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Knote Point Method in Extrapolation of Boundary Measurement Results

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Abstract
The new measurement technique that relies on distinguishing the ranges in object's space in which the searched quantities are approximated by multiparameter continuous functions has been introduced. The parameters of these functions are calculated by minimalizing the functional describing the distance between different functions determined at the same points and distance between numerical solution and experimental results and assumed 'a priori' theoretical guidelines. Functions' spaces might be determined in any way and might be changed during the computation. This method is esspecialy usefull in extrapolation of the boundary measurement results on the object interior and in joining the local and all object (global) solutions.

1. The essence of the knote method

Knote point method is a new proposition. It is generalization of the known finite elements methods and Galerkin method [4,5], which relies on approximation of calculated quantity by multiparameter functions $A\left(x_i^{(...)}, p_l, \right)$ on which the consistency condition with the theory describing the phenomenon and boundary conditions has been applied.

The consistency condition with the theory is most often formulated as assumed 'a priori' differential equations system, e.g. equations of classical theory of elasticity. Generalization proposed by the author relies on:

- Introduction of experiment's results as an additional boundary condition.
- Separating in the object's space the areas connected together only by the consistency conditions determined at the given points and named the knote points.
- Treating all conditions imposed on the generated solutions as approximated conditions.
- Treating the data set and alogith set as fuzzy sets.

In such a given task the author uses the Tikhonov regularization method [1,3] and some elements of expert systems theory [2]. We get the ability to eliminate the evidently wrong assumptions and to exclude experimental data burdened with gross error. All the boundary conditions are formulated as a distance $||d||$ between the given quantity and the calculated quantity. The distance $||d||$ for the entered condition is defined as:

$$||d_1|| = \sum_j \alpha_j \beta_j (B_j - A_{n,j})^2 \tag{1}$$

where

- $B_j$is the given quantity in the nth knote point (boundary),
- $A_{n,j}$ is a quantity calculated from the nth approximation at the same point,
- β_j is a logical variable describing the credibility of the given condition
- α_j is a number within (0,1) range classifying the importance of the given boundary condition

Logical variable β_j may be bound to any logic, i.e. the Lukasinski logic. It is advised to use continuous variables or multiparameter fuzzy logics. For the condition entered in equation

$$Eq_k\left(x^{(...)}, q_i, A_n(x_i, p_i)\right) = 0, \tag{2}$$

where

- k is an equation number in a data set,
- q_i are the given equation parameters,
- p_i are calculated approximation parameters,
- A_n is approximation of quantity distribution verified by equation,
- $x_i^{(...)}$ are generelalized variables and theirs derivatives.

distance $||d||$ is definied as:

$$||d_2|| = \sum_m \alpha_{k,m}\beta_{k,m}(Eq_k(...))^2 \tag{3}$$

During the initial run of the procedure the author recommends to take the random variable values at a level a=0.5 and then to modify them using the expert system.

2. Approximations

The calculation task of physical quantity distribution based on finite number of input data is an approximation task. Approximations spaces $A_k(...)$ may be assumed freely. It is advisable though that:

- they cover all the object's space,
- every space has common points with at least one space ascribed to another approximation,
- every experimental point belongs to at least one of the approximations.

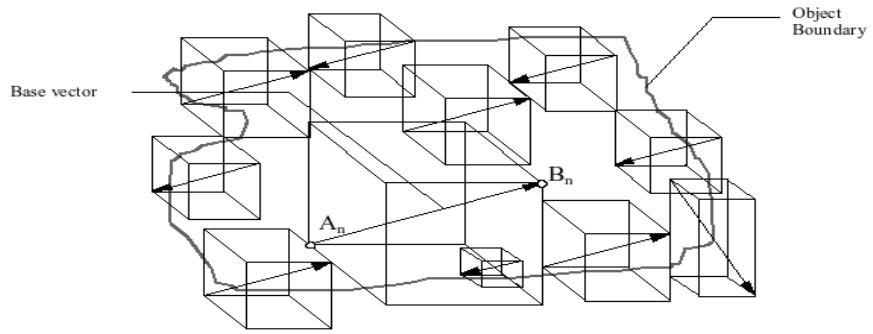

Fig.1. Determination of the function spaces.

Approximations are described by: constant parameters, procedure-determined parameters, space, attributes. Functions $A_k(...)$ can be generated or removed at any computational stage. Removing the function space leads to moving the results from knote points included in it to data set, thus knote points become the new boundary points.

Approximation class selection should make it possible to define all the measured quantities. If there are differential equations of the order N among the input data, then the approximation should be a C_N-class function. The author used the power polynomial of the $N+1$ order. There are following attributes ascribed to every function $A_k(...)$ outside the parameters set:

[1] Number of function spaces.

[2] Weights.

[3] Priorities.

[4] Logical variables.

Any solid can be chosen as a shape of the function space. The easiest choice is to use tetrahedral, cuboidal and spheroidal elements.

Tetrahedron is defined by location all of four vertex and membeship condition of point in (x_i, yi, zi) amounts to check the four inequality system:

$$b_{i,x}x_n + b_{i,y}y_n + b_{i,z}z_n \leq 0. \tag{4}$$

These inequalities describe the half spaces determined by specific wall elements and vertex opposite to them.

Spherical element is defined by center $S_k(x_s, y_s, z_s)$ location and radius r_k and membership condition of point in (x_i, yi, zi) amounts to check the following inequality:

$$(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2 \leq r_k. \tag{5}$$

Cuboidal element is defined by an attached vector $\overline{A_nB_n}$, which one end belongs to the object and another is freely located. The following inequalities define the interior of the element:

$$\begin{aligned} x_A \leq x \leq x_B \quad \vee \quad x_A \geq x \geq x_B \\ y_A \leq y \leq y_B \quad \vee \quad y_A \geq y \geq y_B \\ z_A \leq z \leq z_B \quad \vee \quad z_A \geq z \geq z_B \end{aligned} \tag{6}$$

where (x_A, y_A, z_A); (x_B, y_B, z_B) are the co-ordinates of the vector ends.

We can choose any number of $A(...)$ approximations spaces on the object's area. The spaces should be partly overlapping on each other. Figure 1 shows the example of overlapping the object by the cuboidal elements and Figure 2 shows the typical common region where the knote points are generated. Common regions and consistency condition of the approximated quantities at the knote points act in this method similary to continuity conditions in finite element method.

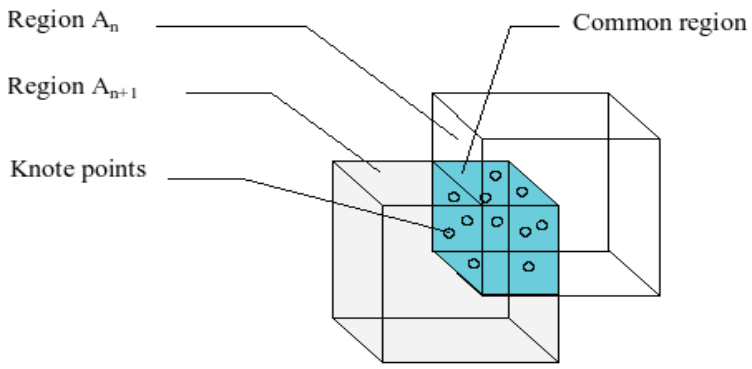

Fig. 2. Distribution of the knot points.

3. Knote points

There are the so-called knote points determined in the common regions for two or more function spaces $A_N(...)$ (Fig. 2). Every knote point has got its own:

[1] co-ordinates

[2] membership regions

[3] generation number

[4] logical variables

Membership region is a vector $W_m(w_1, ..., w_n, w_N)$ with the size equal to number of functions $A_N(...)$ and which components are the following:

$w_n = 0$ – when the point is located outside the function space$A_n(...)$

$w_n = p_m \cdot \alpha_n \cdot \beta_n$ – when the point is located inside the function space$A_n(...)$

Number p_m is a priority given to common region by expert system. At the initial run $p = 1$. At every next run the priority is lowered

as the same common region is added with next knote points. The knote points are generated automatically. The co-ordinates (x, y, z) are generated using any algorithm (e.g. random-number generator). Next, the membership to function spaces $A_n(...)$ is checked and the priority p is set. The point is counted to the set if the sum of the components of the vector w is greater than assumed treshold value

$$\sum_i w_i \geq w_{gr}. \tag{7}$$

Treshold value similary to priority p may be changed during the calculation process. Logical variables determine the point usefulnes in the further calculation process and may remove it from some operations.

Knote points are used to bind together the approximation parameters in various spaces. The distance between functions for every point (x, y, z) is calculated, which numbers k belong to the attribute set. For example, it can be defined as follows:

$$||d_3|| = \sum_k \left\{ \alpha_k \beta_k A_k(...) - \frac{\sum_k \alpha_k \beta_k A_k(...)}{\sum_k 1} \right\}^2 . \tag{8}$$

4. Boundary points

The boundary points are special kind of knote points. These points are points determined by the user or internal procedure in which we know object's state parameters. It may have values taken 'a priori' (e.g. zero normal stress on solid's surface) or empirical values. Priority of these points is a logical variable given by the expert system controlling the program and depending on credibility of data entered into the system.

5. Input data

Input data are the following:

[1] Approximation class $A_n(...)$

[2] Area in which the function spaces can be generated

[3] Spaces (imposed by the user)

[4] Knot points generation areas

[5] Boundary points or procedures of their generation provided by the user

[6] Conditions set 'a priori' at boundary points (e.g. zero displacement in a solid fulcrum)

[7] Equations and procedures imposed on functions $A_n(...)$ (e.g. theory of elasticity equations, fluid mechanics equations)

[8] Parameters of equations in item 7 imposed by the user (e.g. Young's modulus, Poisson's ratio)

[9] Measurement results

[10] Results at the knote points obtained from the previous calculation stage

[11] Approximation parameters values calculated at the previous stage

[12] Approximations, common region and knote points attributes from the previous calculation stage

[13] Expert system initial data base

Certain logical variable called a likelihood number is ascribed to every element mentioned above. In effect of the carried out calculation process it can accept different value than the imposed one at the initial run time of the program. In extreme situation some input data could be rejected (e.g. wrong Young's modulus accepted). According to the structure of likelihood numbers ascribed to input data the same procedure may work as a numerical simulator, black box analyzing experimental data only or hybrid system generating and experimentaly verifying computational models using all the accessible knowledge.

6. Tikhonov potential

For every boundary point and every knote point there are determined distances between approximation connected to them or approximation distances from the given conditions (imposed data and experimantal data). The sum of these distances measured for all the considered points creates certain potential F dependent on parameters $p_{j,n}$ of approximation $A_n(...)$

$$F = \sum ||d||. \tag{9}$$

Solution to the task reduces to determination of potential minimum, i.e. solution of the system of equations

$$\frac{\partial F}{\partial p_{j,n}}. \tag{10}$$

The system consists of

$$M = n_{max} \cdot j_{max} \tag{11}$$

equations and may be the so-called wrong conditioned system. Instability mainly results from wrong teoretical assumptions, imprecise boundary conditions and entering measured data burdened with errors. Nevertheles to achieve the result the author has proposed to use the modificated Tikhonov's idea. The idea rely on joining the potential F additional with specific element, which is a product of certain measure determined on the parameters vector p and small positive number d called regulariztion parameter. The following potential will be subject to minimization

$$T = F + \delta \cdot ||\bar{p}|| \tag{12}$$

what leads to new system of equations:

$$\frac{\partial T}{\partial p_{j,n}} = \frac{\partial F}{\partial p_{j,n}} + \delta \cdot \frac{\partial ||\bar{p}||}{\partial p_{j,n}}. \tag{13}$$

We can prove that for sufficiently large value of regularization parameter δ system (13) had got stable solution. The regularization parameter and measure built on the parameter vector do not have the clear physical interpretation. Therefore, the author has introduced in

the Tikhonov potential definition the parameter sequence connected with $||\bar{p}||_i$ measure of different definition and determined on the p vector.

$$T = F + \sum_i \delta_i \cdot ||\bar{p}||_i \tag{14}$$

For example we may assume

$$\begin{aligned}
\delta_1 \cdot ||\bar{p}||_1 &= \delta_1 \cdot \sum_{j,n} (p_{j,n})^2, \\
\delta_2 \cdot ||\bar{p}||_2 &= \delta_2 \cdot \sum_{j,n} (p_{j,n} - p_{j,n}^{alt})^2, \\
\delta_3 \cdot ||\bar{p}||_3 &= \delta_3 \cdot \sum_{j,n} \left(\frac{p_{j,n}}{p_{j,n}^{alt}} - 1 \right)^2, \\
\delta_4 \cdot ||\bar{p}||_4 &= \delta_4 \cdot \sum_{j,n} \left(\frac{p_{j,n}}{p_{j,n}^{wzr}} - 1 \right)^2,
\end{aligned} \tag{15}$$

where

$||\bar{p}||_1$	determines the sensitivity on numerical instability
$||\bar{p}||_2; ||\bar{p}||_3$	determine the distance between the solution and achieved results gained on the same object type
$||\bar{p}||_4$	determines the distance between results and the solution scheme applied

The solution of this problem is a limit of a sequence of solutions got for different regularization parameters, when $\delta_i \to 0$.

7. Joining the results

The final calculation results are values and derivatives of approximations $A_n(...)$ and values of functions built on them at individual points in object space. If the point belongs simultaneuosly to several funcion spaces, the weighted average of the ascribed approximations is provided to resulting procedure

$$A_{sr}(...) = \frac{\sum \alpha_n \cdot \beta_n \cdot A_n(...)}{\sum \alpha_n \cdot \beta_n}. \tag{16}$$

The weights α_n ascribed to individual approximations An(...) may be constants. But there is a risk of appearing in the final solution the groundless jumps of the approximated value on the boundary between spaces. It is though better to take $\alpha_n = \alpha_n(x, y, z)$ as a function, which approach zero at the boundary of spaces. For the cuboidal elements it may be a function like this:

$$\alpha_n(x, y, z) = |(x - x_{A_n})(y - y_{A_n})(z - z_{A_n})|. \tag{17}$$

8. Algorythm frame

[1] Determine roughly object space, ascribing 1-10 approximation spaces $A_n(...)$ to it.

[2] Determine approximation class (function or procedure definition).

[3] Generate the knote point set of the 1-st layer.

[4] Take the first potion of data (equations, 'a priori' boundary conditionsk experimental data, archival data).

[5] Calculate analytical derivatives $\delta A_n(...)/\delta p_i$.

[6] Define the distances in the Tikhonov potential $||\bar{p}||_i$.

[7] Calculate analytical derivatives $\delta_i||\bar{p}||_i/\delta p_i$.

[8] Build and solve the system of equations (13).

[9] Enter the next approximation.

[10] Enter the next portion of boundary data.

[11] Take the solutions from item 8 as an additional boundary conditions.

[12] Build and solve system (13) for new entered data.

[13] Add the received solution to the input data set.

[14] Add the equations determined in item 8 to the equations determined for previous layer.

[15] Solve the received system of equations.

[16] Go to item 9 or 17.

[17] Join results from the individual function spaces together.

[18] Archive or edit calculation results.

Results received in consecutive computational layers at most of the points are very close to the results from the previous layer. Therefore in the solution of system of equations (13) we may use the iterative algorithm, e.g. the Stefan algorithm.

The proposed procedure contains many points, in which the logical variables are ascribed to certain data, procedures, attributes and parameters. It fulfils the classical expert system role. Instead of building the own program we may only define the individual components (algorithms as procedures and date, atributes or parameters as facts) and control the whole using the typical expert software, e.g. the system CLIPS.

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