Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XXIII (2018)
57–64
DOI http://dx.doi.org/10.16926/m.2018.23.05

SOME REMARKS ABOUT K-CONTINUITY OF K-SUPERQUADRATIC MULTIFUNCTIONS

KATARZYNA TROCZKA-PAWELEC

Abstract

Let $X = (X, +)$ be an arbitrary topological group. The set-valued function $F\colon X \to n(Y)$ is called K-superquadratic iff

$$F(x+y) + F(x-y) \subset 2F(x) + 2F(y) + K,$$

for all $x, y \in X$, where Y denotes a topological vector space and K is a cone.

In this paper the $K-$continuity problem of multifunctions of this kind will be considered with respect to $K-$boundedness. The case where $Y = \mathbb{R}^N$ will be considered separately.

1. Introduction

Let $X = (X, +)$ be an arbitrary topological group. A real-valued function f is called superquadratic, if it fulfils inequality

$$(1) \qquad 2f(x) + 2f(y) \leq f(x+y) + f(x-y), \qquad x, y \in X.$$

If the sign " $\leq$ " in (1) is replaced by " $\geq$ ", then f is called subquadratic. The continuity problem of functions of this kind was considered in [2]. This problem was also considered in the class of set-valued functions. By the set-valued functions we understand functions of the type $F\colon X \to 2^Y$, where X and Y are given sets. Throughout this paper set-valued functions will be always denoted by capital letters. A set-valued function F is called superquadratic if it satisfies inclusion

$$(2) \qquad 2F(x) + 2F(y) \subset F(x+y) + F(x-y), \qquad x, y \in X,$$

and subquadratic set-valued function, if it satisfies inclusion defined in this form

$$(3) \qquad F(x+y) + F(x-y) \subset 2F(x) + 2F(y), \qquad x, y \in X.$$

• *Katarzyna Troczka-Pawelec* — e-mail: k.troczka@ujd.edu.pl
Jan Długosz University in Częstochowa.

For single-valued real functions properties of subquadratic and superquadratic functions are quite analogous and, in view of the fact that if a function f is subquadratic, then the function $-f$ is superquadratic and conversely, it is not necessary to investigate functions of these two kinds individually. In the case of set-valued functions the situation is different. Even if properties of subquadratic and superquadratic set-valued functions are similar, we have to proved them separately. If the sign " $\subset$ " in the inclusions above is replaced by " $=$ ", then F is called quadratic set-valued function. The class of quadratic set-valued functions is an important subclass of the class of subquadratic and superquadratic set-valued functions. Quadratic set-valued functions have already extensive bibliography (see W. Smajdor [5], D. Henney [1] and K. Nikodem [4]). The continuity problem of subquadratic and superquadratic set-valued functions was considered in [6] and [7].

Adding a cone K in the space of values of a set-valued function F lets us consider a K-superquadratic set-valued function , that is solution of the inclusion

$$F(x+y)+F(x-y)\subset 2F(x)+2F(y)+K, \qquad x,y\in X. \tag{4}$$

The concept of K-superquadraticity is related to real-valued superquadratic functions. Note, in the case when F is a single-valued real function and $K=[0,\infty)$, we obtain the standard definition of superquadratic functionals (1). Similarly, if a set-valued function F satisfies the following inclusion

$$2F(x)+2F(y)\subset F(x+y)+F(x-y)+K, \qquad x,y\in X \tag{5}$$

then it is called K-subquadratic. The K-continuity problem of multifunction of this kind was considered in [9]. In this paper we will consider the K-continuity problem for K-superquadratic set-valued functions. Likewise as in functional analysis we can look for connections between K-boundedness and K-semicontinuity of set-valued functions of this kind.

Assuming $K=\{0\}$ in (4) and (5) we obtain the inclusions (2) and (3).

Let us start with the notations used in this paper. Let Y be a topological vector space. We consider the family $n(Y)$ of all non-empty subsets of as a topological space with the Hausdorff topology. In this topology the set

$$N_W(A) := \{B\in n(Y) : A\subset B+W, B\subset A+W\}$$

where W runs the base of neighbourhoods of zero in Y, form a base of neighbourhoods of a set $A\in n(Y)$. By $cc(Y)$ we denote the family of all compact and convex members of $n(Y)$. The term set-valued function will be abbreviated to the form s.v.f.

Now we present here some definitions for the sake of completeness. Recall that a set $K\subset Y$ is called a cone iff $K+K\subset K$ and $sK\subset K$ for all $s\in(0,\infty)$.

Definition 1. (cf. [3]) *A cone K in a topological vector space Y is said to be a normal cone iff there exists a base $\mathfrak{W}$ of zero in Y such that*

$$W = (W + K) \cap (W - K)$$

for all $W \in \mathfrak{W}$.

Definition 2. (cf. [3]) *An s.v.f. $F\colon X \to n(Y)$ is said to be K-upper semi-continuous (abbreviated $K-$u.s.c.) at $x_0 \in X$ iff for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that*

$$F(x) \subset F(x_0) + V + K$$

for every $x \in x_0 + U$.

Definition 3. (cf. [3]) *An s.v.f. $F\colon X \to n(Y)$ is said to be K-lower semi-continuous (abbreviated $K-$l.s.c.) at $x_0 \in X$ iff for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that*

$$F(x_0) \subset F(x) + V + K$$

for every $x \in x_0 + U$.

Definition 4. (cf. [3]) *An s.v.f. $F\colon X \to n(Y)$ is said to be $K-$continuous at $x_0 \in X$ iff it is both $K-$u.s.c. and $K-$l.s.c. at x_0. It is said to be $K-$continuous iff it is $K-$continuous at each point of X.*

Note that in the case where $K = \{0\}$ the $K-$continuity of F means its continuity with respect to the Hausdorff topology on $n(Y)$.

In the proof of the main theorems we will use some known lemmas (see Lemma 1.1, Lemma 1.3, Lemma 1.6 and Lemma 1.9 in [3]). The first lemma says that for a convex subset A of an arbitrary real vector space Y the equality $(s + t)A = sA + tA$ holds for every $s, t \geq 0$ or (s,t<0). The second lemma says that in a real vector space Y for two convex subsets A, B the set $A + B$ is also convex. The next lemma says that if $A \subset Y$ is a closed set and $B \subset Y$ is a compact set, where Y denotes a real topological vector space, then the set $A + B$ is closed. For any sets $A, B \subset Y$, where Y denotes the same space as above, the inclusion $\overline{A} + \overline{B} \subset \overline{A + B}$ holds and equality holds if and only if the set $\overline{A} + \overline{B}$ is closed.

Let us adopt the following three definitions which are natural extension of the concept of the boundedness for real-valued functions.

Definition 5. *An s.v. f. $F\colon X \to n(Y)$ is said to be $K-$lower bounded on a set $A \subset X$ iff there exists a bounded set $B \subset Y$ such that $F(x) \subset B + K$ for all $x \in A$. An s.v. f. $F\colon X \to n(Y)$ is said to be $K-$lower bounded at a point $x \in X$ iff there exists a neighbourhood U_x of zero in X such that F is $K-$lower bounded on a set $x + U_x$*

Definition 6. *An s.v. f.* $F\colon X \to n(Y)$ *is said to be* $K-$*upper bounded on a set* $A \subset X$ *iff there exists a bounded set* $B \subset Y$ *such that* $F(x) \subset B - K$ *for all* $x \in A$. *An s.v. f.* $F\colon X \to n(Y)$ *is said to be* $K-$*upper bounded at a point* $x \in X$ *iff there exists a neighbourhood* U_x *of zero in* X *such that* F *is* $K-$*upper bounded on a set* $x + U_x$

Definition 7. *An s.v. function* $F\colon X \to n(Y)$ *is said to be locally* K*-lower (upper) bounded in* X *if for every* $x \in X$ *there exists a neighbourhood* U_x *of zero in* X *such that* F *is* K*-lower (upper) bounded on a set* $x + U_x$. *It is said to be locally* K*-bounded in* X *if it is both locally* K*-lower and locally* K*-upper bounded in* X.

Definition 8. *We say that 2-divisible topological group* X *has the property* $(\frac{1}{2})$ *iff for every neighbourhood* V *of zero there exists a neighbourhood* W *of zero such that* $\frac{1}{2}W \subset W \subset V$.

For the $K-$superquadratic set-valued functions the following two theorems hold.

Theorem 1. (cf. [8]) *Let* X *be a* 2*-divisible topological group with property* $(\frac{1}{2})$, Y *locally convex topological real vector space and* $K \subset Y$ *a closed normal cone. If a* K*-superquadratic s.v.f.* $F\colon X \to cc(Y)$ *is* K*-u.s.c. at zero,* $F(0) = \{0\}$ *and locally* K*- bounded in* X, *then it is* K*-u.s.c. in* X.

Theorem 2. (cf. [10]) *Let* X *be a* 2*-divisible topological group,* Y *locally convex topological real vector space and* $K \subset Y$ *a closed normal cone. If a* K*-superquadratic s.v.f.* $F\colon X \to cc(Y)$ *is* K*-u.s.c. at zero,* $F(0) = \{0\}$ *and locally* K*- bounded in* X *then it is* K*-l.s.c. in* X.

Let us note, that Theorem 1 and Theorem 2, by Definition 4, yield directly the following main theorem for K-superquadratic multifunctions.

Theorem 3. *Let* X *be a* 2*-divisible topological group with property* $(\frac{1}{2})$, Y *locally convex topological real vector space and* $K \subset Y$ *a closed normal cone. If a* K*-superquadratic s.v.f.* $F\colon X \to cc(Y)$ *is* K*-u.s.c. at zero,* $F(0) = \{0\}$ *and locally* K*- bounded in* X, *then it is* K*-continuous in* X.

Let us introduce the following definitions.

Definition 9. *An s.v. f.* $F\colon X \to n(Y)$ *is said to be weakly* $K-$*lower bounded on a set* $A \subset X$ *iff there exists a bounded set* $B \subset Y$ *such that* $F(x) \bigcap (B + K) \neq \emptyset$ *for all* $x \in A$.

Definition 10. *An s.v. f.* $F\colon X \to n(Y)$ *is said to be weakly* $K-$*upper bounded on a set* $A \subset X$ *iff there exists a bounded set* $B \subset Y$ *such that* $F(x) \bigcap (B - K) \neq \emptyset$ *for all* $x \in A$.

Definition 11. *An s.v. f.* $F\colon X \to n(Y)$ *is said to be locally weakly* $K-$*upper bounded in* X *iff for every* $x \in X$ *there exists a neighbourhood* U_x *of zero in* X *such that* F *is* $K-$*upper bounded on a set* $x + U_x$.

Definition 12. *An s.v. f.* $F\colon X \to n(Y)$ *is said to be locally weakly* $K-$*lower bounded in* X *iff for every* $x \in X$ *there exists a neighbourhood* U_x *of zero in* X *such that* F *is* $K-$*lower bounded on a set* $x + U_x$.

Definition 13. *An s.v. f.* $F\colon X \to n(Y)$ *is said to be locally weakly* $K-$*bounded in* X *iff for every* $x \in X$ *there exists a neighbourhood* U_x *of zero in* X *such that* F *is weakly* $K-$*lower and weakly* $K-$*upper bounded on a set* $x + U_x$.

Clearly, if F is $K-$upper ($K-$lower) bounded on a set A, then it is weakly $K-$upper ($K-$lower) bounded on a set A. In the case of single-valued functions these definitions coincide.

For the $K-$superquadratic set-valued functions the following lemma holds.

Lemma 1. *Let* X *be a* $2-$*divisible topological group satisfying condition* $\left(\frac{1}{2}\right)$, Y *topological vector space and* $K \subset Y$ *a cone. Let* $F\colon X \to B(Y)$ *be a* $K-$*superquadratic s.v.f. , such that* $F(0) = \{0\}$ *and* $G\colon X \to n(Y)$ *be an s.v.f. with*

$$G(x) \subset F(x) + K \tag{6}$$

for all $x \in X$.

If F *is* $K-$*lower bounded at zero and* G *is locally weakly* K*-upper bounded in* X *, then* F *is locally* $K-$*lower bounded in* X.

Proof. Let $x \in X$. There exist a bounded set $B_1 \subset Y$ and a symmetric neighbourhood U_1 of zero in X such that

$$G(x-t) \cap (B_1 - K) \neq \emptyset, \quad t \in U_1,$$

which implies that that for all $t \in U_1$ there exists $a \in G(x-t)$ and $a \in (B_1 - K)$. Consequently, we get

$$0 = a - a \in G(x-t) - B_1 + K \tag{7}$$

for all $t \in U_1$. Since F is $K-$lower bounded at zero, there exist a symmetric neighbourhood U_2 of zero in X and a bounded set $B_2 \subset Y$ such that

$$F(t) \subset B_2 + K, \quad t \in U_2. \tag{8}$$

Let $\widetilde{U}$ be a symmetric neighbourhood of zero in X with $\frac{1}{2}\widetilde{U} \subset \widetilde{U} \subset U_1 \cap U_2$. Let $t \in \frac{1}{2}\widetilde{U}$. Using (6), (7) i (8), we obtain

$$F(x+t)+0 \subset F(x+t)+G(x-t)-B_1+K \subset F(x+t)+F(x-t)-B_1+K \subset$$

$$\subset 2F(x) + 2F(t) - B_1 + K \subset 2F(x) + 2B_2 - B_1 + K.$$

Define $\widetilde{B} := 2F(x) + 2B_2 - B_1$. Since $F(x)$ is a bounded set, then the set $\widetilde{B}$ is also bounded as the sum of bounded sets. Therefore

$$F(x+t) \subset \widetilde{B} + K, \quad t \in \frac{1}{2}\widetilde{U},$$

which means that F is locally $K-$lower bounded in X. □

In the case of K-superquadratic multifunctions we require Y space to be locally bounded topological vector space. Then the following theorem holds.

Theorem 4. *Let X be a 2-divisible topological group with property $(\frac{1}{2})$, Y locally convex topological vector space and $K \subset Y$ a closed normal cone. If a K-superquadratic s.v.f. $F\colon X \to cc(Y)$ is K-u.s.c. at zero, $F(0) = \{0\}$ and locally K- upper bounded in X, then it is K-continous in X.*

Proof. Let W be a bounded neighbourhood of zero in Y. Since F is K-u.s.c. at zero and $F(0) = \{0\}$, then there exists a neighbourhood U of zero in X such that

$$F(t) \subset V + K$$

for all $t \in U$, which means that F is K-lower bounded at zero. The condition of locally K-upper boundedness in X implies F is locally K-weakly upper bounded in X. By Lemma 1 ($G = F$) F is locally K-lower bounded in X. Consequently by Theorem 3 F is $K-$continuous at each point of X. □

2. The case $n(\mathbb{R}^N)$

Now we consider the case where the space of values is $n(\mathbb{R}^N)$. In our next proof, we will use known following lemma.

Lemma 2. (cf. [9]) *Let Y be a topological vector space and K be a cone in Y. Let A, B, C be non-empty subsets of Y such that $A + C \subset B + C + K$. If B is convex and C is bounded then $A \subset \overline{B + K}$.*

For the $K-$superquadratic set-valued functions the following lemma holds.

Lemma 3. *Let X be a topological group and K a closed cone in $\mathbb{R}^N$. Let $F\colon X \to cc(\mathbb{R}^N)$ be a $K-$superquadratic s.v.f. with $F(0) = \{0\}$. If F is K-l.s.c. at some point $x_0 \in X$, then it is K-l.s.c. at zero.*

Proof. Let W be a neighbourhood of zero in Y.There exists a convex neighbourhood V of zero in Y such that the set $\overline{V}$ is compact with $3\overline{V} \subset W$. Since F is K-l.s.c. at $x_0 \in X$ then there exists a symmetric neighbourhood U of zero in X such that

$$F(x_0) \subset F(x_0 + t) + V + K, \tag{9}$$

$$F(x_0) \subset F(x_0 - t) + V + K, \tag{10}$$

for all $t \in U$.
Let $t \in U$. By convexity of the set $F(x_0)$ and by (9) i (10), we obtain

$$2F(x_0) \subset F(x_0 + t) + F(x_0 - t) + 2V + K \subset 2F(x_0) + 2F(t) + 2V + K.$$

Then

$$F(x_0) + \{0\} \subset F(x_o) + F(t) + \overline{V} + K \quad t \in U. \tag{11}$$

Since $F(x_0)$ is a bounded set and $F(t) + \overline{V}$ is a convex set, then by Lemma 2, we have

$$\{0\} \subset \overline{\overline{V} + F(t) + K}$$

for all $t \in U$. Note that the set $\overline{V} + F(t) + K$ is closed as a sum of compact and closed set. Consequently, by condition $F(0) = \{0\}$, we obtain

$$F(0) \subset \overline{V} + F(t) + K \subset F(t) + W + K$$

for all $t \in U$, which means F is K-l.s.c. at zero. □

This article is the introduction to the discussion on the K-continuity problem for K-superquadratic set-valued functions. In the theory of K-subquadratic and K-superquadratic set-valued functions an important role is played by theorems giving possibly weak conditions under which such multifunctions are K-continuous.

References

[1] Henney D., *Quadratic set-valued functions.* Ark. Mat. 4, 1962

[2] Kominek Z.,Troczka-Pawelec K., *Continuity of real valued subquadratic functions.* Commentationes Mathematicae, Vol. 51, No. 1 (2011), 71-75

[3] Nikodem K., *K-convex and K-concave set-valued functions.* Publ. Math. Debrecen 30, 1983

[4] Nikodem K., *On quadratic set-valued functions.* Zeszyty Naukowe Politechniki Łódzkiej, nr 559, Łódź 1989

[5] Smajdor W., *Subadditive and subquadratic set-valued functions.* Prace Naukowe Uniwersytetu Ślaskiego w Katowicach, nr 889, Katowice 1987

[6] Troczka-Pawelec K. *Continuity of superquadrqtic set-valued functions.* Scientific Issues Jan Długosz University in Częstochowa, Mathematics XVII, 2012

[7] Troczka-Pawelec K. *Continuity of subquadrqtic set-valued functions.* Demonstratio Mathematica, vol. XLV, no 4, 2012, 939-946

[8] Troczka-Pawelec K. *K-continuity problem of K-superquadratic set-valued functions.*Scientific Issues Jan Długosz University in Częstochowa, Mathematics XIX, 2014

[9] Troczka-Pawelec K. *K-continuity of K-subquadratic set-valued functions.*Scientific Issues Jan Długosz University in Częstochowa, Mathematics XIX, 2014

[10] Troczka-Pawelec K. *On K-superquadratic set-valued functions.*Journal of Applied Mathematics and Computational Mechanics, Volume 14, Issue 1, 2015

Received: November 2018

Katarzyna Troczka–Pawelec
Jan Długosz University in Częstochowa,
Institute of Mathematics and Computer Science,
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
Email address: k.troczka@ujd.edu.pl