Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XXIII (2018)
11–24
DOI http://dx.doi.org/10.16926/m.2018.23.01

ASYMMETRY IN REAL FUNCTIONS THEORY

JACEK MAREK JĘDRZEJEWSKI

Abstract

Since the beginning of the XX century many authors considered characterizations of local properties for real functions of a real variable which have been defined as global properties. We present a short survey of local properties of the well known global ones and consider of how small/big the set of asymmetrical behaviour of a function must be.

1. Introduction

We shall consider only real functions defined in an open interval. When we use topological terminology, then it is applied in the sense of natural topology in the set of real numbers (or in its subsets).

Limit numbers of a real function defined in subsets of $\mathbb{R}$ have been considered in many articles by many mathematicians. Starting from the classical result of W. H. Young [20] concerning asymmetry of functions through problems of usual limit numbers, J. M. Jędrzejewski and W. Wilczyński [12], approximate limit numbers discussed by M. Kulbacka [14], L. Belowska [1], W. Wilczyński [18] and others, problems of qualitative limit numbers (W. Wilczyński [19]) $\mathcal{B}$-limit numbers (J. M. Jędrzejewski [7], [8], J. M. Jędrzejewski together with W. Wilczyński [13]) one can come up to a big monograph on local systems by B. S. Thomson [17].

The first part of our considerations deals with the asymmetry of functions with respect to limit numbers of different kinds.

Some properties of functions (continuity, Darboux condition and others) can be characterized globally and locally. For many of those properties we have theorems which say that a function has this global property if and only if it has its adequate local property. The second part of the article deals with some of such properties.

• *Jacek M. Jędrzejewski* — jacek.m.jedrzejewski@gmail.com
Retired from Pomeranian Academy in Słupsk.

The last part of the paper is devoted to results obtained by T. Świątkowski in view of general approach to limit numbers considered originally by B. S. Thomson and me.

2. Asymmetry of Sets of Limit Numbers

2.1. **Limit Numbers of a Real Function.** We shall start with the classical problem called Rome's Theorem. The theorem was probably the first one which dealt with arbitrary function. Let us remind necessary definitions and properties.

Definition 1. (W. H. Young [20]) *Let a real function f be defined in an open interval (a,b). Then a number g (or $+\infty$ or $-\infty$) is called the limit number of f at a point x_0 from (a,b) if there exists a sequence $(t_n)_{n=1}^{\infty}$ such that*

(1) $t_n \neq x_0$, *for each positive integer n,*
(2) $\lim_{n\to\infty} t_n = x_0$,
(3) $\lim_{n\to\infty} f(t_n) = g$.

If the inequality $t_n \neq x_0$ is replaced by $t_n > x_0$, then such a limit number is called the right limit number of f at x_0.

If the inequality $t_n \neq x_0$ is replaced by $t_n < x_0$, then such a limit number is called the left limit number of f at x_0.

- By $L^+(f,x_0)$ we denote the set of all right limit numbers of f at x_0.
- By $L^-(f,x_0)$ we denote the set of all left limit numbers of f at x_0.
- By $L(f,x_0)$ we denote the set of all limit numbers of f at x_0.

Let us remark that limit numbers can be equivalently defined in the following way:

Theorem 1. *Let a real function f be defined in an open interval (a,b). Then a number g (or $+\infty$ or $-\infty$) is a limit number of f at a point x_0 from (a,b) if and only if the set*

$$\left\{x \in (a,b) \,:\, f^{-1}(U_g) \cap [(x_0-\varepsilon, x_0+\varepsilon) \setminus \{x_0\}]\right\}$$

is non-empty for each positive ε and each neighbourhood U_g of the point g.

It is obvious that:

Theorem 2. *The sets $L^-(f,x_0)$, $L^+(f,x_0)$ and $L(f,x_0)$ are non-empty and closed, moreover*

$$L(f,x_0) = L^-(f,x_0) \cup L^+(f,x_0)$$

for any function $f\colon (a,b) \longrightarrow \mathbb{R}$ and any $x \in (a,b)$.

The main theorem which was announced in Rome at the congress of mathematicians is stated as follows:

Theorem 3. Rome's Theorem on Asymmetry (W. H. Young, 1906) *For any function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *the set*

$$\left\{x \in (a,b) \,:\, L^{-}(f,x) \neq L^{+}(f,x)\right\}$$

is at most countable.

Quite similarly one can say that:

Theorem 4. *For any function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *the set*

$$\{x \in (a,b) \,:\, f(x_0) \notin L(f,x)\}$$

is at most countable.

Let us remark that for each countable set E in $\mathbb{R}$ there exists a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ for which

$$E = \left\{x \in (a,b) \,:\, L^{-}(f,x) \neq L^{+}(f,x)\right\}.$$

It is quite obvious if the set E is finite; if it is infinite it is possible to define a monotone function, which fulfils the required condition. We shall construct such a function.

Example 1. Monotone function with infinite set of asymmetry.

Let $E = (x_n)_{n=1}^{\infty}$ and the sequence of positive numbers $(\alpha_n)_{n=1}^{\infty}$ be such that the series $\sum\limits_{n=1}^{\infty} \alpha_n$ is convergent. The function

$$f(x) = \sum_{\{n\,:\,x_n<x\}} \alpha_n$$

fulfils all the required properties.

2.2. **Qualitative Limit Numbers.** Following the way as in Theorem 1. one can define other kinds of limit numbers as qualitative (W. Wilczyński [19]) or approximative limit numbers (L. Belowska [1], M. Kulbacka [14], J. Jaskuła [5] and W. Wilczyński [18]) when we define limit numbers using the above mentioned property.

Definition 2. *A number* g *or* $+\infty$ *or* $-\infty$ *is called the qualitative limit number of a function* f *at a point* x_0 *if the set*

$$\left\{x \in (a,b) \,:\, f^{-1}(U_g) \cap (x_0-\varepsilon, x_0+\varepsilon)\right\}$$

is of the second category for each positive ε *and arbitrary neighbourhood* U_g *of the point* g.

Definition 3. *If the set*

$$\{x \in (a,b) \,:\, f^{-1}(U(g)) \cap (x_0-\varepsilon, x_0)\}$$

is of the second second category for each positive ε, then g is called the left qualitative limit number of a function f at the point x_0.

Similarly, g is called the right qualitative limit number of a function f at a point x_0 if the set

$$\{x \in (a,b) \,:\, f^{-1}(U(g)) \cap (x_0, x_0+\varepsilon)\}$$

is of the second category for each positive ε and each neighbourhood U_g of the point g.

- By $L_q^+(f,x_0)$ we denote the set of all right qualitative limit numbers of f at x_0.
- By $L_q^-(f,x_0)$ we denote the set of all left qualitative limit numbers of f at x_0.
- By $L_q(f,x_0)$ we denote the set of all qualitative limit numbers of f at x_0.

Then, similarly as for usual limit numbers one can state:

Theorem 5. *For arbitrary real function f on the interval (a,b) and any x_0 from (a,b) the sets $L_q(f,x_0)$, $L_q^-(f,x_0)$ and $L_q^+(f,x_0)$ are non-empty, closed and*

$$L_q(f,x_0) = L_q^-(f,x_0) \cup L_q^+(f,x_0).$$

Considering the sets of qualitative limit numbers we can get the analogue of Rome's Theorem, namely:

Theorem 6. *For any function $f\colon (a,b) \longrightarrow \mathbb{R}$ the set*

$$\{x \in (a,b) \,:\, L_q^-(f,x) \neq L_q^+(f,x)\}$$

is at most countable.

We can observe that the considered sets are at most countable, it means that they are rather small with natural topology in the set of real numbers. The quantity of such sets will be of our main interest. Unfortunately not always such sets must be countable.

2.3. **Approximate Limit Numbers.** Several mathematicians considered approximate limit numbers but we remind basic definitions and properties.

Definition 4. *A number g or $+\infty$ or $-\infty$ is called the approximate limit number of a function f at a point x_0 if the set*

$$\{x \in (a,b) \,:\, f^{-1}(U(g)) \cap [(x_0-\varepsilon, x_0+\varepsilon)]\}$$

has positive upper exterior density at x_0 *for every open neighbourhood* U_g *of the point* g *and each positive* ε.

Definition 5. *A number* g *or* $+\infty$ *or* $-\infty$ *is called the left approximate limit number of a function* f *at a point* x_0 *if the set*

$$\{x \in (a,b) \,:\, f^{-1}(U(g)) \cap [(x_0 - \varepsilon, x_0)]\}$$

has positive upper exterior density at x_0 *for every open neighbourhood* U_g *of the point* g *and each positive* ε.

And similarly, a number g (or $+\infty$, $-\infty$) is called the right approximate limit number of a function f at a point x_0 if the set

$$\{x \in (a,b) \,:\, f^{-1}(U(g)) \cap [(x_0, x_0 + \varepsilon)]\}$$

has positive upper exterior density at x_0 for every open neighbourhood U_g of the point g and each positive ε.

- By $L_a^+(f, x_0)$ we denote the set of all right approximate limit numbers of f at x_0.
- By $L_a^-(f, x_0)$ we denote the set of all left approximate limit numbers of f at x_0.
- By $L_a(f, x_0)$ we denote the set of all approximate limit numbers of f at x_0.

Then, similarly as for usual limit numbers one can state:

Theorem 7. *For arbitrary real function* f *on the interval* (a,b) *and any* x_0 *from* (a,b) *the sets* $L_a(f,x_0)$, $L_a^-(f,x_0)$ *and* $L_a^+(f,x_0)$ *are non-empty, closed and*

$$L_a(f,x_0) = L_a^-(f,x_0) \cup L_a^+(f,x_0).$$

Now considering the sets of approximate limit numbers we can get the analogue of Rome's Theorem, but:

Theorem 8. (M. Kulbacka [14]). *For any function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *the set*

$$\{x \in (a,b) \,:\, L_a^-(f,x) \neq L_a^+(f,x)\}$$

is first category set and has measure 0.

This time the sets of the first category which have measure 0 do not characterize the set of asymmetry of functions. J. Jaskuła gave some additional properties for the set of approximate asymmetry.

Theorem 9. (J. Jaskuła [5]) *For any function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *the set*

$$\{x \in (a,b) \,:\, L_a^-(f,x) \neq L_a^+(f,x)\}$$

is first category and has measure 0, *moreover it is of type* $F_{\sigma\delta\sigma}$.[1]

[1] W.Wilczyński informed me that the results of J. Jaskuła were a big deeper, i.e. the set approximate asymmetry is also σ-porous.

2.4. **Generalized Limit Numbers.** Let us observe that the class of sets which are of the first category at the point x_0 and the class of positive upper external density at that point have common properties. When we denote such a class by $\mathcal{B}$ then this class fulfils:

(1) If $B \in \mathcal{B}$ and $E \supset B$, then $E \in \mathcal{B}$,
(2) If $B_1 \cup B_2 \in \mathcal{B}$ then $B_1 \in \mathcal{B}$ or $B_2 \in \mathcal{B}$,
(3) If $B \in \mathcal{B}$ and $\varepsilon > 0$ then $B \cap (x_0 - \varepsilon, x_0 + \varepsilon) \in \mathcal{B}$.

The class of sets which are uncountable in each $(x_0 - \varepsilon, x_0 + \varepsilon)$ or have positive outer measure in each such interval and many other classes of sets have the previously pointed properties. The articles on this topic are as follows: J. Jędrzejewski [7], [8], J. Jędrzejewski with W. Wilczyński [13], J. Jędrzejewski with S. Kowalczyk [10] and [11].

Let us start now from the beginning:

Definition 6. *For each $x \in \mathbb{R}$ let $\mathfrak{B}_x^+$ be a class of non-empty sets fulfilling the following conditions:*

(1) $B_1 \cup B_2 \in \mathfrak{B}_x^+ \iff (B_1 \in \mathfrak{B}_x^+ \vee B_2 \in \mathfrak{B}_x^+)$,
(2) $B \cap (x, x+t) \in \mathfrak{B}_x^+$ *for each* $B \in \mathfrak{B}_x^+$ *and* $t > 0$.

For each $x \in \mathbb{R}$ let $\mathfrak{B}_x^-$ be a class of non-empty sets fulfilling the following conditions:

(1) $B_1 \cup B_2 \in \mathfrak{B}_x^- \iff (B_1 \in \mathfrak{B}_x^- \vee B_2 \in \mathfrak{B}_x^-)$,
(2) $B \cap (x, x+t) \in \mathfrak{B}_x^-$ *for each* $B \in \mathfrak{B}_x^-$ *and* $t > 0$.

Let $\mathfrak{B}_x = \mathfrak{B}_x^- \cup \mathfrak{B}_x^+$.

Definition 7. *If f defined in some (a, b) is a real function, then a number (or $+\infty$ or $-\infty$) is called $\mathfrak{B}$-limit number of f at x_0 from (a, b) if*

$$\left\{x \in (a, b) \,:\, f^{-1}(U_g)\right\} \in \mathfrak{B}_{x_0}$$

for any neighbourhood U_g of the point g.

Definition 8. *If*

$$\left\{x \in (a, b) \,:\, f^{-1}(U_g) \in \mathfrak{B}_{x_0}^-\right\}$$

for any neighbourhood U_g of the point g, then g is called the left $\mathfrak{B}$-limit number of a function f at a point x_0.

Similarly we define right $\mathfrak{B}$-limit numbers of a function f at a point x_0.

- By $L_{\mathfrak{B}}^+(f, x_0)$ we denote the set of all right $\mathfrak{B}$-limit numbers of f at x_0.
- By $L_{\mathfrak{B}}^-(f, x_0)$ we denote the set of all left $\mathfrak{B}$-limit numbers of f at x_0.
- By $L_{\mathfrak{B}}(f, x_0)$ we denote the set of all $\mathfrak{B}$-limit numbers of f at x_0.

Then, as for usual limit numbers, one can state:

Theorem 10. *For arbitrary real function f on the interval (a,b) and any x_0 from (a,b) the sets $L_{\mathfrak{B}}(f,x_0)$, $L^-_{\mathfrak{B}}(f,x_0)$ and $L^+_{\mathfrak{B}}(f,x_0)$ are non-empty, closed and*

$$L_{\mathfrak{B}}(f,x_0) = L^-_{\mathfrak{B}}(f,x_0) \cup L^+_{\mathfrak{B}}(f,x_0).$$

Considering the sets of $\mathfrak{B}$-limit numbers we are not able to get the analogue of Rome's Theorem. The situation depends on the class $\mathfrak{B}$. But if we add a special condition for the family $\mathfrak{B}$, we can get adequate analogue of Young's theorem.

Definition 9. *We say that the class $\mathfrak{B}$ fulfils condition $\mathscr{M}$ if*

$$\bigcup_{n=1}^{\infty} E_n \in \mathfrak{B}_{x_0}$$

for any: $x_0 \in (a,b)$, sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 and every sequence of sets $(E_n)_{n=1}^{\infty}$ such that $E_n \in \mathfrak{B}_{x_n}$.

This condition permits us to state:

Theorem 11. *If the class $\mathfrak{B}$ fulfils condition $\mathscr{M}$, then*

$$\left\{x \in (a,b) \,:\, L^-_{\mathfrak{B}}(f,x) \neq L^+_{\mathfrak{B}}(f,x)\right\}$$

is at most countable set for any function $f\colon (a,b) \longrightarrow \mathbb{R}$.

3. Asymmetry for Some Classes of Functions

3.1. **Differentiation of Functions.** Everybody knows:

Theorem 12. *The set of all those points at which left derivative of a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is different from the right derivative of this function is at most countable.*

3.2. **Continuity of Functions.** One can get that the set of points at which a function is continuous from exactly one side as a quite simple corollary of Young's Theorem.

Theorem 13. *For any function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ the set of all points at which f is continuous from the only one side is at most countable.*

3.3. **Darboux Condition of Functions.** As before: everybody knows that Darboux condition has been originally defined as a global condition of a function. It sounds like this: the function f fulfils Darboux condition if it takes all values in between; exactly:

Definition 10. *We say that a function $f\colon (a,b) \longrightarrow \mathbb{R}$ fulfils Darboux condition if for any x_1 and x_2 such that $f(x_1) \neq f(x_2)$ and any number c lying between $f(x_1)$ and $f(x_2)$ there exists a point x lying (strictly) between x_1 and x_2 such that $f(x) = c$.*

This condition can be replaced by the one that function f transforms connected sets onto connected sets.

But still this condition is not good enough to say about asymmetry. We should define this condition locally, even more it must be defined separately for both sides. Let's start to do it, what was done by A. Bruckner and J. Ceder in 1965. For simplicity, let us assume that all the discussed functions are bounded.

Definition 11. (A. Bruckner, J. Ceder) [2]) *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is said to be Darboux from the left side at a point $x_0 \in (a,b)$ if*

(1) $f(x_0) \in L^-(f, x_0)$,
(2) *for each $c \in (\inf L^-(f,x_0), \sup L^-(f,x_0))$ and for each $t > 0$ there exists a point $x \in (x_0 - t, x_0)$ such that $f(x) = c$.*

Similarly,

Definition 12. *We say that a function $f\colon (a,b) \longrightarrow \mathbb{R}$ is Darboux from the right side at a point $x_0 \in (a,b)$ if*

(1) $f(x_0) \in L^+(f, x_0)$,
(2) *for each $c \in (\inf L^+(f,x_0), \sup L^+(f,x_0))$ and for each $t > 0$ there is a point $x \in (x_0, x_0 + t)$ such that $f(x) = c$.*

In the end:

Definition 13. *We say that a function $f\colon (a,b) \longrightarrow \mathbb{R}$ is Darboux at a point $x_0 \in (a,b)$ if it is Darboux from both sides at x_0.*

These definitions would not be good enough if the next theorem is false. But luckily it is not so.

Theorem 14. *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is Darboux if and only if it is Darboux at each point $x_0 \in (a,b)$.*

And now we can say about Darboux asymmetry.

Theorem 15. [9]. *For each function $f\colon (a,b) \longrightarrow \mathbb{R}$ the set of all those points at which f Darboux from exactly one side is at most countable.*

3.4. **Connectedness of Functions.** Next class of functions we want to discuss is the class of functions with connected graphs. They are called connected functions, however they can be defined in each topological spaces we shall consider only real functions defined in an interval. The adequate characterization has been given by B. D. Garret, D. Nelms and K. R. Kellum [3].

Definition 14. *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is called connected if its graph is a connected set on the plane.*

As before this definition is a global one, we have to find a local definition which will be as good as to get that local and global characterizations coincide.

As before, we assume that all discussed functions are bounded.

Definition 15. (B. D. Garret, D. Nelms, K. R. Kellum) [3]) *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is connected from the left side at a point $x_0 \in (a,b)$ if*

(1) *$f(x_0) \in L^-(f,x_0)$,*
(2) *for each continuum K (connected and compact set) such that*

$$\mathrm{proj}_x(K) = [x_0 - t, x_0] \quad \text{for some} \quad t > 0$$

and

$$\mathrm{proj}_y(K) \subset \big(\inf L^-(f,x_0), \sup L^-(f,x_0)\big)$$

the (graph) function f has common point with K.

Similarly:

Definition 16. *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is connected from the right side at a point $x_0 \in (a,b)$ if*

(1) *$f(x_0) \in L^+(f,x_0)$,*
(2) *for each continuum K such that*

$$\mathrm{proj}_x(K) = [x_0, x_0 + t] \quad \text{for some} \quad t > 0$$

and

$$\mathrm{proj}_y(K) \subset \big(\inf L^+(f,x_0), \sup L^+(f,x_0)\big)$$

the (graph) function f has common point with K.

Definition 17. *We say that a function $f\colon (a,b) \longrightarrow \mathbb{R}$ is connected at a point $x_0 \in (a,b)$ if it is connected from both sides at x_0.*

And of course:

Theorem 16. *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is connected if and only if it is connected at each point $x_0 \in (a,b)$.*

Finally, we are able to formulate theorem on connectivity asymmetry.

Theorem 17. *For each function $f\colon (a,b) \longrightarrow \mathbb{R}$ the set of all those points at which f is connected from exactly one side is at most countable.*

3.5. **Almost Continuity of Functions.** The last class of functions we want to discuss is the class of almost continuous functions. The adequate local characterization has been given by J. M. Jastrzębski, T. Natkaniec and J. Jędrzejewski [6].

Definition 18. *A function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *is called almost continuous if each neighbourhood of its graph contains some continuous function defined in* (a,b).

As before this definition is a global one, we have to find a local definition which will be as good as to get that local and global characterizations coincide.

We assume that all discussed functions are bounded.

Definition 19. *A function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *is almost continuous from the left side at a point* $x_0 \in (a,b)$ *if*

(1) $f(x_0) \in L^-(f, x_0)$,
(2) *there is a positive* ε *such that for each open neighbourhood of* $f_{|(x,\infty)}$ *arbitrary* $y \in (\inf L^-(f,x_0), \sup L^-(f,x_0))$, *arbitrary neighbourhood* G *of the point* $(x,y) \in \mathbb{R}^2$ *and arbitrary* $t \in (x_0, x_0+\varepsilon)$ *there is a continuous function* $g\colon (x_0, x_0+\varepsilon) \longrightarrow \mathbb{R}$ *such that* $g \subset U \cup G$ *and* $g(x_0) = y$, $g(t) = f(t)$.

Similarly one can define almost continuity from the right side at a point $x_0 \in (a,b)$.

Definition 20. *We say that a function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *is almost continuous at a point* $x_0 \in (a,b)$ *if it is almost continuous from both sides at* x_0.

And of course:

Theorem 18. *A function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *is almost continuous if and only if it is almost continuous at each point* $x_0 \in (a,b)$.

Finally, one can state:

Theorem 19. *For each function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *the set of all those points at which* f *is almost continuous from exactly one side is at most countable.*

4. General Approach to Asymmetry of Functions

Some general theorems were discussed in previous parts of the article. Let us come to Thomson's monograph. B. S. Thomson gathered several ideas in one theory. He defined local systems which contain $\mathfrak{B}$ classes and $\mathfrak{B}^*$ classes that have been defined in [7]. For sake of completeness let us remind the basic notions.

4.1. **Local Systems.**

Definition 21. B. S. Thomson [17].
By a local system in $\mathbb{R}$ *we mean a class* $\mathcal{S}$ *consisting of non-empty collections* $\mathcal{S}(x)$ *for each real number* x, *fulfilling the following conditions:*

$$\{x\} \notin \mathcal{S}(x), \tag{1}$$

$$E \in \mathcal{S}(x) \Longrightarrow x \in E, \tag{2}$$

$$(E \in \mathcal{S}(x) \wedge F \supset E) \Longrightarrow F \in \mathcal{S}(x), \tag{3}$$

$$(E \in \mathcal{S}(x) \wedge \delta > 0) \Longrightarrow E \cap (x-\delta, x+\delta) \in \mathcal{S}(x). \tag{4}$$

Definition 22. *By a left local system in $\mathbb{R}$ we mean a class $\mathcal{S}$ consisting of non-empty collections $\mathcal{S}(x)$ for each real number x, fulfilling the following conditions:*

$$\{x\} \notin \mathcal{S}(x), \tag{5}$$

$$E \in \mathcal{S}(x) \Longrightarrow x \in E, \tag{6}$$

$$(E \in \mathcal{S}(x) \wedge F \supset E) \Longrightarrow F \in \mathcal{S}(x), \tag{7}$$

$$(E \in \mathcal{S}(x) \wedge \delta > 0) \Longrightarrow E \cap (x-\delta, x] \in \mathcal{S}(x). \tag{8}$$

Similarly we define right local systems.

A local system is called filtering at a point x if

$$E \cap F \in \mathcal{S}(x) \ \text{ whenever } \ E \in \mathcal{S}(x) \ \text{ and } \ F \in \mathcal{S}(x). \tag{9}$$

A local system is called filtering if it is filtering at each x in $\mathbb{R}$.
A local system is called bilateral if

$$E \cap (x-\delta, x) \neq \varnothing \quad \text{and} \quad E \cap (x, x+\delta) \neq \varnothing$$

for each $x \in \mathbb{R}$, $E \in \mathcal{S}(x)$ and $\delta > 0$.

Let us observe that those definitions are very close to Definition 6. When B. S. Thomson assumes that dual system for $\mathcal{S}$ is filtering, then $\mathcal{S}$ fulfils all requirements of Definition 6. The only difference lays in the belonging of the point x to every set from the class $\mathcal{S}_x$.

Definition 23. *A number g is called $\mathcal{S}$-limit of a function f at a point x if*

$$f^{-1}(g-\varepsilon, g+\varepsilon) \cup \{x\} \in \mathcal{S}(x)$$

for each positive ε.

We shall write then

$$g = (\mathcal{S}) \lim_{t \to x} f(t).$$

The set of all $(\mathcal{S})$-limits are denoted by $\Lambda_{\mathcal{S}}(f, x)$.

For each local system $\mathcal{S}$ there is a system $\mathcal{S}^*$ which is also a local system, that is defined by:

$$E \in \mathcal{S}^*(x) \iff \big(x \in E \wedge \big[(\mathbb{R} \setminus E) \cup \{x\}\big] \notin \mathcal{S}(x)\big)\,.$$

This system is called dual system for $\mathcal{S}$.

A system $\mathcal{S}$ is called filtering if $E_1 \cap E_2 \in \mathcal{S}(x)$ for every sets $E_1 \in \mathcal{S}(x)$ and $E_2 \in \mathcal{S}(x)$ and each $x \in \mathbb{R}$.

Definition 24. *We say that two systems $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy a joint intersection condition if for any choices $\{E_x : x \in \mathbb{R}\}$ and $\{D_x : x \in \mathbb{R}\}$ such that $E_x \in \mathcal{S}_1(x)$, $D_x \in \mathcal{S}_2(x)$ there exists a gauge δ on $\mathbb{R}$ so that if $0 < |x - y| < \min\{\delta(x), \delta(y)\}$ then at least one of the sets $E_x \cap D_y$ or $D_x \cap E_y$ contains points other than x and y.*

By a gauge on the set $\mathbb{R}$ we mean a positive function defined in $\mathbb{R}$.

And now we are able to formulate the asymmetry theorem given by Thomson.

Theorem 20. *Let $\mathcal{S}^1$, $\mathcal{S}^2$ be local systems such that both of them are filtering and that the pair $(\mathcal{S}_1, \mathcal{S}_2)$ has the joint intersection condition. Then for any function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ the set*

$$\{x \in \mathbb{R} \,:\, \Lambda_{\mathcal{S}^1}(f,x) \neq \Lambda_{\mathcal{S}^2}(f,x)\}$$

is at most countable.

Example 2.

Let $\mathcal{S}^1_x$ be the class consisting of all sets E for which $E \cap (x-\varepsilon, x+\varepsilon)$ is of the first category.

Let $\mathcal{S}^2_x$ be the class consisting of all sets D for which $D \cap (x-\varepsilon, x+\varepsilon)$ has positive outer measure.

There are two sets A and B such that $A \cap B = \varnothing$, $A \cup B = (0,1)$, A is of the first category in $(0,1)$, and B has measure 1.

Let $f\colon (0,1) \longrightarrow \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in A, \\ 1 & \text{if} \quad x \in B. \end{cases}$$

For this function, all points from $(0,1)$ are points of $\left(\mathcal{S}^1, \mathcal{S}^2\right)$-asymmetry.

4.2. **Świątkowski Approach to Asymmetry.**

Definition 25. (T. Świątkowski [15]) *Let T be a stronger topology in $\mathbb{R}$ than the natural one. For a subset E of $\mathbb{R}$ the symbol E'_T denote the set of all accumulation points with respect to topology T. Let moreover $L_x = (-\infty, x)$ and $P_x = (x, \infty)$ for any real number x. Consider now the function φ in the following way:*

$$x \in \varphi(A) \quad if \quad x \in (A \cap L_x)'_T \triangle (A \cap P_x)'_T$$

for any subset A of $\mathbb{R}$.

Each point from the set $(A \cap L_x)'_T \triangle (A \cap P_x)'_T$ is called T-asymmetry point of the set A.

Definition 26. *Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be arbitrary function and x a real number. We say that g is T-limit number of the function f at a point x if*

$$x \in \left(f^{-1}(U)\right)'_T$$

for each neighbourhood U of the point x.

Not every topology is good enough to get the adequate theorem on asymmetry; let us call the property (W) from the article [15].

Definition 27. [15] *Let T be a stronger topology then the natural one in the set $\mathbb{R}$. We say that T fulfils condition (W) if for every $x \in \mathbb{R}$, sequence $(x_n)_{n=1}^{\infty}$ converging to x and every sequence $(E_n)_{n=1}^{\infty}$ such that $x_n \in (E_n)'_T$ the point x belongs to $(\bigcup_{n=1}^{\infty} E_n)'_T$.*

This condition (W) for the topology T described as above is equivalent to the condition (W'):
for an arbitrary $x \in \mathbb{R}$ and its T-neighbourhood U there exists a positive number δ such that $((x-\delta, x+\delta) \setminus U)'_T = \varnothing$.

The condition (W') allows to formulate one of the most general theorems on asymmetry.

Theorem 21. *If T is a stronger than the natural topology in the set $\mathbb{R}$ and fulfils condition (W), then for any function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ the set of asymmetry of f is at most countable.*

It is now easy to observe that:

If T is a natural topology in $\mathbb{R}$, Theorem 21 allows us to obtain the classical Young's Theorem on asymmetry. It is implied from the fact that T fulfils condition (W') (see Theorem 3).

Let us remark that if T is a Hashimoto topology in $\mathbb{R}$ generated by sets of the first category, Theorem 21 allows us to obtain Theorem on qualitative asymmetry of functions. It follows from the fact that T also fulfils (W') (see Theorem 6).

4.3. **Comments on the Three Approaches to Asymmetry.** When we want to compare the three ideas of B. S. Thomson, of T. Świątkowski and J. Jędrzejewski, we can observe that some local systems $\mathcal{S}$-limits can be understood as $\mathfrak{B}$-limits, some systems can be understood as systems $\mathfrak{B}$. However, in each theorem where Thomson assumes that the dual system for a system $\mathcal{S}$ is filtering, then the system fulfils all conditions for the system $\mathfrak{B}$. Świątkowski's condition and mine called $\mathscr{W}$ or $\mathscr{M}$ are equivalent, so Thomson's theorems are almost the same as Świątkowski's and mine ones. The only difference lays on different approaches to the problem.

References

[1] Belowska L., *Résolution d'un problème de M. Z. Zahorski sur les limites approximatives*, Fund. Math. 48 (1960), p. 277 – 286,

[2] Bruckner A., Ceder J., *Darboux continuity*, Jber. Deutsch. Math. Verein. 67 (1965), p. 93 – 117,

[3] Garret B. D., Nelms D., Kellum K. R., *Characterizations of connected real functions* Jber. Deutsch. Math. Verein. 79 (1971), p. 131 – 137,

[4] Hashimoto H., On the *topology and application, Fund. Math. XCI, (1976), p. 5–10,

[5] Jaskuła J., *Doctor's Thesis*, Uniwersytet Łódzki, 1971,

[6] Jastrzębski J. M., Jędrzejewski J. M., Natkaniec T., *Points of Almost Continuity of Real Functions*, Real Anal. Ex. 16 (1990–91),

[7] Jędrzejewski J. M., *On limit numbers of real functions*, Fund. Math. LXXXIII (1974), p. 269 – 281,

[8] Jędrzejewski J. M., *On the family of sets of* $\mathfrak{B}$*-limit values and Baire's functions*, Acta Univ. Lodz. ser. II 14 (1980), p. 59 – 66,

[9] Jędrzejewski J. M., *On Darboux Asymmetry*, Real Anal. Ex. 7 (1981-82), 172 – 176,

[10] Jędrzejewski J. M., Kowalczyk S., *Generalized Cluster Sets of Real Functions*, Tatra Mount. Math. Publ. 62 (2014), p. 1 – 7, DOI: 10.2478/tmmp-2014-0000.

[11] Jędrzejewski J. M., Kowalczyk S., *Cluster Sets and Topology*, Mathematica Slovaca, in print

[12] Jędrzejewski J., Wilczyński W., *On the family of sets of limit numbers*, Bull. Acad. Polon. Sci. sér. Sci. Math. Astronom. Phys. 18 (8) (1970), p. 453 – 460,

[13] Jędrzejewski J. M., Wilczyński W., *On the family of sets of* $\mathfrak{B}$*-limit numbers*, Zeszyty Nauk. Uniw. Łódzkiego 52 (1973),

[14] Kulbacka M., *Sur l'ensemble des points de l'asymétrie approximative*, Acta Sci. Math. (Szeged) 21 (1960), p. 90 – 95,

[15] Świątkowski T., *On some generalization of the notion of asymmetry of functions*, Coll. Math. 17 (1967), p. 77 – 91,

[16] Świątkowski T., *On a certain generalization of the notion of derivative*, Zeszyty Naukowe PŁ, 149, Mat. (1972), 89 – 103,

[17] Thomson B. S., *Real Functions*, Lectures Notes in Mathematics, No 1170, Springer Verlag, Berlin-Heidelberg-New York-Tokyo 1985,

[18] Wilczyński W., *On the Family of Sets of Qualitative Limit Numbers*, Rev. Roum. de Math. Pures et App., XVIII, (1973), p. 1297 – 1302,

[19] Wilczyński W., *On the Family of Sets of Approximate Limit Numbers*, Fund. Math. LXXV (1972), p. 169 – 174,

[20] Young W. H., *La symétrie de structure des fonctions des variables réelles* Bull. Sci. Math. 52 (2) (1928), p. 265 – 280,

Received: July 2018

Email address: `jacek.m.jedrzejewski@gmail.com`