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EXAMPLES OF INVESTIGATIONS FOR BEGINNERS

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Abstract. There are examples of several types of investigations available for beginners: 1. search for patterns, 2. iterating a certain procedure and analysing the results, 3. looking for exceptions, or special cases in a pattern, 4. generalizing given problem.

Investigating a pattern of numbers or shapes can lead to discoveries and may raise a number of challenging questions. Investigative work is a suitable introduction to the art of problem solving. There are examples of several types of investigations available for beginners:

1. search for patterns;
2. iterating a certain procedure and analysing the results;
3. looking for exceptions, or special cases in a pattern;
4. generalizing given problem.

We want to demonstrate these types of investigations with the help of examples.

Example 1: Investigate sums of the first odd natural numbers.

Solution:

Systematic experimentation:

$1,\ 1+3=4,\ 1+3+5=9,\ 1+3+5+7=16,\ 1+\ldots+9=25$

Our sums are square numbers.

Conjecture: $(\forall n \in \mathbb{N})\ 1+3+5+\ldots+e_n = k^2$, where $k \in \mathbb{N}$ and e_n is the nth odd number.

Other systematic experimentation:

$1+3=2^2,\ 1+3+5=3^2,\ 1+3+5+7=4^2$, and also $1=1^2$

New conjecture: $(\forall n \in \mathbb{N})\ 1+3+5+\ldots+e_n = n^2$

Proof can be done by mathematical induction. After that we have

Mathematical Theorem: $(\forall n \in \mathbb{N})\ 1+3+5+\ldots+e_n = n^2$

If we use geometrical way of investigation, we get the following figure (see Fig. 1)

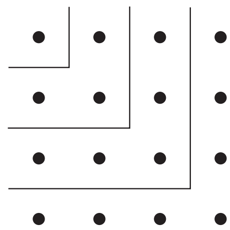

Fig. 1:

Example 2: Let T_0 be an equilateral triangle of unit area. Divide T_0 into 4 equilateral triangles T_1 by joining the midpoints of the sides of T_0. Now remove the central triangle. Treat the remaining 3 triangles in the same way and repeat the process. It means iterace – n-times. Find out:

a) What is the sum S_n of the removed triangles after n-step?

b) What happens to S_n as n tends to infinity?

Solution: Iteration of our procedure leads to

1. step ..... 1 removed triangle T_1 of area $1/4$
2. step ..... 3 removed triangles T_2 of area $(1/4)^2$
3. step ..... 3^2 removed triangles T_3 of area $(1/4)^3$
4. step ..... 3^3 removed triangles T_4 of area $(1/4)^4$

Generalization:

At the nth step

$$3^{n-1} \text{ removed triangles } T_n \text{ of area } (1/4)^n$$

a) $S_n = 1 \cdot \frac{1}{4} + 3 \cdot \left(\frac{1}{4}\right)^2 + 3^2 \cdot \left(\frac{1}{4}\right)^3 + \ldots + 3^{n-1} \cdot \left(\frac{1}{4}\right)^n = \frac{1}{4} \cdot \frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}} = 1 - \left(\frac{3}{4}\right)^n$

b) If n tends to infinity, $\left(\frac{3}{4}\right)^n$ tends to 0 and S_n tends to 1.

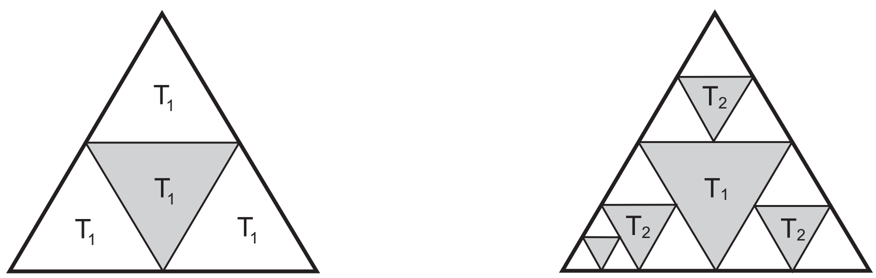

Fig. 2:

Remark: Consideration in b) is very good preparation for limits of sequences and functions.

Problem 3: Figure 3 shows three mirrors of length l forming a triangle ABC. A light source is placed at a point S of AB, at a distance d from A. A light ray, emerging from S at an angle of 60° with SB, gets reflected from the sides of triangle ABC until it returns to S. Find the length of the light ray's path in terms of l.

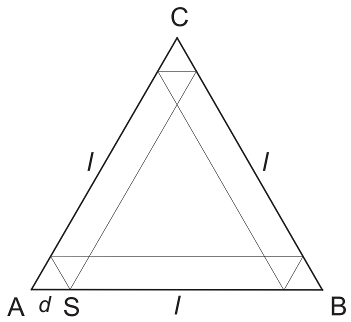

Fig. 3:

Solution: It is not difficult to prove that the length of the light ray's is $3\,l$. This answer does not involve the distance d of S from A. So it seems that the length of the light ray's path is the same for all positions of S on AB. But there is one exception. If S is at the midpoint of AB, the length of the path is only $3/2\,l$.

Problem 4 (Pappus problem)**:** ABC is an arbitrary triangle and ACDE and CBFG are arbitrary parallelograms constructed on two of the sides. Lines ED and FG meet in H.

Construct a parallelogram ALKB on the third side AB such that AL and KB are equal to CH and parallel to it (see Fig. 4).
Prove that the area of ALKB is the sum of the areas of ACDE and BFGC.

Solution: On EH construct the point O such that AO is parallel to CH, and on FH construct P such that BP is parallel to CH. Line HC meets AB in N and LK in M (see Fig. 5).
ACHO is a parallelogam having the same base AC and the same corresponding height as the parallelogram ACDE. ACHO and ALMN are also parallelograms with equal bases (CH = NM) and equal corresponding heights. It means

$$\text{Area ACDE} = \text{Area ACHO} = \text{Area ALMN}$$

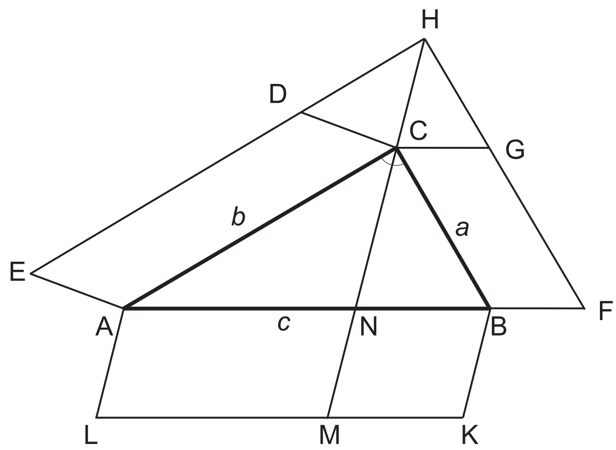

Fig. 4:

Similarly: Area BFGC = Area BPHC = Area NMKB
This implies that: Area ALKB = Area ACDE + Area BFGC

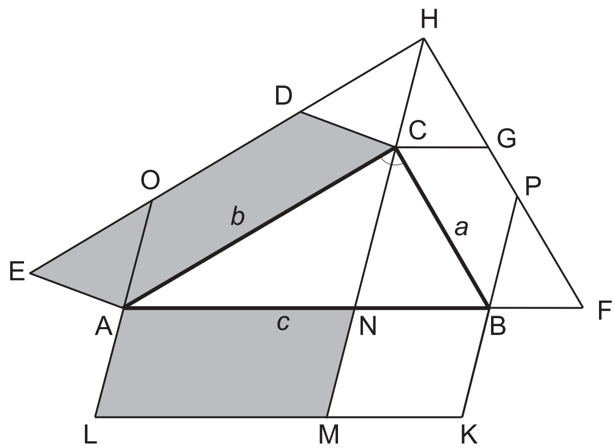

Fig. 5:

Remark: Problem 4 is one possible generalization of Pythagoras Theorem.

References

[1] J. Kopka. Hrozny problémů ve školské matematice. Ústí nad Labem, UJEP, 1999.
[2] J. Kopka. Zkoumaní ve školské matematice. Ružomberok, Katolícka univerzita v Ružomberku, 2006.