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HOW MANY AND WHAT KIND OF STOOLS CAN BE BUILT BY A CARPENTER? – MEANING HOW PEDAGOGICS STUDENTS SOLVED CERTAIN PROBLEMS

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Abstract Uncommon mathematical problems play an important role in childrens' education in mathematics. These exercises inspire creativity in children and help them develop a sense of divergent thinking. Pedagogics students, as future teachers, must not only recognize the value of such mathematical problems, but must also be able to solve them. This article is a presentation of the skills of the students in this regard.

1. Introduction

An important part of integrated education is to form a person who is creative and has the mental disposition to enable him to come up with many ideas. His traits should include fluency, flexibility and original thinking, as well as the ability to analyze, synthesize, generalize, compare, classify, deduct, abstract, define, step-by-step problem solve and so forth. We care about creating a pupil who is open and adaptable. We can reach this goal by encouraging solving text mathematical problems with many solutions.

What makes these problems so valuable is that there are a very specific problematical situation and the ability to solve them will be of practical use, the solutions becoming a pattern (paradigm) that will come in handy in any difficult situation (problematical).

Solving problems (as long as it is self-reliant) can be a splendid learning experience, teaching both creativity and criticism, work-organizing, searching for effective problem solving strategies, the ability to code and present solutions, self-control of progress and elimination of mistakes.

2. Research

How pupils deal with problem solving and if they become creative in the progress depends greatly on the teacher's style of teaching. The success of his students is dependant on the teacher's own knowledge and pedagogic skills. The teacher should be characterized by a creative attitude and should give his students as many occasions as possible to solve open and atypical problems. If the teacher has trouble with solving certain problems, he definitively should not assign his students to solve them.

I have decided to assess the competency of pedagogics teachers of early education in the regard of solving atypical problems.

In January of 2007, I conducted tests among 163 teachers, who had received bachelor degrees (from different institutions) in preschool and early school pedagogics and had started master degrees in preschool and early school pedagogics in The Pedagogics Academy in Cracow.

Each one of them was asked to solve the following problem in writing:

Problem: A carpenter builds stools with 3 and 4 legs. He has 30 legs for the stools. How many and what kind of stools will he be able to build?

The mathematical model of this problem is a diophantine equation $3x + 4y = 30$, where x is the number of three-legged stools and y is the number of four-legged stools. The equation has three solutions: $(10; 0)$, $(6; 3)$, and $(2; 6)$.

It is worth to discuss the amount of solutions taking into account the following questions:

[1] Is the answer $(10; 0)$ correct? It does not include any four-legged stools.

[2] Must the carpenter make use of all of the legs? The content of the problem does not mention that. Therefore, it is worth to take into consideration the solutions, which will involve a smaller than 30 number of legs. In this situation the mathematical model of this problem is the equation $3x+4y = n$, for $n \leq 30$, which, together with the $(x; y) = (0; 0)$, has 48 solutions. In this situation the number of unused legs is any whole number from 0 to 27 or the number 30. There cannot be 29 nor 28 legs, because a stool cannot be built with neither 1 nor 2 legs. The following table shows all solutions for $n > 25$. The solutions have been numbered (the first row) with numbers from 1 to 13, which will be discussed in the later part of the article.

nr solution	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	etc.
x	10	6	2	7	3	8	4	0	9	5	1	6	2	etc.
y	6	3	6	2	5	1	4	7	0	3	6	2	5	etc.
n	30			29		28			27			26		etc.

[3] If not all the legs are to be used, does it make any sense to leave more than 2 legs? If there are 3 legs left, they can be used to build another thee-legged stool and none will be left. If there are 4 legs, a four-legged stool can be built and a three-legged one, leaving only 1 leg. When 5 are left … and so forth.

[4] Can a carpenter build stools having only legs? The content of the problem does not specify the presence of other indispensable in the construction of stools elements such as, for example, the seat.

Let us take notice that, depending on the answers to the former questions, the number of solutions changes. It is therefore difficult to state if the student gave all the answers or not. It is expected that the teachers understand that to solve this problem means to discuss all of the possible variants and find an answers in each of these specific cases.

None of the tested students presented all of the answers for n from 0 to 30. The highest number of solutions was 8. They were all of the cases for $n = 28, 29, 30$, which are the answers marked from 1 to 8. The students were evidently looking for the largest number of stools. From every 3 or 4 legs they "made" more stools.

Analyzing the work of teacher-students I took notice not only of the number of correct solutions, but also of the following a) how the tested students understood the content of the problem, b) what the term "solving the problem" meant to them, c) what strategies were used to find solutions and d) how they presented their results.

3. Test results

3.1. Understanding the content of the problem

All tested students understood the content of the problem very well and the question therein. Some expressed doubt if the problem is properly formulated because:

- *It was not explained if all legs must be used.*
- *There was no information instructing that there must be as many stools as possible.*

- *No stools can be made without seats!*
- *They thought the problem should be formulated more precisely.*

3.2. What does "solving the problem" mean?

The following table illustrates a summary of the answers taking into account the number of solutions and the type of solutions (numbered solutions). In the table the solutions with the help of illustrations are separated for ones given simply arithmetically (without illustrations). The analysis of the facts compiled in the table allow one to suspect how the tested students understood "solving the problem". This understanding may vary greatly.

<table>
<tr><th colspan="2">Type of solution</th><th>Numbered solutions</th><th>With illustration</th><th>Without illustration</th><th>Together</th><th>Summary</th></tr>
<tr><td colspan="2">Trivial: dividing $30:3=10$ & $30:4=7$ s 2</td><td>1 and 8</td><td>0</td><td>5</td><td>5</td><td>5</td></tr>
<tr><td colspan="2" rowspan="3">Only one possibility</td><td>1</td><td>0</td><td>1</td><td>1</td><td rowspan="3">8</td></tr>
<tr><td>2</td><td>3</td><td>0</td><td>3</td></tr>
<tr><td>3</td><td>4</td><td>0</td><td>4</td></tr>
<tr><td rowspan="4">Two possibilities</td><td rowspan="3">One trivial ($30:3=10$) & one „special"</td><td>1 & one with spare legs</td><td>5</td><td>1</td><td rowspan="3">25</td><td rowspan="4">36</td></tr>
<tr><td>1 & 2</td><td>4</td><td>0</td></tr>
<tr><td>1 & 3</td><td>15</td><td>0</td></tr>
<tr><td>$n=30$ & $x\neq 0$ i $y\neq 0$</td><td>2 & 3</td><td>11</td><td>0</td><td>11</td></tr>
<tr><td colspan="2">Three possibilities — all solutions for $n=30$</td><td>1 & 2 & 3</td><td>35</td><td>9</td><td>44</td><td>44</td></tr>
<tr><td colspan="2">Four possibilities: all for $n=30$ & $30:4=7$ spare 2</td><td>1 & 2 & 3 & 8</td><td>14</td><td>2</td><td>16</td><td>16</td></tr>
<tr><td colspan="2">More than 4 solutions</td><td>1 & 2 & 3 & one with spare legs</td><td>7</td><td>2</td><td>9</td><td rowspan="2">11</td></tr>
<tr><td colspan="2">Almost all with spare legs smaller than 3 (one possibility missing)</td><td>from 1 to 8 without one example</td><td>1</td><td>1</td><td>2</td></tr>
<tr><td colspan="2">All for $n=28,29,30$</td><td>from 1 to 8</td><td>1</td><td>8</td><td>9</td><td>9</td></tr>
<tr><td colspan="2">Strange calculations (including correct ones) but without an answer</td><td></td><td>0</td><td>4</td><td>4</td><td>4</td></tr>
<tr><td colspan="3">Summary</td><td>118</td><td>45</td><td></td><td>163</td></tr>
</table>

a) There were some (5 people), that, without reflecting on the different possibilities of building stools, divided $30 : 3 = 10$ and $30 : 4 = 7$ spare 2 answering that the carpenter can build 10 three-legged stools or 7 four-legged ones. For them, "solving the problem" means operating arithmetically on numbers and giving the results as solutions.

b) There were also those for whom "solving the problem" involved finding the one correct answer (8 people). One can suspect that these people either do not see other possibilities or (which is more likely) associate solving a problem with finding only one correct answer.

c) For many the term "How many and what kind of stools will he be able to build?" implied that there were many possibilities and give more than one answer but not all of them. Two possibilities were given by 36 people, in which for 11 of them they were all none-zero solutions of the $3x + 4y = 30$ equation. I will elaborate on this group in my next point.

d) For many, the important part of solving the problem is to find and present "all" of the possibilities and that was what they did (64 people), although for some "all" meant those in which:

   – There would be no legs left and both types of stools would be made;
   – There would be no legs left and not both types had to be made;
   – Legs would be left, but less than necessary to build a stool (nr from 1 to 8; 9 people).

   The various understandings of solving the problem were expressed in comments:

   $30 : 3 = 10$ *If he were to build only three-legged stools, yet he has to make them with 3 and 4 legs!;*

   $30 : 4 = 7$ spare 2 *He could build 7 four-legged stools, but he would have 2 legs left, so that cannot be;*

   The first comment indicates that the student looks only for solutions in which $x \neq 0$ and $y \neq 0$, disregarding other solutions, the second indicates, that the student looks for possibilities in using the total number of 30 legs (no unused leg can remain).

e) There were some who in solving the problem tried to find and present many possibilities, many even wanted to give "all" the possibilities. Often, in the course of looking for solutions for $n = 30$ they noticed that there can be a different situation ($n = 28$ or 29) which they presented as another possibility. In 16 of those works there was an additional solution (apart from solutions nr 1, 2, 3) being the result of dividing 30 by

4 meaning solution nr 8. In the remaining 9 works, apart from solutions 1, 2 and 3, there are also cases in which 1 leg or 2 legs are left, yet they are not the only possibilities of that kind. In 2 works there are up to 7 solutions (without 1 out of the 8 for $n = 28, 29, 30$).

f) None of the tested students tried to find all of the solutions meaning all of the solutions to the $3x + 4y = 30$ equation, for $n \leq 30$. There were however some that found solutions for $n = 30$ explaining for example:

This problem has many solutions. Every answer (when not more than 30 legs are used) will be correct as long as there will be enough seats.

This sort of comment is evidence of the awareness of many solutions, however, most likely, the tested student either did not feel like searching for them, or, even more likely, did not know how to do it.

3.3. Solving strategies

In the works of the students one can discover many solving strategies. Some use arithmetical methods, other prefer illustrations. Others first perform an illustrated simulation, then confirm their liability with appropriate calculations and vice-versa: first they find a solution through arithmetical methods, then use an illustration. Some illustrations are very realistic, other schematic. Most commonly, 30 lines and dots were drawn and grouped in threes and fours. Some presented their results in tables. Others subsequently subtracted the number 3 or 4 from 30 and analyzed the divisibility of the result by the other of the two numbers. Nonetheless, sometimes miscalculations occurred, which did not allow to obtain all solutions even for $n = 30$.

In many works one could observe a “fairness” in splitting an even amount of legs among the type of stools. The latter students would apply a form of symmetry being the division of 30 legs into 2 equal parts ($30 : 2 = 15$) and building from each part one type of stools. Another form of this kind is to build an equal amount of three and four-legged stools.

Among works were those that presented ready solutions without a trace of any sort or arithmetical calculations or illustrations. It is not out of the question that these solutions were guessed or found through trial-and-error method without presenting the method.

In a few solutions a number axis was employed on which waves of 3 units and 4 units were applied. Such an illustration on a number axis allows one to notice the common multiples of the numbers 3 and 4 (being 12 and 24) and, moreover, that each of the 12 legs (there are two 12s) can be used to build either 3 four-legged stools or 4 three-legged stools. From the remaining 6 legs one can build for example 2 three-legged stools. The analysis of such an illustration allows one to find many solutions.

3.4. Answers

Some tested students, after finding "all" – in their own estimation – of the solutions to the problem, answered in full sentence by listing all of the solutions that they found, thus closing the problem solving process. Others answered by adding short sentences to each found solution. There were even some that would not formulate answers and only underline obtained results. Finally, it seemed some "worked for themselves". After finding the solution, they felt no need to formulate any sort of answer plainly because they found the answers already.

5. Summary

The tested students showed varied abilities in dealing with the aforementioned problem. Out of 163 people merely 9 gave all of the solutions for $n = 28, 29, 30$. No one gave solutions for the remaining n.

There were miscalculations in many of solutions given by the students, which I chose not to present because of the length of this article. Some had trouble writing the proper formula for dividing with a spare. They were not able to manage to find solutions. They limited themselves solely to dividing. Or to present a guessed idea. In their search for solutions they were very often rather clumsy. After finding a solution, they were not always capable of presenting it.

Nevertheless, many tested students were resourceful and creative. Many used illustrated simulations. They used their problem solving strategies rationally and, even though they did not find all of the solutions, those strategies could be used in the search for successive solutions.

The tested students are indeed qualified educators and work in schools, at the same time they decided to start complementary mathematics studies. One can hope that those who have difficulties in solving atypical problems will better their abilities in this regard. Otherwise I doubt they will be able to achieve the goal I mentioned in the beginning. If the teachers will not be creative themselves, it is difficult to expect them to teach creativity to their own students.

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