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ANALOGICAL PROBLEMS IN THE PLANE AND SPACE

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Abstract. It is possible to form analogical problems concerning 2D objects in the space for 3D objects. As for students profitable is to learn how to formulate and solve this 3D problems, especially those ones whose solution requests non-trivial modification of the solution of the original problem.

1. Formulation and solving analogical problems

When students learn to formulate to a given problem the analogical one in the space of another dimension, they create the spectrum of analogical conceptions in these spaces. They treat with not only the spectrum of basic objects *point – line – plane – space* but also another spectrum, for example *half line, half plane, semi-space* and subsequently then *segment* (intersection of two half lines), *triangle* (intersection of three half planes), *tetrahedron* (intersection of four semi-spaces), or *segment, square* and *cube* (like sets of points, whose coordinates form one-dimensional, two-dimensional or three-dimensional interval), or *pair of points, circle* and *sphere* (like set points that have given distance from a fixed point). Solving analogical problem in higher dimension can sometimes use the benefit from the same intellectual draft like the original problem, another time is solving of the analogical problem in new dimension non-trivial, however. The series of these problems is known, for instance some simple numerical problems. If we have for example in the plane the square with the side of the length n cm and if we both ćolorąll its sides and cut it to squares with sides of 1 cm, we can then calculate, how much of them has colored two sides or one or no side. Analogical task we can formulate in the space for cube, which has colored all walls.

2. Problems related to triangle and tetrahedron

Investigation of medians and median points can culminate by symbolic equations for median points of segment, triangle and tetrahedron in the form

$$T = \tfrac{A+B}{2}, \text{ respectively } T = \tfrac{A+B+C}{3}, \text{ respectively } T = \tfrac{A+B+C+D}{4}.$$

The construction of a triangle circumscribed circle by means of bisectors of sides is known. Then students analogously search sets of all points in the space, that are equidistant from two, three or four vertexes of tetrahedron. They find out also interesting piece of knowledge, that the join of centre of the circumsphere with centres of circles circumscribed to walls are perpendicular to this walls. Similarly: the construction of a triangle inscribed circle by means of bisectors of sides is known, then students analogously search sets of all points in space, that are equidistant from two, three or four planes, determined by sides of tetrahedron. Further, they can for example derive relation between radius of r triangle inscribed circle and its area S and circumference o

$$S = S_a + S_b + S_c = \frac{a \cdot r}{2} + \frac{b \cdot r}{2} + \frac{c \cdot r}{2} = \frac{o \cdot r}{2} \Rightarrow r = \frac{2 \cdot S}{o},$$

and then analogously derive analogical identity for tetrahedron: $r = \frac{3 \cdot V}{S}$. By the investigation of heights of a tetrahedron students will find out that in case general the heights can be skew. All of four heights cut in one point if and only if when pair of opposite edges of tetrahedron lie in respectively perpendicular lines.

3. Generalization of Pascal's triangle

The numeral configuration, which is called Pascal's triangle, can be defined in this way, that numbers represent the number of ways, which lead from the origin $[0; 0]$ to a given point $[x; y]$. Students reveal through the use of extension ways by a unit the recurrent formulation

$$C(0, y) = C(x, 0) = 1 \wedge C(x, y+1) + C(x+1, y) = C(x+1, y+1),$$

and subsequently derive other characteristics of this numbers, namely that they determine the number of subsets, appearing in the binomial theorem and have explicit formula in the form:

$$C(x, y) = \frac{(x+y)!}{x! \cdot y!}.$$

How can generalization of this conception look like, if we proceed to three-dimensional space? We will count the ways that lead from the origin $[0; 0; 0]$ to a given point $[x; y; z]$ again.

In co-ordinal planes three Pascal's triangles originate and every number inside the octant originates as sum of three earlier calculated numbers. Then recurrent formula for all the numbers has then the form:

$$C(x,y+1,z+1)+C(x+1,y,z+1)+C(x+1,y+1,z)=C(x+1,y+1,z+1),$$

whereas initial values are given as follows:

$$C(x,y,0)=\frac{(x+y)!}{x!\cdot y!}\wedge C(x,0,z)=\frac{(x+z)!}{x!\cdot z!}\wedge C(0,y,z)=\frac{(y+z)!}{y!\cdot z!}.$$

Another meanings of this numbers are numbers of ordered pairs of subsets, occuring in terms like $(p+h+d)^n$ and having explicit formulation in the form: $C(x,y,z)=\frac{(x+y+z)!}{x!\cdot y!\cdot z!}$. This "geometrical"method of generalization follows the line of permutations with repetition, so then combination without repetition are special case of permutations with repetition.

4. Filling plane with circles and space with spheres

The question "Which part of the plane can we fill with suitable organized consistent circles?ćan bring us to nice problems. It's evident, that quotient can be quantified as quotient of the area of one circle and appropriate ćell", which originates from it by "inflationóf all circles at once.

Understandably the shape of the cell depends on general circles layout, in the first case the cell is quadratic, in the second case it's hexagonal. Calculations aren't difficult:

$$p=\frac{\pi r^2}{S_B}=\frac{\pi r^2}{4r^2}=\frac{\pi}{4}\approx 0,7854$$

$$p=\frac{\pi r^2}{S_B}=\frac{\pi r^2}{2\sqrt{3}\cdot r^2}=\frac{\pi}{2\sqrt{3}}\approx 0,9069.$$

How to formulate and solve analogical problem in space? If spheres concern by their "polesąnd by four points on ȩquator", appropriate cubic cell and ratio will be:

$$p=\frac{\frac{4}{3}\pi r^3}{V_B}=\frac{\frac{4}{3}\pi r^3}{8r^3}=\frac{\pi}{6}\approx 0,5236.$$

More effective spheres' ordering can be done in two ways, when upper spheres lie either on four or on three spheres of the underlay.

It's very interesting, that for both of this ordering the appropriate cell is the same (so-called rhombic dodecahedron) and wanted quotient has value

$$p=\frac{\frac{4}{3}\pi r^3}{V_B}=\frac{\frac{4}{3}\pi r^3}{4\sqrt{2}r^3}=\frac{\pi}{3\sqrt{2}}\approx 0,7405.$$

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