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On Some Special Morphisms Between Groups and Algebras

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1. Introduction

A classical trigonometric identity states that

$$(\sin x)^2 + (\cos x)^2 = 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

Replacing here the sine and the cosine by functions $f, g : \mathbb{R} \to \mathbb{R}$, and 2 by a natural number k, we receive the functional equation:

$$\text{(I)} \qquad (f(x))^k + (g(x))^k = 1$$

with $x \in \mathbb{R}$, which was studied by R. Tardiff in [3], in connection with some trigonometrical considerations. In my papers [1] and [2] I gave a partial answer to the question of the professor Roman Ger concerning addition formulas: $f(x + y)$ and $g(x + y)$ for the function f, g satisfying (I), which would correspond to the well known representations of $\cos(x + y)$ and $\sin(x + y)$ then $x, y \in \mathbb{R}$ in the case where $k = 2$. In [1] I introduced such formulas in the case of even k, the case of odd k was considered in [2]. Though in both cases these results have common features, the latter case requires more sophisticated tools. The results contained in the paper [1] yield the starting point for considerations of the present paper. Therefore, we repeat briefly these results from [1], which we compare with some more general results presented in what follows.

Let X be an arbitrary non-empty set and let k be a fixed natural number.

Lemma 1. Two real functions f, g defined on X satisfy functional equation (I) for $x \in X$ if and only if there exists a function $t : X \to \mathbb{R}$ such that

(1)
$$f(x) = \frac{\cos t(x)}{\sqrt[k]{(\cos t(x))^k + (\sin t(x))^k}}, \quad g(x) = \frac{\sin t(x)}{\sqrt[k]{(\cos t(x))^k + (\sin t(x))^k}}$$

for $x \in X$.

Now, suppose that X is a non-empty subset of a group $(G, +)$.

Theorem 1. Suppose that functions $f, g : X \to \mathbb{R}$ satisfy functional equation (I) for $x \in X$, and let a function $t : X \to \mathbb{R}$ be such that the relations (1) hold in X. If t is an invertible function on X, then, for every $x, y \in X$ such that $x + y \in X$ and $z := t^{-1}(t(x+y) - t(x)) \in X$, the following system of functional equations is satisfied:

(2)
$$\begin{cases} f(x+y) = \dfrac{f(x)f(z) - g(x)g(z)}{\sqrt[k]{(f(x)f(z) - g(x)g(z))^k + (f(x)g(z) + f(z)g(x))^k}} \\[2ex] g(x+y) = \dfrac{f(x)g(z) + f(z)g(x)}{\sqrt[k]{(f(x)f(z) - g(x)g(z))^k + (f(x)g(z) + f(z)g(x))^k}} \end{cases}$$

Moreover, if t is an additive and invertible function, then $z = y$.

In the subsequent text we assume that $(X, +)$ is a group with a neutral element denoted by θ, and k is a fixed even natural number.

For any functions $f, g : X \to \mathbb{R}$ we define the function $w : X^2 \to \langle 0, +\infty)$ by the formula:

$$(3) \quad w(x, y) = \sqrt[k]{(f(x)f(y) - g(x)g(y))^k + (f(x)g(y) + f(y)g(x))^k}$$

for $(x, y) \in X^2$.

We say that the functions f, g do not vanish simultaneously in X if and only if $(f(x))^2 + (g(x))^2 > 0$ for all $x \in X$. It is clear that $w(x, y) > 0$ for $x, y \in X$ if f, g do not vanish simultaneously in X. Therefore we can consider the following system of functional equations:

$$\text{(II)} \qquad \begin{cases} f(x+y) = \dfrac{f(x)f(y) - g(x)g(y)}{w(x,y)} \\[2ex] g(x+y) = \dfrac{f(x)g(y) + f(y)g(x)}{w(x,y)} \end{cases}$$

for $x, y \in X$. Let us notice that system (II) states nothing else then system (2) with $z = y$.

In what follows we denote the multiplicative group of the unit circle with the multiplication of the complex numbers by $(T, \bullet)$ ($T = \{z \in \mathbb{C} : |z| = 1\}$), whereas $H(X, T)$ stands for the family of all homomorphisms mapping the group $(X, +)$ into the group $(T, \bullet)$ (they will be called: *characters*). Now, let us recall the following theorem:

Theorem 2. Functions $f, g : X \to \mathbb{R}$ do not vanish simultaneously and satisfy the system (II) for all $x, y \in X$ if and only if there exists a character $h \in H(X, T)$ with $U := \operatorname{Re} h$, $V := \operatorname{Im} h$ such that:

$$\text{(4)} \qquad f(x) = \frac{U(x)}{\sqrt[k]{(U(x))^k + (V(x))^k}}, \quad g(x) = \frac{V(x)}{\sqrt[k]{(U(x))^k + (V(x))^k}}$$

for $x \in X$.

Moreover, if functions $f, g : X \to \mathbb{R}$ satisfy the system (II) or the conditions (4), then they satisfy the equation (I) on X, where f is even, g is odd, $f(\theta) = 1$, and $g(\theta) = 0$.

Corollary 1. If $a : X \to \mathbb{R}$ is an additive function and $f, g : X \to \mathbb{R}$ are defined by the formulas:

(5)

$$f(x) = \frac{\cos a(x)}{\sqrt[k]{(\cos a(x))^k + (\sin a(x))^k}}, \quad g(x) = \frac{\sin a(x)}{\sqrt[k]{(\cos a(x))^k + (\sin a(x))^k}}$$

for $x \in X$, then f, g do not vanish simultaneously and satisfy the system (II) and equation (I).

In what follows functions f and g defined by formulas (4) will be called *the generalized cosine* on X and *the generalized sine* on X respectively.

In the present paper we will show that system (II) becomes a special case of more general functional equation. We will present this equation and show that its solutions preserve basic properties of generalized sine and cosine functions.

2. Some generalized trigonometric maps

We continue to assume that $(X,+)$ is a group with a neutral element denoted by θ. Moreower, let $(A, K, +, \bullet, \cdot)$ be a commutative algebra with a unity over the field K of real or complex numbers. The multiplication of vectors will be denoted by the symbol "$\bullet$". The unity of this algebra will be denoted by $\mathbf{1}$, and its zero-element by $\mathbf{0}$.

Remark 1. Let $\varphi : X \to A$ be a non-zero morphism of a group $(X,+)$ into the semigroup $(A,\bullet)$, which means that: $\varphi(x+y) = \varphi(x) \bullet \varphi(y)$ for all $x, y \in X$. Then the following conditions are satisfied:

(i) $\varphi(x) \neq 0$ for all $x \in X$;

(ii) if the algebra A has no zero-divisors, (i.e. $a \bullet b = 0$ if and only if $a = 0$ or $b = 0$, for $a, b \in A$), then $\varphi(\theta) = 1$;

(iii) $\varphi(x+y) = \varphi(y+x)$ for all $x, y \in X$.

In what follows we assume that $n : A \to \langle 0, +\infty)$ is a non-negative function such that:

(a) $n(a) = 0$ if and only if $a = \mathbf{0}$,

(b) $n(\lambda a) = \lambda n(a)$ for all $\lambda > 0$, $a \in A$.

Obviously, every norm in the space A satisfies the conditions (a) and (b).

Put $S := \{a \in A : n(a) = 1\}$ and let $u : A\backslash\{\mathbf{0}\} \to S$ be a function defined by the formula:

$$u(a) = \frac{a}{n(a)} \quad \text{for all} \quad a \in A\backslash\{\mathbf{0}\}. \tag{6}$$

Suppose that there exists a non-zero morphism $\varphi : X \to A$ from a group $(X, +)$ into the semigroup $(A, \bullet)$ such that:

$$u(\varphi(X)) = S. \tag{7}$$

Now we assume that $F : X \to A$ is a function satisfying the following condition:

$$n(F(x)) = 1 \quad \text{for all } \ x \in X; \tag{8}$$

this means that $F : X \to S$. We will show that the function F yields a solution of a functional equation comparable, in a sense, with the system of equations (2).

We observe that conditions (7) and (8) imply the following condition:

$$\forall_{x\in X} \exists_{y\in X} F(x) = u(\varphi(y)). \tag{9}$$

So, condition (9) defines a function $t : X \to X$ such that:

$$F(x) = u(\varphi(t(x))) = \frac{\varphi(t(x))}{n(\varphi(t(x)))} \quad \text{for} \quad x \in X. \tag{10}$$

Now, observe that:

$$F(x)\bullet F(z) = \frac{\varphi(t(x))}{n(\varphi(t(x)))}\bullet\frac{\varphi(t(z))}{n(\varphi(t(z)))} = \frac{\varphi(t(x)+t(z))}{n(\varphi(t(x)))n(\varphi(t(z)))} \quad \text{for } x, z \in X,$$

whence

$$n(F(x) \bullet F(z)) = \frac{n(\varphi(t(x)+t(z)))}{n(\varphi(t(x)))n(\varphi(t(z)))} \quad \text{for } \ x, z \in X.$$

From last two equalities we deduce that

$$\frac{F(x) \bullet F(z)}{n(F(x) \bullet F(z))} = \frac{\varphi(t(x)+t(z))}{n(\varphi(t(x)+t(z)))} \quad \text{for } \ x, z \in X. \tag{11}$$

Moreover, from (10) we get the following result:

$$F(x+y) = \frac{\varphi(t(x+y))}{n(\varphi(t(x+y)))} \quad \text{for } \ x, y \in X. \tag{12}$$

Let us now consider two situations which can occur:

(**A**) If t is an additive function on X, then conditions (11) and (12) imply the following equation:

$$\text{(T)} \qquad F(x+y) = \frac{F(x) \bullet F(y)}{n(F(x) \bullet F(y))} \quad \text{for} \quad x, y \in X.$$

Moreover, from condition (10) and the additivity of the function t it follows that

$$\text{(13)} \qquad F(x) = \frac{\varphi(t(x))}{n(\varphi(t(x)))} = \frac{\psi(x)}{n(\psi(x))} \quad \text{for} \quad x \in X,$$

where $\psi = \varphi \circ t : X \to A$ is a new non-zero morphism from the group $(X, +)$ into the semigroup $(A, \bullet)$

$\big(\psi(x+y) = \varphi(t(x+y)) = \varphi(t(x)+t(y)) = \varphi(t(x)) \bullet \varphi(t(y)) = \psi(x) \bullet \psi(y)$ for $x, y \in X\big)$. (**B**) If there exists the inverse function $t^{-1} : X \to X$, then from conditions (11) and (12) we obtain the equation

$$\text{(aT),} \qquad F(x+y) = \frac{F(x) \bullet F(z)}{n(F(x) \bullet F(z))}$$

where $z = t^{-1}(t(x+y) - t(x))$ for $x, y \in X$; in fact, if $z = t^{-1}(t(x+y) - t(x))$, $x, y \in X$, then $t(z) = t(x+y) - t(x)$, $t(z) + t(x) = t(x+y)$. From this and from the condition (iii) of Remark 1 we obtain the following equalities: $\varphi(t(x) + t(z)) = \varphi(t(z) + t(x)) = \varphi(t(x+y))$. Now, a comparison of the right hand sides of formulas (11), (12) leads to the equation (aT).

Now we will show that the system of equations (2) yields a special case of equation (aT), while the equation (II) is a special case of equation (T).

Remark 2. In the real space $\mathbb{R}^2$ we can introduce an associative operation "$\bullet$" of a multiplication of vectors in accordance with the usual multiplication of complex numbers:

(14)

$$\forall_{\substack{(u,y) \in \mathbb{R}^2 \\ (r,s) \in \mathbb{R}^2}} (u, v) \bullet (r, s) = \Big(\mathrm{Re}((u+iv)(r+is)), \mathrm{Im}((u+iv)(r+is))\Big) =$$

$$= (ur - vs, us + vr).$$

Then $(\mathbb{R}^2, \mathbb{R}, +, \bullet, \cdot)$ is a real commutative algebra with a unity $\mathbf{1} = (1,0)$. Moreover this is a Banach algebra with a Euclidean norm: $\|u,v\| = |u+iv|$ satisfying the consistency condition with the equality, i.e.:

$$\text{(15)} \qquad \|(u,v)\bullet(r,s)\| = \|(u,v)\|\cdot\|(r,s)\| \quad \text{for} \quad (u,v),(r,s)\in\mathbb{R}^2.$$

Remark 3. Let $n : \mathbb{R}^2 \to \langle 0, +\infty)$ be a function defined by the formula:

(16) $n(u,v) = \sqrt[k]{u^k + v^k}$ for $(u,v) \in \mathbb{R}^2$ (still k is an arbitrarily fixed even natural number).

Then following conditions hold:

(17) n is a norm in real space $\mathbb{R}^2$;

(18) $n(1,0) = 1$;

(19) $n(u,-v) = n(u,v)$ for $(u,v) \in \mathbb{R}^2$.

Remark 4. Suppose that $n : \mathbb{R}^2 \to \langle 0, +\infty)$ is defined by formula (16) and $F = (f,g) : X \to \mathbb{R}^2$ is a nowhere vanishing map. Then the real coordinates f, g of the map F satisfy the system of equations (II) in X if and only if the map F satisfies the functional equation (T) in X. Moreover, if F satisfies (T) in X, then $F(\theta) = (1,0)$ and the set of values of F is contained in a curve S symmetric with respect to $(0,0)$ and defined by the formula:

$$\text{(20)} \qquad S = \left\{(u,v)\in\mathbb{R}^2 : n(u,v) = \sqrt[k]{u^k+v^k} = 1\right\}$$

Proof. To show the equivalence of the system (II) with equation (T) it suffices to observe that $F(x)\bullet F(y) = (f(x), g(x))\bullet(f(y), g(y)) = (f(x)f(y) - g(x)g(y), f(x)g(y) + f(y)g(x))$ for $(x,y) \in X$. Therefore $n(F(x)\bullet F(y)) = w(x,y)$ for $x, y \in X$ where w is defined by (3).

Fig. 1 presents curves S defined by formula (20) for $k = 2$ and $k = 4$.

Fig. 1.

Arguments similar to those used in the proof of Remark 4 show that if functions f, g and t satisfy the assumptions of Theorem 1, $F = (f, g)$, and n is the function defined by (16), then the system of equations (2) assumes the form (aT).

Now we will try to solve the equation (T) under some additional conditions.

Theorem 3. Suppose that $(X, +)$ is a group with a neutral element θ and $(A, K, +, \bullet, \cdot)$ is a commutative algebra with an unity $\mathbf{1}$ over the field K of real or complex numbers (the symbol "$\bullet$" denotes multiplication of vectors and $\mathbf{0}$ stands for the zero element of algebra). Moreover, assume that $n : A \to \langle 0, +\infty)$ is a function satisfying the conditions:

(a) $n(a) = 0$ if and only if $a = \mathbf{0}$,

(b) $n(\lambda a) = \lambda n(a)$ for all $\lambda > 0$, $a \in A$, as well as

(c) $n(\mathbf{1}) = 1$.

Then a map $F : X \to A$ satisfies the functional equation

$$F(x+y) = \frac{F(x) \bullet F(y)}{n(F(x) \bullet F(y))} \quad \text{for} \quad x, y \in X \tag{T}$$

jointly with the following conditions:

$$\text{(W)} \qquad n(F(x) \bullet F(y)) = n(F(-x) \bullet F(-y)) \quad \text{for all} \quad x, y \in X,$$

$$\text{(Z)} \qquad F(\theta) = \mathbf{1}$$

if and only if there exists a non-zero morphism $\varphi : X \to A$ from the group $(X, +)$ into the multiplicative semigroup $(A, \bullet)$ ie. $\varphi(x+y) = \varphi(x) \bullet \varphi(y)$ for all $x, y \in X$ such that

$$\varphi(\theta) = \mathbf{1}, \quad n(\varphi(x)) = n(\varphi(-x)) \quad \text{and} \quad F(x) = \frac{\varphi(x)}{n(\varphi(x))} \quad \text{for} \quad x \in X.$$

Before the proof of **Theorem 3,** observe that every norm in the space A such that $n(\mathbf{1}) = 1$ satisfies the assumptions of this theorem. Moreover, from the form of equation (T) and from assumptions on the function n it follows that its solution $F : X \to A$ necessarily has to satisfy the inequality $F(x) \bullet F(y) \neq \mathbf{0}$ for all $x, y \in X$, as well as the equality $n(F(x)) = 1$ for every $x \in X$ (to see this it suffices to put $y = \theta$ in (T)).

Proof of **Theorem 3.** Suppose that $F : X \to A$ is a map satisfying equation (T) jointly with conditions (W) and (Z). Then for $x \in X$ and $y = -x$ we obtain

$$\text{(21)} \qquad \mathbf{1} = F(\theta) = \frac{F(x) \bullet F(-x)}{n(F(x) \bullet F(-x))}.$$

Let $\varphi : X \to A$ be a map defined by the formula:

$$\text{(22)} \qquad \varphi(x) := \frac{F(x)}{\sqrt{n(F(x) \bullet F(-x))}} \quad \text{for} \quad x \in X.$$

Then we get

$$\varphi(x+y) = \frac{F(x+y)}{\sqrt{n(F(x+y) \bullet F(-y-x))}} =$$

$$= \frac{\dfrac{F(x) \bullet F(y)}{n(F(x) \bullet F(y))}}{\sqrt{n\left(\dfrac{F(x) \bullet F(y)}{n(F(x) \bullet F(y))} \bullet \dfrac{F(-y) \bullet F(-x)}{n(F(-y) \bullet F(-x))}\right)}} =$$

$$= \frac{F(x) \bullet F(y)}{n(F(x) \bullet F(y))\sqrt{\frac{n([F(x) \bullet F(-x)] \bullet [F(y) \bullet F(-y)])}{n(F(x) \bullet F(y))n((F(-x) \bullet F(-y))}}} =$$

$$= \frac{F(x) \bullet F(y)}{\sqrt{\frac{n(F(x) \bullet F(y))}{n(F(-x) \bullet F(-y))} n([F(x) \bullet F(-x)] \bullet [F(y) \bullet F(-y)])}} =$$

$$= \frac{\frac{F(x)}{\sqrt{n(F(x) \bullet F(-x))}} \bullet \frac{F(y)}{\sqrt{n(F(y) \bullet F(-y))}}}{\sqrt{\frac{n(F(x) \bullet F(y))}{n(F(-x) \bullet F(-y))} \frac{n([F(x) \bullet F(-x)] \bullet [F(y) \bullet F(-y)])}{n(F(x) \bullet F(-x))n(F(y) \bullet F(-y))}}} =$$

$$= \frac{\varphi(x) \bullet \varphi(y)}{\sqrt{\frac{n(F(x) \bullet F(y))}{n(F(-x) \bullet (F(-y))} \; n\left(\frac{F(x) \bullet F(-x)}{n(F(x) \bullet F(-x))} \bullet \frac{F(y) \bullet F(-y)}{n(F(y) \bullet F(-y))}\right)}}$$

for all $x, y \in X$.
Taking into account conditions (W) and (21) and the equality $n(\mathbf{1}) = 1$ we obtain the following results:

$$\frac{n(F(x) \bullet F(y))}{n(F(-x) \bullet F(-y))} = 1, \quad \frac{(F(x) \bullet F(-x))}{n(F(x) \bullet F(-x))} = \mathbf{1}, \quad \frac{(F(y) \bullet F(-y))}{n(F(y) \bullet F(-y))} = \mathbf{1}$$

and $n(\mathbf{1}\bullet\mathbf{1}) = n(\mathbf{1}) = 1$. Therefore, $\varphi(x+y) = \varphi(x)\bullet\varphi(y)$ for $x, y \in X$. Moreover, $\varphi(\theta) = \frac{F(\theta)}{\sqrt{n(F(\theta)\bullet F(-\theta)}} = \mathbf{1}$, hence, φ is a non-zero morphism. Additionally, the following equalities:

$$n(F(x)) = n(F(x+\theta)) = n\left(\frac{F(x) \bullet F(\theta)}{n(F(x) \bullet F(\theta))}\right) = \frac{n(F(x) \bullet F(\theta))}{n(F(x) \bullet F(\theta))} = 1$$

imply that

$$n(\varphi(x)) = \frac{n(F(x))}{\sqrt{n(F(x) \bullet F(-x))}} = \frac{1}{\sqrt{n(F(x) \bullet F(-x))}} \tag{23}$$

for all $x \in X$.

From here and from (22) it follows that

$$F(x) = \varphi(x) \cdot \sqrt{n(F(x) \bullet F(-x))} = \frac{\varphi(x)}{n(\varphi(x))} \quad \text{for all} \quad x \in X.$$

Condition (23) also implies that

$$n(\varphi(-x)) = \frac{1}{\sqrt{n(F(-x) \bullet F(x))}} = \frac{1}{\sqrt{n(F(x) \bullet F(-x))}} = n(\varphi(x))$$

for all $x \in X$.

Conversely, suppose now that $\varphi : X \to A$ is a morphism satisfying the conditions formulated in the assertion of this theorem. Since φ is a non-zero morphism, by Remark 1, the inequality $\varphi(x) \neq \mathbf{0}$ holds for all $x \in X$ and we can define a map $F : X \to A$ by the formula: $F(x) = \frac{\varphi(x)}{n(\varphi(x))}$ for all $x \in X$. We will show that such F satisfies functional equation (T) jointly with conditions (W) and (Z). Indeed,

$$\frac{F(x) \bullet F(y)}{n(F(x) \bullet F(y))} = \left(\frac{\varphi(x)}{n(\varphi(x))} \bullet \frac{\varphi(y)}{n(\varphi(y))}\right) \cdot \frac{1}{n\left(\frac{\varphi(x)}{n(\varphi(x))} \bullet \frac{\varphi(y)}{n(\varphi(y))}\right)} =$$

$$\frac{\varphi(x+y)}{n(\varphi(x))n(\varphi(y))} \cdot \frac{1}{\frac{n(\varphi(x+y))}{n(\varphi(x))n(\varphi(y))}} = \frac{\varphi(x+y)}{n(\varphi(x+y))} = F(x+y)$$

for all $x, y \in X$. Moreover,

$$n(F(x) \bullet F(y)) = n\left(\frac{\varphi(x)}{n(\varphi(x))} \bullet \frac{\varphi(y)}{n(\varphi(y))}\right) = \frac{n(\varphi(x+y))}{n(\varphi(x))n(\varphi(y))} =$$

$$\frac{n(\varphi(-y-x))}{n(\varphi(-x))n(\varphi(-y))} = n\left(\frac{\varphi(-x)}{n(\varphi(-x))} \bullet \frac{\varphi(-y)}{n(\varphi(-y))}\right) = n(F(-x) \bullet F(-y))$$

for all $x, y \in X$. Finally, note that $F(\theta) = \frac{\varphi(\theta)}{n(\varphi(\theta))} = \mathbf{1}$, because $\varphi(\theta) = \mathbf{1}$. This completes the proof.

Definition. Suppose that $(X, +)$ is a group and $(A, K, +, \bullet, \cdot)$ is a commutative algebra with a unity $\mathbf{1}$ over the field K of real or complex numbers and let $n : A \to \langle 0, +\infty)$ be a function satisfying the conditions (a), (b) and (c). A map $F : X \to A$ will be called *a generalized trigonometric map* if and only if it satisfies the functional equation (T) jointly with the conditions (W) and (Z).

Now, we shall show that Theorem 2 is a special case of Theorem 3. To see this, it suffices to prove the following remark.

Remark 5. Let $(X,+)$ is a group (with a neutral element θ) and let $(A, K, +, \bullet, \cdot)$ stand for the Banach algebra $(\mathbb{R}^2, \mathbb{R}, +, \bullet, \cdot)$ with the multiplication of vectors defined by (14) and with the Euclidean norm. Suppose that $n : \mathbb{R}^2 \to \langle, +\infty)$ is the function defined by (16) (with an arbitrarily fixed even natural number k) and $F = (f, g) : X \to \mathbb{R}^2$ is a nowhere vanishing map.
Then the following conditions are equivalent:

(24) Functions $f,\ g$ satisfy equation (II);

(25) F satisfies equation (T) jointly with the conditions (W) and (Z);

(26) There exists a character $h \in H(X, T)$ with $U = \operatorname{Re} h$ and $V = \operatorname{Im} h$ such that

$$f(x) = \frac{U(x)}{n(U(x), V(x))}, \quad g(x) = \frac{V(x)}{n(U(x), V(x))} \quad \text{for all } \ x \in X;$$

(27) There exists a non-zero morphism $\varphi : X \to \mathbb{R}^2$ from a group $(X,+)$ into the multiplicative semigroup $(\mathbb{R}^2, \bullet)$ such that $\varphi(\theta) = (1, 0)$,
$n(\varphi(x)) = n(\varphi(-x))$ and

$$F(x) = \frac{\varphi(x)}{n(\varphi(x))} \quad \text{for all} \quad x \in X.$$

Proof. The equivalence of conditions (24) and (26) results from Theorem 2, whereas conditions (25) and (27) are equivalent taking account of Theorem 4. Thus, it suffices to show that conditions (26) and (27) are equivalent.

Assume that condition (26) holds. Now we put $\varphi = (U, V)$. Since $h = U + iV$ is a character, then $U(\theta) = 1, V(\theta) = 0, U$ is an even function and V is odd. From here and from (19) ($n(u, -v) = n(u, v)$ for $(u, v) \in \mathbb{R}^2$) follows condition (27).

Conversely, suppose that condition (27) is satisfied. Let $\varphi = (\varphi_1, \varphi_2)$, where φ_1, φ_2 are coordinates of the morphism φ. Taking $U(x) = \frac{\varphi_1(x)}{\|\varphi(x)\|}$, $V(x) = \frac{\varphi_2(x)}{\|\varphi(x)\|}$ for $x \in X$ and $h = U + iV$ we obtain:

$$|h(x)| = \|U(x), V(x)\| = \left\| \frac{\varphi_1(x)}{\|\varphi(x)\|}, \ \frac{\varphi_2(x)}{\|\varphi(x)\|} \right\| =$$

$$= \frac{\|(\varphi_1(x), \varphi_2(x))\|}{\|\varphi(x)\|} = \frac{\|\varphi(x)\|}{\|\varphi(x)\|} = 1$$

for $x \in X$. Since

$$(\varphi_1(x+y), \varphi_2(x+y)) = \varphi(x+y) = \varphi(x) \bullet \varphi(y) = (\varphi_1(x), \varphi_2(x)) \bullet (\varphi_1(y), \varphi_2(y)) =$$

$$= (\mathrm{Re}((\varphi_1(x)+i\varphi_2(x))(\varphi_1(y)+i\varphi_2(y))), \mathrm{Im}((\varphi_1(x)+i\varphi_2(x))(\varphi_1(y)+i\varphi_2(y))))$$

for $x, y \in X$ and the Euclidean norm satisfies the consistency condition (15), we have

$$h(x+y) = U(x+y) + iV(x+y) = \frac{\varphi_1(x+y)}{\|\varphi(x+y)\|} + i\frac{\varphi_2(x+y)}{\|\varphi(x+y)\|} =$$

$$= \frac{(\varphi_1(x) + i\varphi_2(x))(\varphi_1(y) + i\varphi_2(y))}{\|\varphi(x)\varphi(y)\|} = \frac{\varphi_1(x) + i\varphi_2(x)}{\|\varphi(x)\|} \cdot \frac{\varphi_1(y) + i\varphi_2(y)}{\|\varphi(y)\|} =$$

$$= (U(x) + iV(x))(U(y) + iV(y)) = h(x)h(y)$$

for $x, y \in X$. Consequently $h \in H(X, T)$. Moreover,

$$f(x) = \frac{\varphi_1(x)}{n(\varphi(x))} = \frac{\frac{\varphi_1(x)}{\|\varphi(x)\|}}{n\left(\frac{\varphi_1(x)}{\|\varphi(x)\|}, \frac{\varphi_2(x)}{\|\varphi(x)\|}\right)} = \frac{U(x)}{n(U(x), V(x))} \quad \text{for} \quad x \in X$$

and analogously

$$g(x) = \frac{V(x)}{n(U(x), V(x))} \quad \text{for} \quad x \in X.$$

This completes the proof of Remark 5.

A careful inspection of the proof of Remark 5 shows that for the proof of the equivalence of conditions (26) and (27) the form of the function
$n : \mathbb{R}^2 \to \langle 0, +\infty)$ is irrelevant, but the fact that this function satisfies conditions (a), (b) and (19) is essential. Therefore, we conclude that the following remark is true.

Remark 6. Suppose that $(X, +)$ is a group (with a neutral element θ), $F = (f, g) : X \to \mathbb{R}^2$ is a nowhere vanishing map and $n : \mathbb{R}^2 \to \langle 0, +\infty)$ is a function satisfying the conditions: (a), (b) and (19). Then conditions (26) and (27) are equivalent.

3. Examples and corollaries

Now, we shall present several examples of functions $n : \mathbb{R}^2 \to \langle 0, +\infty)$ which satisfy the conditions (a), (b), (c) and (19). Some of them are norms, the others are not. Moreover, for every such function n we will draw a curve S defined by the formula:

$$S = \{(u, v) \in \mathbb{R}^2 : n(u, v) = 1\}. \tag{28}$$

Example 1. Let $n : \mathbb{R}^2 \to \langle 0, +\infty)$ be a function defined by the formula:

$$n(u, v) = |u| + |v| + \sqrt{|uv|} \quad \text{for} \quad (u, v) \in \mathbb{R}^2. \tag{29}$$

Then n is not a norm and Fig. 2 presents the curve S.

Fig. 2.

Example 2. Let $n : \mathbb{R}^2 \to \langle 0, +\infty)$ be a function defined by the formula:

$$n(u, v) = \begin{cases} \dfrac{u^2 + v^2}{|u| + |v|} & \text{for} \quad (u, v) \neq (0, 0) \\ 0 & \text{for} \quad (u, v) = (0, 0) \end{cases}. \tag{30}$$

Then n is not a norm and Picture 3 presents the curve S.

Fig. 3.

Example 3. Let $p > 0$ be an arbirarily fixed positive number and let $n : \mathbb{R}^2 \to \langle 0, +\infty)$ be a function defined by the formula:

$$n(u,v) = (|u|^p + |v|^p)^{\frac{1}{p}} \quad \text{for} \quad (u,v) \in \mathbb{R}^2. \tag{31}$$

Then n is a norm if and only if $p \geq 1$. Figures 4 and 5 illustrate the curve S in the case where $p = 3$ and $p = \frac{1}{2}$, respectively.

Fig. 4. Fig. 5.

In the following example we shall present a certain "graphic" method of obtaining some functions $n : \mathbb{R}^2 \to \langle 0, +\infty)$ satisfying conditions (a), (b), (c) and (19).

Example 4. Suppose that a closed curve $K \subset \mathbb{R}^2$ on a plane is given in the implicit form: $\Phi(x, y) = 0$, where $\Phi : G \to \mathbb{R}$ is a function defined on a set $G \subset R^2$ which is symmetric with respect to the $0x$ axis. Additionally, assume that a set D is bounded by this curve K, the point $(0, 0)$ belongs to the interior of $D, (1, 0) \in K, K$ is symmetric with respect to the $0x$ axis, and any ray with the origin in $(0, 0)$ crosses this curve K exactly at one point. Clearly, the point $(0, 0)$ can not be a point of this intersection because it do not belong to K. Therefore the following condition is satisfied:

(32) for every point $(x, y) \notin \mathbb{R}^2 \setminus \{(0, 0)\}$ there exists exactly one number $t(x, y) > 0$ such that $\Phi(t(x, y)(x, y)) = 0$.

Fig. 6 illustrates this situation.

Fig. 6.

Let $n : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by the formula:

$$n(x,y) = \begin{cases} \dfrac{1}{t(x,y)} & \text{for} \quad (x,y) \neq (0,0) \\ 0 & \text{for} \quad (x,y) = (0,0) \end{cases} \tag{33}$$

Then n satisfies the conditions (a), (b). Indeed, for every $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and $\lambda > 0$ we have: $\Phi(\lambda t(\lambda x, \lambda y)(x, y)) =$

$\Phi(t(\lambda x, \lambda y)(\lambda x, \lambda y)) = 0$. From here and from (32) it follows that $t(x, y) = \lambda t(\lambda x, \lambda y)$, and, consequently, the following equality holds: $\frac{1}{t(\lambda x,\lambda y)} \ \frac{\lambda}{t(x,y)}$. In view of (33) we obtain: $n(\lambda(x, y)) = n(\lambda x, \lambda y) = \lambda n(x, y)$.

Moreover, note that

$$S = \{(x, y) \in \mathbb{R}^2 : n(x, y) = 1\} =$$

$$= \{(x, y) \in \mathbb{R}^2 : \Phi(1(x, y)) = \Phi(x, y) = 0\} = K.$$

Since $(1, 0) \in K$, we have $n(1, 0) = 1$ and, therefore, n satisfies the condition (c). Moreover, the symmetry of K with respect to the $0x$ axis implies that if $\Phi(x, y) = 0$, then $\Phi(x, -y) = 0$ for $(x, y) \in G$. Thus, we obtain:

$$0 = \Phi(t(x, y)(x, y)) = \Phi(t(x, y)x, t(x, y)y) = \Phi(t(x, y)x, -t(x, y)y) =$$

$$= \Phi(t(x, y)(x, -y))$$

for all $(x, y) \in \mathbb{R}^2$, whence $t(x, -y) = t(x, y)$ for $(x, y) \in \mathbb{R}^2$. Therefore $n(x, -y) = n(x, y)$ for $(x, y) \in \mathbb{R}^2$, and, consequently, n satisfies condition (19), too.

Now, we shall show some functions n for which the equation (T) is equivalent to the function of the conditions (W), (Z) jointly with the equation (T). In this way we shall show, for which functions n an analogue of Theorem 2 remains valid.

Remark 7. Let $(X, +)$ be a group and let $(A, K, +, \bullet, \cdot) = (\mathbb{R}^2, \mathbb{R}, +\bullet, \cdot)$. Suppose that $n : \mathbb{R}^2 \to \langle 0, +\infty)$ is a function satisfying the conditions: (a), (b), (c), and (19). If a map $F = (f, g) : X \to \mathbb{R}^2$ satisfies equation (T), then F satisfies the conditions (W) and (Z).

Proof. We define functions $m : X \to \boldsymbol{C}$ and $\eta : \boldsymbol{C} \to \langle 0, +\infty)$ by the formulas:

(34) $m(x) = f(x) + ig(x) \quad \text{for} \quad x \in X,$

(35) $\eta(z) = n(\mathrm{Re}(z), \mathrm{Im}(z)) = n(u, v) \;\text{ for }\; z = u + iv \in \mathbb{C}, (u, v \in \mathbb{R}).$

Clearly, a function m satisfies the following functional equation

$$\text{(Tc)} \qquad m(x+y) = \frac{m(x)m(y)}{\eta(m(x)m(y))} \quad \text{for} \quad x, y \in X.$$

Moreover, the function η has the following properties:

(36) $\eta(z) = 0$ if and only if $z = 0$;

(37) $\eta(\lambda z) = \lambda \eta(z)$ for all $\lambda > 0, z \in \mathbb{C}$;

(38) $\eta(1) = 1$;

(39) $\eta(\overline{z}) = \eta(z)$ for $z \in \mathbb{C}$.
Puting $y = \theta$ (θ is the neutral element of $(X, +)$) in (Tc) we obtain that

$$\text{(40)} \qquad \eta(m(x)) = 1 \ \text{ for } x \in X.$$

Thus, $m(x) \neq 0$ for every $x \in X$, by (36). Now, equation (Tc) implies that

$$\frac{m(x+y)}{m(x)m(y)} = \frac{1}{\eta(m(x)m(y))} > 0 \quad \text{for} \quad x, y \in X.$$

Hence,

$$\frac{m(x+y)}{m(x)m(y)} = \left| \frac{m(x+y)}{m(x)m(y)} \right| = \frac{|m(x+y)|}{|m(x)||m(y)|} \quad \text{for all } x, y \in X.$$

From here it follows that

$$\text{(41)} \qquad \frac{m(x+y)}{|m(x+y)|} \ \frac{m(x)}{|m(x)|} \cdot \frac{m(y)}{|m(y)|} \quad \text{for all } x, y \in X.$$

Now we define a function $h : X \to \mathbb{C}$ by the formula:

$$h(x) = \frac{m(x)}{|m(x)|} \quad \text{for} \quad x \in X.$$

Condition (41) implies that

$$h(x+y) = h(x)h(y) \quad \text{for} \quad x, y \in X.$$

Moreover, $|h(x)| = 1$ for $x \in X$. Therefore, h is a homomorphism from $(X, +)$ into the multiplicative group of the unit circle $(\mathbf{T}, \cdot)$.

Since $1 = h(\theta) = \frac{m(\theta)}{|m(\theta)|}$, we have $1 = \eta(1) = \eta\left(\frac{m(\theta)}{|m(\theta)|}\right) = \frac{\eta(m(\theta))}{|m(\theta)|} = \frac{1}{|m(\theta)|}$, by (38) and (40). From here it follows that $|m(\theta)| = 1$ and therefore $m(\theta) = 1$. Consequently $f(\theta) = 1$ and $g(\theta) = 0$. i.e. $F(\theta) = 1$; thus the condition (Z) holds true.

Now, observe that $h(-x) = \overline{h(x)}$ for $x \in X$, whence $\frac{m(-x)}{|m(-x)|} = \frac{\overline{m(x)}}{vertm(x)|}$ for $x \in X$. The last equality and conditions (39) and (40) imply that

$$\frac{1}{|m(x)|} = \frac{\eta(n(x))}{|m(x)|} = \frac{\eta\left(\overline{m(x)}\right)}{|m(x)|} = \eta\left(\frac{\overline{m(x)}}{|m(x)|}\right) = \eta\left(\frac{m(-x)}{|m(-x)|}\right) =$$

$$= \frac{\eta(m(-x))}{|m(-x)|} = \frac{1}{|m(-x)|}$$

for $x \in X$. Thus, $|m(x)| = |m(-x)|$ for $x \in X$. From here and from the inequality:

$$\eta(m(x)m(y)) = \frac{m(x)m(y)}{m(x+y)} > 0 \quad \text{for all} \quad x, y \in X,$$

we infer that:

$$\eta(m(x)m(y)) = \left|\frac{m(x)m(y)}{m(x+y)}\right| \quad \text{for} \quad x, y \in X.$$

Therefore,

$$\eta(m(x)m(y)) = \left|\frac{m(x)m(y)}{m(x+y)}\right| = \frac{|m(x)||m(y)|}{|m(x+y)|} = \frac{|m(-x)||m(-y)|}{|m(-y-x)|} =$$

$$\left|\frac{m(-y)m(-x)}{m(-y-x)}\right| = \eta(m(-y)m(-x)) = \eta(m(-x)m(-y))$$

for $x, y \in X$ and, consequently,

$$n(F(x) \bullet F(y)) = n((f(x), g(x)) \bullet (f(y), g(y))) =$$
$$= n(\mathrm{Re}(m(x)m(y)), \mathrm{Im}(m(x)m(y))) = \eta(m(x)m(y)) = \eta(m(-x)m(-y)) =$$
$$= n(F(-x) \bullet F(-y))$$

for all $x, y \in X$; thus the condition (W) holds true.

Remarks 6 and 7 jointly with Theorem 3 imlpy the following

Corollary 2. Let $(X,+)$ be a group and let $n:\mathbb{R}^2 \to \langle 0,+\infty)$ be a function satisfing the conditions (a), (b), (c) and (19). Suppose that $F=(f,g): X \to \mathbb{R}^2$ is nowhere vanishing. Then F satisfies functional equation (T) on X if and only if there exists a character $h \in H(X,T)$ with $U = \operatorname{Re} h$ and $V = \operatorname{Im} h$ such that

$$f(x)\frac{U(x)}{n(U(x),V(x))} \quad \text{and} \quad g(x)=\frac{V(x)}{n(U(x),V(x))} \quad \text{for all} \quad x \in X.$$

4. Parametric equations of some curves

Let $S \subset \mathbb{R}^2$ be a closed curve symmetric with respect to the Ox axis and such that the point $(0,0)$ belongs to the interior of a set which is bounded by this curve. Additionally, suppose that

$$S=\left\{(u,v)\in\mathbb{R}^2 : n(u,v)=1\right\},$$

where $n:\mathbb{R}^2 \to \langle 0,+\infty)$ is a function satisfying the conditions (a), (b), (c) and (19) (recall that $(\mathbb{R}^2,\mathbb{R},+,\bullet,\cdot)$ is an algebra). From (c) it follows that $(1,0)\in S$. Now, we may consider the functional equation

$$\text{(T)} \qquad F(x+y)=\frac{F(x)\bullet F(y)}{n(F(x)\bullet F(y))},$$

where $F=(f,g):\mathbb{R}\to\mathbb{R}^2$, $x,y\in\mathbb{R}$. By Remark 7 and by the observation made after Theorem 3 we infer that every solution F of equation (T) satisfies conditions (W), (Z), and its values belong to the curve S. Since a map $\varphi:\mathbb{R}\to\mathbb{R}^2$ defined by the formula: $\varphi(x)=(\cos x,\sin x)$ for $x\in\mathbb{R}$ establishes a homomorphism between the group $(\mathbb{R},+)$ and the multiplicative semigroup of the unit circle in $\mathbb{R}^2$, then, by Theorem 3, the map $F:\frac{\varphi}{n(\varphi)}$ such that: $F(x)=\frac{(\cos x,\sin x)}{n(\cos x,\sin s)}$ for $x\in\mathbb{R}$ yields a solution of equation (T). Then the argument x measures the slope of the vector $\overline{(0,0),F(x)}$ to the axis $0x$, this means that x is an argument of the complex value $m(x)$ of the function $m=f+ig$ defined on $\mathbb{R}$ with its values on S, provided that S is treated as a complex curve. In what follows, we assume that any argument of a complex number $z=u+iv$ stands for the argument of the point $(u,v)\in\mathbb{R}^2$. By the condition (b), we observe that there exists exactly one point on the curve S with a fixed argument x (it means that any ray with the origin in $(0,0)$ crosses the curve S exactly once). Indeed, if points $p_1 = (r_1 \cos x, r_1 \sin x)$ and $p_2 = (r_2 \cos x, r_2 \sin x)$ with the radiuses $r_1, r_2 > 0$, respectively, and with the same argument x belong to the curve S, then:

$$r_1 n(\cos x, \sin x) = n(r_1 \cos x, r_1 \sin x) = 1 = n(r_2 \cos x, r_2 \sin x) =$$

$$= r_2 n(\cos x, \sin x).$$

From here it follows that $r_1 = r_2$ and, consequently, $p_1 = p_2$. Therefore, the point $F(x)$ (corresponding to the number $m(x)$, provided S is treated as a complex curve) is a unique point with the argument x which belongs to S. Since every real number x is an argument of the solution F, then its values $F(x)$ fill up the curve S. Likewise we obtain some parametric equations for the curve S :

(42)
$$u = f(x) = \frac{\cos x}{n(\cos x, \sin x)}, \; v = g(x) = \frac{\sin x}{n(\cos x, \sin x)} \quad \text{for } x \in \mathbb{R}.$$

Moreover, we can show a recurrent sequence $(s_k)_{k\in\mathbb{N}}$ of points belonging to S such that the argument of the next point differs from that of the previous point for a constant value x.
Of course, we can try to determine this sequence by above parametric equations (42), but finding the values of trigonometric functions not always is simple task. We will use the functional equation

$$\text{(Tc)} \qquad m(x+y) = \frac{m(x)m(y)}{\eta(m(x)m(y))} \quad \text{for all} \quad x, y \in \mathbb{R},$$

where $m = f + ig$ and $\eta(z) = n(\mathrm{Re}\,(z), \mathrm{Im}\,(z))$ for $z \in \mathbb{C}$.

Put $s_k = (\mathrm{Re}\, m_k, \mathrm{Im}\, m_k)$ for $k \in \mathbb{N}$, where $(m_k)_{k\in\mathbb{N}}$ is a sequence of complex numbers belonging to the curve S (provided S is treated as a complex curve), defined by the following way: we find a point $(u+iv) = m_1 \neq 1$, with some argument x, directly from the presented system of parametric equations (42) or by the solution of the following system of equations:

$$\begin{cases} n(u, v) = 1 \\ v = au \end{cases}$$

where, for example: $u > 0, \alpha = \mathrm{tg}\, x > 0$. The value x is not known, but for sufficiently small positive values α, the points obtained will lie densely on S. Clearly, the point m_1 obtained has the argument x and, therefore, $m_1 = m(x)$. From equation (Tc) we find the next point m_2 by the following way:

$$m_2 = \frac{(m_1)^2}{\eta((m_1)^2)} = \frac{m(x)m(x)}{\eta(m(x)m(x))} = m(2x).$$

Suppose that we have found a point m_k for any $k \in \mathbb{N}$ and $m_k = m(kx)$. Then we get the next point m_{k+1} according to the rule:

$$m_{k+1} = \frac{m_k \cdot m_1}{\eta(m_k \cdot m_1)} = \frac{m(kx)m(x)}{\eta(m(kx)m(x))} = m((k+1)x).$$

In this way we obtain the required sequence of the points on the curve S. Below we present the set of the points on the curve
$S = \left\{(u,v) \in \mathbb{R}^2 : \left(\sqrt{|u|} + \sqrt{|v|}\right)^2 = 1\right\}$

Fig. 7.

5. Another example

To terminate this paper we will give another example of a solution of the equation (T) jointly with the conditions (W) and (Z) in the case where $(A, K, +, \bullet, \cdot)$ is a function algebra.

Example 5. Let $(X,+) = (\mathbb{R},+)$ be a group and let $(A,K,+,\bullet,\cdot) = (C(\langle -a,a\rangle),\mathbb{R},+,\bullet,\cdot)$ be the algebra of real continuous functions defined on the interval $\langle -a,a\rangle, (a>0)$ with the standard addition and multiplication: $(f+g)(t) = f(t)+g(t), (f\bullet g)(t) = f(t)\cdot g(t)$, for $t\in\langle -a,a\rangle$. Then A is a real commutative algebra with a unity $\mathbf{1}$, where $\mathbf{1}(t) = 1$ for $t\in\langle -a,a\rangle$. Assume that $\|\bullet\| : C(\langle -a,a\rangle)\to\langle 0,+\infty)$ stands for the uniform convergence norm in the linear space $(C(\langle -a,a\rangle),\mathbb{R},+,\cdot)$, i.e.: $\|f\| = \sup\{|f(t)| : -a\le t\le a\}$ for $f\in C(\langle -a,a\rangle)$. Obviously, we have $\|\mathbf{1}\| = 1$. Now, let us consider the following functional equation:

$$F(x+y) = \frac{F(x)\bullet F(y)}{\|F(x)\bullet F(y)\|}, \tag{Tn}$$

where $F:\mathbb{R}\to C(\langle -a,a\rangle),\ x,y\in\mathbb{R}$. Define a map $\varphi:\mathbb{R}\to\langle 0,+\infty)$ by the formula:

$$\forall_{x\in\mathbb{R}}\ \forall_{t\in\langle -a,a\rangle}(\varphi(x))(t) = e^{xt}.$$

Then φ is a morphism from the group $(\mathbb{R},+)$ into the multiplicative semigroup $(C(\langle -a,a\rangle),\bullet)$. Indeed, $(\varphi(0))(t) = e^{0t} = 1$ for $t\in\langle -a,a\rangle$, thus $\varphi(0) = \mathbf{1}$. Moreover

$$(\varphi(x{+}y))(t) = e^{(x+y)t} = e^{xt}e^{yt} = (\varphi(x))(t){\cdot}(\varphi(y))(t) \quad \text{for } x,y\in\mathbb{R},\ t\in\langle -a,a\rangle.$$

From here it follows that

$$\varphi(x+y) = \varphi(x)\bullet\varphi(y) \quad \text{for} \quad x,y\in\mathbb{R}.$$

Additionally,

$$\forall_{x\in\langle -a,a\rangle}\|\varphi(x)\| = \sup\left\{e^{xt} : -a\le t\le a\right\} = e^{|x|a},$$

whence $\|\varphi(x)\| = \|\varphi(-x)\|$ for $x\in\langle -a,a\rangle$. By Theorem 3 we infer that the map $F:\mathbb{R}\to C(\langle -a,a\rangle)$ given by

$$F(x) = \frac{\varphi(x)}{\|\varphi(x)\|} \quad \text{for} \quad x\in\langle -a,a\rangle,$$

i.e.

$$\forall_{x\in\mathbb{R}}\ \forall_{t\in\langle -a,a\rangle}(F(x))(t) = \frac{e^{xt}}{e^{|x|a}} = e^{xt-|x|a},$$

satisfies functional equation (Tn) in $\mathbb{R}$ jointly with (W) and (Z).

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