Jan Długosz University of Częstochowa
Scientific Issues, Mathematics XII, Częstochowa 2007

REMARKS ON CONNECTIVITY AND I-CONNECTIVITY

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The following definition has been introduced by Jadwiga Knop and Małgorzata Wróbel in 2006 (see [3]).

Definition 1. (J. Knop & M. Wróbel – 2006) A subset A of a topological space X is called to be i-connected if it is connected and $\mathrm{Int}\,(A)$ is nonempty and connected.

Of course this definition requires much more from a set as for usual connectedness. However, in the space of real numbers endowed by natural topology each connected set fulfils the condition from definition 1. Some of properties of i-connected sets were described in that article. We want to discuss the problem, in what kinds of spaces each connected set is also i-connected.

If $\mathscr{L}$ is any ideal of sets which does not contain any nonempty open set, then Hashimoto topology generated by this ideal fulfils our requirement.

It is not difficult to observe that any space which is homeomorphic to the space of real numbers endowed by the described Hashimoto topology fulfils our requirements i.e. each connected set with nonempty interior is i-connected as well.

Another example of such spaces is the so called "long line" ([4]) and each space which is homeomorphic to that space.

One can observe that a circle on a plane with topology generated by Euclidean space $\mathbb{R}^2$ also fulfils the considered condition.

Now we will consider some of necessary conditions for a topological space to fulfill the considered condition. From now on $\mathscr{J}$ will denote the class of topological spaces in which every connected set with nonempty interior is i-connected. Before we formulate next theorem we will remind the denotation. For a subset E of a topological space X by A^d, $\mathrm{Int}\,(E)$ and $\overline{E}$ we will denote the set of all accumulation (adjoint) points of the set E, the interior of the set E and closure of the set E, respectively.

We defined so called k-connected subsets of a topological space, which are a little different than i-connected sets, but have some interesting properties.

Definition 2. [1] A subset A of a topological space X is called to be k-connected if its interior is connected and $A \subset \overline{\mathrm{Int}\,(A)}$.

Of course, each k-connected set is i-connected as well.

It is quite easy to see that for euclidean space of real numbers $\mathbb{R}$, a subset A is k-connected if and only if it is connected.

By $\mathscr{K}$ we will denote the class of topological spaces in which every connected set with nonempty interior is k-connected. Hence $\mathscr{K} \subset \mathscr{J}$.

As we could observe, k-connected sets are similar to i-connected ones and we will make use of this notion, so it is worth to compare those kinds of sets.

Theorem 1. [1] A subset A of a topological space is i-connected if and only if it can be represented in the form

$$A = B \cup C,$$

such that B is k-connected and

$$B \cap C = \emptyset, \quad \mathrm{Int}\,(C) = \emptyset$$

and each component of C is not separated with B.

Theorem 2. Let X be a topological space in which each connected set with a nonempty interior is i-connected. Then:

(1) for each nonempty, open and disjoint and connected subsets A, B and C of the space X

$$\overline{A} \cap \overline{B} \cap \overline{C} = \emptyset.$$

Proof. Let $X \in \mathscr{J}$ and suppose that condition (1) is not fulfilled. There exist then nonempty disjoint open and connected sets A, B and C such that

$$\overline{A} \cap \overline{B} \cap \overline{C} \neq \emptyset.$$

Let $x \in \overline{A} \cap \overline{B} \cap \overline{C}$. Consider the set $A \cup B \cup \{x\}$ and denote it by E. Since the set A is not separated from $\{x\}$ and the set B is not separated from $\{x\}$ then E is a connected set.

From condition $x \in \overline{C}$ it follows that $U \cap C \neq \emptyset$ for each neighbourhood U of the point x. Hence

$$x \notin \mathrm{Int}\,(A \cup B)\,.$$

From here we infer that

$$\operatorname{Int}(E) = A \cup B.$$

The set $\operatorname{Int}(E)$ is not connected and in consequence E is connected has a nonempty interior and is not i-connected. Contradiction completes the proof. $\square$

By a cut point of a connected space X we mean a point x such that $X \setminus \{x\}$ is not connected. A cut point is called strong cut point of a connected space X if the set $X \setminus \{x\}$ has 2 components.

One can see that not every point belonging to a space X from the class $\mathscr{J}$ is a cut point. It may happen that there is no cut point of such a space. However:

Corollary 1. If $X \in \mathscr{J}$ and x is a cut point, then it is a strong cut point of X.

Theorem 3. Let X be a topological space in which each connected set with a nonempty interior is i-connected. Then:

(2) for each nonempty, open and disjoint and connected subsets A and B of X

$$\left(\overline{A} \cap \overline{B}\right)^d = \emptyset.$$

Proof. Let $X \in \mathscr{J}$ and suppose that there exist two disjoint nonempty open and connected sets A and B such that

$$\left(\overline{A} \cap \overline{B}\right)^d \neq \emptyset.$$

Let $x \in \left(\overline{A} \cap \overline{B}\right)^d$ and $E = A \cup B \cup \{x\}$.

Since the sets A and B are connected and not separated from $\{x\}$, then the set E is connected. As before it is not difficult to notice that $x \notin \operatorname{Int}(E)$. It follows then that $\operatorname{Int}(E)$ is not connected, i.e. E is not i-connected, what completes the proof of condition (2). $\square$

Theorem 4. If X is a locally connected Hausdorff space in which every i-connected set is k-connected, then

(3) there is no connected set A having at least two elements such that $\operatorname{Int}(A) = \emptyset$.

Proof. Suppose that there exists a connected set A and points x and y such that

$$\operatorname{Int}(A) = \emptyset, \quad x \in A, \quad y \in A, \quad x \neq y.$$

It follows from local connectedness of X that there exists a neighbourhood U of x such that $y \notin U$.

Let $B = U \cup A$. The set B is connected because of both set A and U are connected and non-disjoint.

Now we will show that

(4) $\mathrm{Int}\,(B) \subset \overline{U}$.

Suppose that $a \in \mathrm{Int}\,(B) \setminus \overline{U}$. There exists a neighbourhood V of the point a such that

$$a \in V \quad \text{and} \quad V \subset B \setminus \overline{U}.$$

On the other hand:

$$V = \mathrm{Int}\,(V) = \mathrm{Int}\,(V \cap B) = \mathrm{Int}\,(V \cap (U \cup A)) =$$

$$= \mathrm{Int}\,((V \cap U) \cup (V \cap A)) = \mathrm{Int}\,(V \cap A) = \emptyset.$$

Contradiction completes the proof of the required inclusion (4). Inclusions $U \subset B$ and (4) imply inclusions

$$U \subset \mathrm{Int}\,(B) \subset \overline{U},$$

which prove, that $\mathrm{Int}\,(B)$ is connected.

In that way we have proved that the set B is i-connected. Hence it is k-connected as well. Thus

$$B \subset \overline{\mathrm{Int}\,(B)} \subset \overline{U}$$

what is impossible in view of $y \in B$ and $y \notin \overline{U}$. □

We remind that a topological space is called totally disconnected if its every component is a singleton.

Theorem 5. If X is a topological space such that there is no connected set A having at least two elements such that $\mathrm{Int}\,(A) = \emptyset$, then any i-connected set E can be represented in the form $E = B \cup C$ such that B is k-connected, C is totally disconnected and $B \cap C = \emptyset$.

PROOF.

Let E be i-connected set which is not k-connected. Then E and $\mathrm{Int}\,(E)$ are connected and

$$E \not\subset \overline{\mathrm{Int}\,(E)}.$$

Let A be any component of the set $E \setminus \overline{\mathrm{Int}\,(E)}$. The set A is nonempty. Moreover $\mathrm{Int}\,(A) = \emptyset$, since

$$\mathrm{Int}\,(A) \subset \mathrm{Int}\left(E \cap (X \setminus \overline{\mathrm{Int}\,(E)}\right) = \mathrm{Int}\,(E) \setminus \overline{\mathrm{Int}\,(E)} = \emptyset.$$

In view of assumptions the set A must not have more than one element, hence it is a singleton. In that way we proved that

$$E = \left(E \cap \overline{\operatorname{Int}(E)}\right) \cup \left(E \setminus \overline{\operatorname{Int}(E)}\right),$$

where $B = E \cap \overline{\operatorname{Int}(E)}$ is k-connected, C, where $C = E \setminus \overline{\operatorname{Int}(E)}$, is totally disconnected and $B \cap C = \emptyset$. □

Theorem 6. If X is a locally connected Hausdorff space and each component of X belongs to the class $\mathscr{J}$, then $X \in \mathscr{J}$.

Proof. If E is a connected set in X with a nonempty interior, then it is contained in one of the components of X, say C. Since $C \in \mathscr{J}$ then E is i-connected in C. The set C is open then

$$\operatorname{Int}_C(E) = C \cap \operatorname{Int}_X(E) = \operatorname{Int}_X(E).$$

Moreover $\operatorname{Int}_C(E)$ is connected, thus $\operatorname{Int}_X(E)$ is connected.

In this way we have proved that E is i-connected. □

It is quite obvious that if a locally connected Hausdorff space X is in the class $\mathscr{J}$, then each component of X also belongs to the class $\mathscr{J}$. Thus:

Corollary 2. If X is a locally connected Hausdorff space, then $X \in \mathscr{J}$ if and only if each component of X belongs to the class $\mathscr{J}$.

Local connectedness of the space X is necessary, since if

$$X = \bigcup_{n=0}^{\infty} C_n,$$

where

$$C_n = \left\{[0,1] \times \left\{\frac{1}{n}\right\} : n \in \mathbb{N}_+\right\}, \quad n \in \mathbb{N}_+$$

and

$$C_0 = [-1,2] \times \{0\},$$

then X is not locally connected and each of the components belongs to the class $\mathscr{J}$, but X does not, since C_0 is a connected subset of X and $\operatorname{Int}(C_0) = ([-1,0) \cup (1,2]) \times \{0\}$ is nonempty and is not connected.

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