Jan Długosz University of Częstochowa
Scientific Issues, Mathematics XII, Częstochowa 2007

ABOUT SOME DIFFICULTIES IN DOING PROOFS ENCOUNTERED BY THE STUDENTS FROM THE FIRST YEAR OF MATHEMATICS

Justyna Hawro

Institute of Mathematics, University of Rzeszów
al. Rejtana 16C, 35-959 Rzeszów, Poland
e-mail: jhawro@poczta.onet.pl

Abstract. This paper contains some results of diagnostic research on the difficulties that students who begin studies at the tertiary level encounter when doing proofs from a section devoted to applying definitions in proofs. The considerations concern the issues connected with understanding of the role of definition and understanding of the texts of definitions by students. In these considerations the examples of solutions given by students to two diagnostic tasks applied in the research are used.

1. Introduction

The movement from elementary to advanced mathematical thinking, as Tall (1992) describes, involves a transition from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions (p.20). Students frequently struggle in making this transition what is especially seen as they tackle the first year of university. As a university teacher I often observe that my students experience many difficulties in their first encounters with definitions, theorems and proofs in undergraduate mathematics courses. The biggest problems seem to be connected with analysis and construction of mathematical proofs. It motivated me to undertake more detailed research in order to answer the question: *What didactic interventions and instructions could be introduced during classes to help students both to overcome their difficulties with proving and to develop their skills in that area?* There is also another justification for the necessity of conducting this kind of studies – there has been relatively little research on the teaching and learning mathematical proof having dealt with university students (Moore, 1994).

Before starting to plan activities, instructions aimed at eliminating students' difficulties with proofs I deemed as necessary to conduct more detailed studies on what are these difficulties in analysis and construction proofs. I would like to present some their results.

2. The study

My diagnostic research concerning the students' difficulties with proofs was conducted during the course "Introduction to Mathematics", which included topics such as logic and proof techniques, set theory, relations and functions, in the winter semester of the 2006/2007 academic year. The research group consisted of 57 students from the first year of mathematics. The research was based on the analysis of solutions of different tasks requiring the analysis of the texts of written proofs or independent construction of proofs, which were presented by students in their works. The required proofs were short, uncomplicated proofs in which inferences were based largely on definitions or previously accepted theorems.

Analyzing students' difficulties with proving I considered the issues connected with: (a) understanding the concept of a proof, its role and other methodological aspects, (b) concept understanding, (c) theorem understanding, (d) proof methods, (e) mathematical language and notation, (f) mathematical logic. In this study I will present some results of the diagnostic research from the section connected with the usage of definition of a concept in a proof. Discussing them I will refer to the students' solutions of the following tasks:

Task 1:
Read the definition and do the exercises:

Definition: Function f is increasing in its domain, if

$$\forall_{x_1,x_2\in D_f} (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)).$$

1) Write this definition by words.
2) Using the definition justify that function $f(x)$=$2x+3$ is increasing in its domain.
3) Is function $f(x)$=tgx increasing in its domain? Justify your answer.

Task 2:
Read the definition and do the exercises:

Definition: Set $X \subset R$ is bounded below, if $\exists_{m\in R} \forall_{x\in X}\ x \geq m$.

1) Explain by words when the set is bounded below.
2) Give an example of bounded below set. Justify your choice.
3) Is set $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\}$ bounded below? Justify your answer.

The tasks required among others truth verification or justification of certain statements about the concept, whose definition was given. However, the definition from the Task 1 was familiar to the students from secondary school, the second one was met for the first time. Application of different in this sense definitions was done deliberately – I was curious if and how this difference would influence the way of solving both tasks by the students.

3. Theoretical background

Tall and Vinner (1981) have distinguished between the concept image and the concept definition. The former refers to the *"total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes"(p. 152) and it is built up by individual through different kinds of experiences with the concept. The latter refers to a formal definition which determines the meaning of the concept. Despite the fact, that in the process of solving tasks (also these which require analyzing and constructing proofs) both concept image and concept definition play the crucial role, the correct deduction demands orderly use of the definition. What is more, definitions provide the language – the words and symbols – for writing a proof, suggest the sequence of individual steps and provide the justification for each step in a proof (Moore, 1994). In this connection analyzing the issue if and how the students apply the definition in a proof I was investigating: 1) their understanding of the role of mathematical definition, 2) their understanding of the texts of definitions.*

4. Research results

a) Understanding of the role of mathematical definition

It can be said, that a student understands the role of mathematical definition if he/she consciously and orderly uses definition in reasoning. Considering the issue of understanding this formal discipline by my students I was analyzing their answers to points 2) and 3) in the Task 1.

In point 2) it was clearly said, that the justification should be formulated on the basis of the definition. Two thirds from 27 students, who were solving this task, conformed to this instruction, despite some of them made a mistake – they were checking the veracity of defining condition for the concrete numbers. A few students were applying other arguments using their intuition about the increasing function; they stated e.g. that given linear function is increasing in its domain because *"the raise of arguments is accompanied with the raise of function"*.

In point 3) only one student formulating justification referred to the definition – to show that the function $f(x) = \operatorname{tg} x$ is not raising in its domain he/she was constructing the counter-example. The rest of the students did not undertake solving the task or used different arguments in their answers. The latter ones most often referred to the graph of the function while formulating the justification, despite the fact if they considered the $f(x) = \operatorname{tg} x$ function to be increasing or not. They stated: *"It is not raising, it is visible in the graph"* or *"The function is not raising in its domain because while observing the graph of this function we can see how its particular parts are raising"*. The authors of these statements did not draw the graphs but visualized the function in their mind. The evoked image was a sufficient argument for them; they did not feel the need to verify their answer on the basis of the definition.

From these considerations it can be concluded that there was a lot of hesitation while applying the definition by the students. Many of them used the definition only in one point, the one in which they were asked to do so. In the second point they were referring rather to the concept image. Perhaps it was the result of recognizing such a solution as more simple and/or the lack of a remark about using the definition obligatory in the instruction. But if we want to speak about the methodological understanding of definition as about using it consciously, we would expect trials of referring to definition not only in the case of a clear instruction but also in the situation where it is not clearly stated. Thus it can be assumed that:

Conjecture: A lot of students do not understand the role of definition in proof.

b) Understanding of the texts of definitions

The facts that student is able to: (a) differentiate the name from the defining condition and knows that the defining condition determines the meaning of the name, (b) interpret the definition, express it in a different form, (c) construct examples of referents, (d) use the definition in solving of simple tasks, evidence about the understanding of the text of the definition (Krygowska, 1977). Having these kinds of competencies by my students I was studying with the usage of above quoted tasks.

Perfunctory analysis of solutions of these two tasks showed that more students were trying to answer the questions concerning the concept of increasing function than the concept of bounded below set. However even in the case of the first of these concepts, which was known and applied previously by students, there were answers showing difficulties in understanding the defining condition. These difficulties were revealed e.g. in point 1), where the students were asked to write the definition using words. Here are some examples of the answers:

"Function f is increasing in its domain, if for x_1 and x_2 which belong to the domain of function, x_1 is smaller than x_2 when and only when the function value at argument x_1 is smaller than the value at argument x_2",
"Function f is increasing in its domain, if for each x_1 and x_2 there is such an x_1 smaller than x_2, that $f(x_1)$ is smaller than $f(x_2)$".
These statements show the lack of knowledge and understanding of logical symbols present in the defining condition, what consequently led to incorrect interpretation of the logical structure of this condition. It can be concluded, that misunderstandings of mathematical language caused difficulties with understanding of the definition. However, even correct interpreting of logical symbols did not mean that students understood the concepts which were behind these symbols. It can be doubted, if they understood for example the concept of a general quantifier as 11 students made a verification for a few chosen numbers while justifying in point 2) that given linear function is increasing in its domain.

In the Task 2, where was an unknown definition of a bounded below set, as many as the half of 30 students, who were solving it, did not try to answer any question. A few students giving the reason stated that: *"I don't know this definition. It's unclear"*. These students behaved passively towards the text of the new definition. They did not take any action towards understanding it by themselves – they did not try to interpret the text of the definition, express it in a different way, look for referents of the concept.
Students who tried to analyze the definition individually not always were able to read correctly defining condition given in a symbolic way. In point 1) students wrote e.g. *"The set X is bounded below if for each x from the set X there is m from the set R that $x \geq m$"*. Interpreting definition condition the author of this comment changed the order of quantifiers revealing the lack of understanding that the order of the quantifiers in notation is important. It indicates some deficiencies in his/her logical education.
There was also a group of the students who in the process of solving the task used some associations connected with colloquial sense of the name of the concept without analyzing the text of the defining condition; for example the student in point 2) gave the set {1,2,3,4,...} as an example of a referent, and wrote in his justification: *"because it has a definite beginning"*. He/she acted as if he/she thought that set is bounded below if it starts from the concrete number. In this connection there are some doubts whether he/she understood that the defining condition determines the meaning of the name and that this meaning should be read from it.

From above examples it follows that:

Conjecture: Students experience a lot of difficulties connected with understanding a definition formulated in formal and symbolic language, regardless it is new for them or not.

5. Conclusions

The considerations presented in this paper show that constructing even simple proofs and justifications on the basis of definition can be difficult for students. It is partly because of some misunderstandings connected with methodological aspects, mainly lack of understanding of the role of mathematical definition. When doing proofs the students did not use definition of concept consciously but they referred to their intuition or mental pictures connected with the concept. What is more, intuitive visual arguments were so convincing that the students did not feel the need to verify their correctness on the grounds of the definition.

Another case are difficulties connected with understanding of the text of definition, especially when it is stated in formal, symbolic language. They can result from the deficiencies connected with the knowledge of the field of mathematical logic. Students encountering a new definition did not try to analyze and understand it. They were not able to create individually the mental picture of the concept given by the definition. The lack of intuitive understanding resulted further to the fact that some students did not try to answer the questions concerning the concept.

This paper contains only the fragment of my research on students' competencies in proving. The analysis of collected materials enabled me to indicate a lot of different difficulties in this area and define some relations between them, what has a great meaning during planning didactic activities aimed at eliminating these difficulties.

References

[1] Z. Krygowska. Zarys dydaktyki matematyki, cz.2. WSiP, Warszawa, 1977.

[2] R.C. Moore. Making the transition to formal proof. Educational Studies in Mathematics, **27**, 249–260, 1994.

[3] D. Tall, S. Vinner. Concept image and concept definition in mathematics with particular reference to limits and continuity. Educational Studies in Mathematics, **12**, 151–169, 1981.

[4] D. Tall. The psychology of advanced mathematical thinking. In D. Tall. Advanced Mathematical Thinking. Kluwer, Dordrecht, pp. 3–21, 1992.