Jan Długosz University of Częstochowa
Scientific Issues, Mathematics XI, Częstochowa 2006

About Various Methods of Calculating the Sum $\sum_{k=1}^{n} k^m$

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Abstract
Pupils of secondary school as well as students often have problems with calculating the sums of the mth powers of successive natural numbers. In this paper we present certain methods of finding such sums.

The first method

To find the sum $\sum_{k=1}^{n} k^m$ we use the expansion of $(n+1)^{m+1}$ according to the binomial theorem and next calculate the difference $(n+1)^{m+1} - n^{m+1}$ (see [5]). Thus, we have

$$(n+1)^{m+1} = \binom{m+1}{0}\ n^{m+1} + \binom{m+1}{1} n^m + \binom{m+1}{2} n^{m-1} + \cdots +$$

$$+ \binom{m+1}{m} n^1 + \binom{m+1}{m+1} n^0, \tag{1}$$

$$(n+1)^{m+1} - n^{m+1} = \binom{m+1}{1} n^m + \binom{m+1}{2} n^{m-1} +$$

$$+ \cdots + \binom{m+1}{m} n + 1. \tag{2}$$

Based on Eq. (2) for natural numbers $1, 2, \dots, n$ we obtain

$$2^{m+1} - 1 = \binom{m+1}{1} 1^m + \binom{m+1}{2} 1^{m-1} + \cdots + \binom{m+1}{m} \cdot 1 + 1,$$

$$3^{m+1} - 2^{m+1} = \binom{m+1}{1} 2^m + \binom{m+1}{2} 2^{m-1} + \cdots + \binom{m+1}{m} \cdot 2 + 1,$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$n^{m+1} - (n-1)^{m+1} = \binom{m+1}{1} (n-1)^m + \tag{3}$$

$$+ \binom{m+1}{2} (n-1)^{m-1} + \cdots + \binom{m+1}{m} (n-1) + 1,$$

$$(n+1)^{m+1} - n^{m+1} = \binom{m+1}{1} n^m + \binom{m+1}{2} n^{m-1} +$$

$$\cdots + \binom{m+1}{m} \cdot n + 1.$$

Summing the both sides of Eq. (3) we find

$$(n+1)^{m+1} - 1 = \binom{m+1}{1} \sum_{k=1}^{n} k^m + \binom{m+1}{2} \sum_{k=1}^{n} k^{m-1} + \cdots$$

$$\cdots + \binom{m+1}{m} \sum_{k=1}^{n} k + n. \tag{4}$$

From (4) it is possible to determine $\sum\limits_{k=1}^{n} k^m$ having the sums

$$\sum_{k=1}^{n} k^{m-1}, \quad \sum_{k=1}^{n} k^{m-2}, \ \ldots, \ \sum_{k=1}^{n} k.$$

These sums can also be calculated using the method described above.

For example, the sum $\sum\limits_{k=1}^{n} k^2$ is obtained as follows.

Since $(n+1)^3 - n^3 = 3n^2 + 3n + 1$, then

$$\begin{aligned}
&2^3 - 1 = 3 \cdot 1^2 + 3 \cdot 1 + 1, \\
&3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1, \\
&\quad\vdots \qquad\qquad \vdots \\
&n^3 - (n-1)^3 = 3\,(n-1)^2 + 3\,(n-1) + 1, \\
&(n+1)^3 - n^3 = 3n^2 + 3n + 1.
\end{aligned}$$

Summing the both sides of these equalities we have

$$(n+1)^3 - 1 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n;$$

whence it follows that

$$3\sum_{k=1}^{n} k^2 = (n+1)^3 - 1 - 3\sum_{k=1}^{n} k - n.$$

Since $\sum\limits_{k=1}^{n} k = \frac{1}{2}n\,(n+1)$, then we finally obtain

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n\,(n+1)\,(2n+1)\,.$$

The second method

The sum $\sum\limits_{k=1}^{n} k^m$ is represented as*

$$\sum_{k=1}^{n} k^m = \sum_{k=1}^{n} k\left[k^m - (k+1)^m\right] + n\,(n+1)^m\,. \tag{5}$$

For example, for the sum $\sum\limits_{k=1}^{n} k^3$ we have:

$$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n} k\left[k^3 - (k+1)^3\right] + n\,(n+1)^3 \tag{6}$$

or

$$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n} k\left(-3k^2 - 3k - 1\right) + n\,(n+1)^3\,. \tag{7}$$

Hence,

$$\sum_{k=1}^{n} k^3 = -3\sum_{k=1}^{n} k^3 - 3\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k + n\,(n+1)^3\,. \tag{8}$$

Taking into account the formulae

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n\,(n+1)\,(2n+1)\,, \qquad \sum_{k=1}^{n} k = \frac{1}{2}n\,(n+1)$$

we find that

$$4\sum_{k=1}^{n} k^3 = -\frac{1}{2}n\,(n+1)\,(2n+1) - \frac{1}{2}n\,(n+1) + n\,(n+1)^3\,. \tag{9}$$

*Using the complete induction it can be shown that the following formula is valid for an arbitrary sequence $\{a_n\}$ (see [6]):

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k\,(a_k - a_{k+1}) + n a_{n+1}$$

Therefore, after simple transformation we finally obtain

$$\sum_{k=1}^{n} k^3 = \frac{1}{4} n^2 (n+1)^2 = \left(\sum_{k=1}^{n} k \right)^2 . \tag{10}$$

It should be noted that this method also involves calculation of the previous sums $\sum\limits_{k=1}^{n} k, \sum\limits_{k=1}^{n} k^2, \ldots, \sum\limits_{k=1}^{n} k^{m-1}$ for calculating the required sum $\sum\limits_{k=1}^{n} k^m$.

The third method

To determine the sum $\sum\limits_{k=1}^{n} k^m$ we use the properties of arithmetic sequences of higher degrees. For an arbitrary number sequence $\{a_n\}$ we define the sequences of successive finite differences

$$\begin{aligned} &\Delta^1 a_i = a_{i+1} - a_i \\ &\Delta^{k+1} a_i = \Delta^k a_{i+1} - \Delta^k a_i \end{aligned} \qquad i = 1, 2, \ldots \tag{11}$$

A sequence $\{a_n\}$ is called an arithmetic sequence of the degree m ($m = 1, 2, \ldots$) if and only if the sequence $\{\Delta^m a_n\}$ is constant and $\Delta^m a_n \neq 0$.

The constant sequence is called an arithmetic sequence of the zero degree. It can be proved that:

1. An arbitrary term of an arithmetic sequence $\{a_n\}$ of the degree m is expressed by the following formula

$$a_n = \binom{n-1}{0} a_1 + \binom{n-1}{1} \Delta^1 a_1 + \binom{n-1}{2} \Delta^2 a_1 + \cdots + \binom{n-1}{m} \Delta^m a_1, \tag{12}$$

whereas the sum of n initial terms of this sequence is equal to

$$s_n = \binom{n}{1} a_1 + \binom{n}{2} \Delta^1 a_1 + \binom{n}{3} \Delta^2 a_1 + \cdots + \binom{n}{m+1} \Delta^m a_1. \tag{13}$$

2. If the terms of a sequence $\{a_n\}$ are of the form $a_n = f(n)$, $n = 1, 2, \ldots$, where f is a polynomial of the mth degree ($m \geq 0$), then the given sequence is an arithmetic sequence of the degree m.

It is obvious that the sum of n initial terms of arithmetic sequence of the degree m can be determined having the term a_1 and the differences $\Delta^1 a_1,\ \Delta^2 a_1, \dots, \Delta^m a_1$. To find these differences we fill in the following table

$$\begin{array}{ccccccc} a_1 & & & & \\ & \Delta^1 a_1 & & & \\ a_2 & & \Delta^2 a_1 & & \\ & \Delta^1 a_2 & & \Delta^3 a_1 & \\ a_3 & & \Delta^2 a_2 & & \ddots \\ & \Delta^1 a_3 & & \vdots & \\ a_4 & & \vdots & & \\ & \vdots & & & \\ \vdots & & & & \end{array} \tag{14}$$

As an example, we calculate the sum $\sum\limits_{k=1}^{n} k^3$ using the method of successive differences. Consider a sequence $a_n = n^3$ being an arithmetic sequence of the third degree. For this sequence we fill in a table of successive finite differences

$$\begin{array}{cccccc} 1 & & & & & \\ & 7 & & & & \\ 8 & & 12 & & & \\ & 19 & & 6 & & \\ 27 & & 18 & & 0 & \\ & 37 & & 6 & & \ddots \\ 64 & & 24 & & 0 & \\ & 61 & & 6 & & \\ 125 & & 30 & & \vdots & \\ & 91 & & \vdots & & \\ 216 & & \vdots & & & \\ & \vdots & & & & \\ \vdots & & & & & \end{array}$$

Hence, we have $a_1 = 1$, $\Delta^1 a_1 = 7$, $\Delta^2 a_1 = 12$, $\Delta^3 a_1 = 6$. Using Eq. (13) we obtain

$$s_n = \sum_{k=1}^{n} k^3 = \binom{n}{1} a_1 + \binom{n}{2} \Delta^1 a_1 + \binom{n}{3} \Delta^2 a_1 + \binom{n}{4} \Delta^3 a_1 =$$
$$= \binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4} = \frac{n^2}{4}(n+1)^2 = \left(\sum_{k=1}^{n} k\right)^2 .$$

It should be pointed out that using the method of successive differences for calculating the sum $\sum\limits_{k=1}^{n} k^m$ it is not necessary to calculate the preceding sums $\sum\limits_{k=1}^{n} k$, $\sum\limits_{k=1}^{n} k^2, \dots, \sum\limits_{k=1}^{n} k^{m-1}$.

The fourth method

Since a sequence $\{a_n\}$, where $a_n = n^k$, is an arithmetic sequence of the degree n, then a sequence $\{s_m(n)\}$, where $s_m(n) = \sum\limits_{k=1}^{n} k^m$, is an arithmetic sequence of the degree $(m+1)$. Consider the following polynomial in n:

$$W_{m+1}(n) = c_{m+1}n^{m+1} + c_m n^m + \cdots + c_2 n^2 + c_1 n \tag{15}$$

and the difference

$$R_{m+1}(n) = s_m(n) - W_{m+1}(n)\,. \tag{16}$$

Therefore, we have

$$\begin{aligned} R_{m+1}(n) - R_{m+1}(n+1) &= \left(\sum_{k=1}^{n} k^m - \sum_{i=1}^{m+1} c_i n^i\right) - \\ &- \left(\sum_{k=1}^{n+1} k^m - \sum_{i=1}^{m+1} c_i (n+1)^i\right) = \\ &= \sum_{i=1}^{m+1} c_i \left[(n+1)^i - n^i\right] - (n+1)^m\,. \end{aligned} \tag{17}$$

Using the binomial theorem repeatedly and putting the differences in good order according to decreasing powers of the parameter n we obtain

$$\begin{aligned} &\left[\binom{m+1}{1}c_{m+1} - \binom{m}{0}\right]n^m + \left[\binom{m+1}{2}c_{m+1} + \binom{m}{1}c_m - \binom{m}{1}\right]n^{m-1} + \\ &+ \left[\binom{m+1}{3}c_{m+1} + \binom{m}{2}c_m + \binom{m-1}{1}c_{m-1} - \binom{m}{2}\right]n^{m-2} + \cdots \\ &\cdots + \left[c_{m+1} + c_m + \cdots + c_1 - \binom{m}{m}\right]. \end{aligned} \tag{18}$$

The polynomial $W_{m+1}(n)$ equals the sum $\sum_{k=1}^{n} k^m$ as for arbitrary n $R_{m+1}(n) - R_{m+1}(n+1) = 0$. Equating the coefficients of this difference to zero we arrive at the following set of equations allowing us to obtain the coefficients $c_1, c_2, \ldots, c_m, c_{m+1}$ (compare with [7]):

$$\begin{cases} \binom{m+1}{1} c_{m+1} = \binom{m}{0} \\ \binom{m+1}{2} c_{m+1} + \binom{m}{1} c_m = \binom{m}{1} \\ \binom{m+1}{3} c_{m+1} + \binom{m}{2} c_m + \binom{m-1}{1} c_{m-1} = \binom{m}{2} \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \binom{m+1}{m+1} c_{m+1} + \binom{m}{m} c_m + \binom{m-1}{m-1} c_{m-1} + \binom{m-2}{m-2} c_{m-2} + \cdots \\ \qquad \cdots + \binom{1}{1} c_1 = \binom{m}{m} \end{cases} \tag{19}$$

The principal determinant of this set of equations is nonzero as $\binom{1}{i} \neq 0$ for $i = 1, 2, \ldots, m+1$. Hence, this set has a uniqie solution

$$\begin{aligned} c_{m+1} &= \frac{\binom{m}{0}}{\binom{m+1}{1}} = \frac{1}{m+1}, \\ c_m &= \frac{1}{\binom{m}{1}} \left[\binom{m}{1} - \binom{m+1}{2} \frac{1}{m+1} \right] = \frac{1}{2}, \\ c_{m-1} &= \frac{1}{12} m, \quad c_{m-2} = 0, \quad c_{m-3} = -\frac{1}{720} m (m-1)(m-2), \ldots \end{aligned} \tag{20}$$

To obtain the sum $\sum_{k=1}^{n} k^m$ it is sufficient to put the received coefficients into Eq. (18).

For example, for the sum $\sum_{k=1}^{n} k^4$ the set of equations (19) has the form

$$\begin{cases} \binom{5}{1}c_5 = \binom{4}{0} \\ \binom{5}{2}c_5 + \binom{4}{1}c_4 = \binom{4}{1} \\ \binom{5}{3}c_5 + \binom{4}{2}c_4 + \binom{3}{1}c_3 = \binom{4}{2} \\ \binom{5}{4}c_5 + \binom{4}{3}c_4 + \binom{3}{2}c_3 + \binom{2}{4}c_2 = \binom{4}{3} \\ \binom{5}{5}c_5 + \binom{4}{4}c_4 + \binom{3}{3}c_3 + \binom{2}{2}c_2 + \binom{1}{1}c_1 = \binom{4}{4} \end{cases}$$

Therefore,

$$c_5 = \frac{1}{5}, \qquad c_4 = \frac{1}{2}, \qquad c_3 = \frac{1}{3}, \qquad c_2 = 0, \qquad c_1 = -\frac{1}{30}.$$

Hence, we have:

$$\sum_{k=1}^{n} k^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n.$$

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