)f(t)dt,$$

$$x\in(0,b].$$

Proof. Clearly, the class P_0 all polynomials of order at most n on $[0,b]$ vanishing at zero, is contained in $K_n(b)$; moreover $P_0=-P_0$. Finally for any $p\in P_0$ we have

$$F_{\varphi,\psi}(p)\in K_n(b)\quad\text{and}\quad -F_{\varphi,\psi}(p)=F_{\varphi,\psi}(-p)\in K_n(b),$$

whence

$$\Delta_h^{n+1}F_{\varphi,\psi}(p)(x)=0$$

for all $x\in[0,b]$ and $h>0$ such $x+(n+1)h\in[0,b]$.

It is well-known (see e.g. M. Kuczma [2], L. Székelyhidi [5]) that the only continuous solutions of this equation are polynomials of order at most n. Therefore, the function $F_{\varphi,\psi}(p)$ is a polynomial at most of order n. In particular,

$$w:=F_{\varphi,\psi}(\mathrm{id})$$

is a polynomial of order at most n. Moreover, since $F_{\varphi,\psi}(\mathrm{id}) \in K_n(b)$ we get $w(0) = 0$. In view of the equality

$$\int_0^x t\psi(t)dt = w(x)\varphi(x), \quad x \in (0, b],$$

φ is a C^1-function on the interval $(0, b]$ and

$$x\psi(x) = w'(x)\varphi(x) + w(x)\varphi'(x), \quad x \in (0, b].$$

Consequently, we have

$$\psi(x) = \frac{w'(x)}{x}\varphi(x) + \frac{w(x)}{x}\varphi'(x), \quad x \in (0, b],$$

and putting this formula to (2) we obtain our assertion.

Remark 1. For $n = 1$ Theorem 1 was already proved in [1].

In the sequel, we will present a necessary condition for getting the analytic form of the operator $F_{\varphi,\psi}$ satisfying the preservation required.

Theorem 2. *Let $\varphi, \psi : [0, b] \longrightarrow \mathbb{R}$ be continuous functions enjoying the following properties: $\varphi(0) = 0$, $\varphi(x) > 0$ and $\psi(x) > 0$ for $x \in (0, b]$ and let φ be (right-hand side) differentiable at zero. If the operator $F_{\varphi,\psi} : C(b) \longrightarrow \mathbb{R}^{[0,b]}$ given by (2) transforms the class $K_n(b)$ into itself, then there exist a real constant $k > 0$ and rational functions p and q on the interval $(0, b]$ such that*

$$\varphi(x) = ke^{\int_0^x p(t)dt}, \quad x \in (0, b],$$

$$\psi(x) = q(x)\varphi(x), \quad x \in (0, b].$$

In particular, there exists a rational function $W : (0, b] \longrightarrow \mathbb{R}$ such that $\psi = W\varphi'$.

Proof. Observe that $w_1 := F_{\varphi,\psi}(\mathrm{id})$ is a polynomial of order at most n on the interval $[0, b]$ (see the proof of Theorem 1). Similarly, we obtain that $w_2 := F_{\varphi,\psi}(\mathrm{id}^2)$ is a polynomial of order at most n. Hence

$$\int_0^x t\psi(t)dt = w_1(x)\varphi(x) \quad \text{and} \quad \int_0^x t^2\psi(t)dt = w_2(x)\varphi(x), \quad x \in (0,b].$$

Hence

$$(*) \qquad x\psi(x) = w_1'(x)\varphi(x) + w_1(x)\varphi'(x), \quad x \in (0,b],$$

$$(**) \qquad x^2\psi(x) = w_2'(x)\varphi(x) + w_2(x)\varphi'(x), \quad x \in (0,b].$$

Consequently:

$$xw_1'(x)\varphi(x) + xw_1(x)\varphi'(x) = w_2'(x)\varphi(x) + w_2(x)\varphi'(x)$$

for $x \in (0,b]$ whence we get

$$\left(xw_1'(x) - w_2'(x)\right)\varphi(x) = \left(w_2(x) - xw_1(x)\right)\varphi'(x), \quad x \in (0,b].$$

Clearly, the function $(0,b] \ni x \mapsto xw_1'(x) - w_2'(x)$ yields a polynomial of order at most n. Likewise, the function $(0,b] \ni x \mapsto w_2(x) - xw_1(x)$ yields a polynomial of order at most $n+1$. Moreover, it is easy to see, that $w_2(x) - xw_1(x) \neq 0$ for $x \in (0,b]$. Therefore, we conclude that

$$\frac{\varphi'(x)}{\varphi(x)} = \frac{xw_1'(x) - w_2'(x)}{w_2(x) - xw_1(x)} =: p(x)$$

for $x \in (0,b]$. Obviously, there exists a $c \in \mathbb{R}$ such that $\ln\varphi(x) = \int_0^x p(t)dt + c$ for all $x \in (0,b]$. Setting $k := e^c > 0$ we have $\varphi(x) = ke^{\int_0^x p(t)dt}$, $x \in (0,b]$. Applying this result to equality $(*)$ we get

$$\psi(x) = k\frac{w_1'(x)}{x}e^{\int_0^x p(t)dt} + k\frac{w_1(x)}{x}p(x)e^{\int_0^x p(t)dt}, \quad x \in (0,b],$$

i.e.

$$\psi(x) = ke^{\int_0^x p(t)dt}\frac{w_1'(x) + w_1(x)p(x)}{x}, \quad x \in (0,b].$$

Putting $q(x) := \frac{w_1'(x)+w_1(x)p(x)}{x}$, $\quad x \in (0,b]$ we obtain

$$\psi(x) = kq(x)e^{\int_0^x p(t)dt}, \quad x \in (0,b],$$

which completes the proof.

Remark 2. Under the assumptions of Theorem 2 the operator $F_{\varphi,\psi}$ assumes the form

$$\binom{**}{*} \qquad F_{\varphi,\psi}(f)(x) = e^{-\int_0^x p(t)dt} \int_0^x f(t)q(t)e^{\int_0^t p(s)ds} dt, \quad x \in (0,b],$$

where $q : (0,b] \longrightarrow \mathbb{R}$ is a positive rational function.

The description of the analytic form of the operator considered has obviously a necessary character only.The question whether, given two rational functions $p, q : (0,b] \longrightarrow \mathbb{R}$, the corresponding operator $\binom{**}{*}$ preserves the class of convex functions of higher orders, remains unanswered. This is caused by the fact that in such a case we are faced to tedious calculations, hardly to be performed, connected with finding suitable primitive functions as well as with the problem of determining possible inverse functions or given rational functions.

Remark 3. As a matter of fact, the assumption that the operator $F_{\varphi,\psi}$ transforms the class $K_n(b)$ into itself, occurring in Theorem 2, was used exclusively in order to prove that the functions $F_{\varphi,\psi}(\mathrm{id})$, $F_{\varphi,\psi}(\mathrm{id}^2)$ belong to $K_n(b)$.

References

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