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ABOUT DEFINITION OF A PERIODIC FUNCTION

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Abstract. In this paper we consider various definitions of a periodic function and establish connections between them, in particular, we prove equivalence of some of them. In papers and textbooks one can find different definitions of a periodic function. This raises the question which of them are equivalent.

1. Periodicity of a function $f : X \to \mathbb{R}$ in a domain X ($X \subseteq \mathbb{R}$)

For a function $f : X \to \mathbb{R}$ ($X \subseteq \mathbb{R}$) one can meet with the following definitions of periodicity*:

Definition 1.
A function f is periodic in a domain X (in the α-sense) $\Leftrightarrow$

$$\Leftrightarrow \underset{T\neq 0}{\exists} \underset{x\in X}{\forall} [x \pm T \in X \wedge f(x+T) = f(x)] \qquad [1,10]$$

Definition 2.
A function f is periodic in a domain X (in the β-sense) $\Leftrightarrow$

$$\Leftrightarrow \underset{T>0}{\exists} \underset{x\in X}{\forall} [x \pm T \in X \wedge f(x+T) = f(x)] \qquad [4]$$

*To distinguish particular definitions of periodicity we use terms: periodicity in the α-sense, in the β-sense and in the γ-sense.

Let us prove that definitions 1 and 2 are equivalent.

Theorem 1.
An arbitrary function $f : X \to \mathbb{R}$ $(X \subseteq \mathbb{R})$ is periodic in the α-sense if and only if it is periodic in the β-sense.

Proof:
It is obvious that periodicity in the β-sense implies periodicity in the α-sense.

To prove the inverse implication let us assume that a function f is periodic in the α-sense. According to Theorem 1 there exists T_1 such that

$$T_1 \neq 0, \tag{1}$$

$$\underset{x \in X}{\forall} [x \pm T_1 \in X \wedge f(x + T_1) = f(x)]. \tag{2}$$

As $T_1 \neq 0$, then $T_1 > 0$ or $T_1 < 0$. In the case $T_1 > 0$ from (2) we have:

$$\underset{T>0}{\exists} \ \underset{x \in X}{\forall} [x \pm T \in X \wedge f(x + T) = f(x)].$$

Assume additionally that $T_1 < 0$. Let also $x \in X$. Then on the basis of (2) we have $x - T_1 \in X$ and $f[(x - T_1) + T_1] = f(x - T_1)$, whence it follows that $f(x - T_1) = f(x)$. Introducing notation $T_2 = -T_1$ we obtain: $T_2 > 0$ and $f(x + T_2) = f(x)$ for arbitrary $x \in X$. Furthermore, $x \pm T_2 \in X$. Therefore,

$$\underset{T>0}{\exists} \ \underset{x \in X}{\forall} [x \pm T \in X \wedge f(x + T) = f(x)].$$

If $T_1 > 0$ or $T_1 < 0$, then

$$\underset{T>0}{\exists} \ \underset{x \in X}{\forall} [x \pm T \in X \wedge f(x + T) = f(x)].$$

Hence, on the basis of definition 2 a function f is periodic in the β-sence, which proves the statement.

In view of Theorem 1, to characterize periodicity in a domain X $(X \subseteq \mathbb{R})$ we can use both definitions 1 and 2. Using complete induction one can prove that:

Theorem 2.
If f is a periodic function (in the α-sense) with the primitive period T in a domain X and $x \in X$, then $x \mp nT \in X$ $(n \in N - \{0\})$.

The following definitions are also connected with the notion of periodicity:

Definition 3.
A function f is periodic in a domain X (in the γ-sense) $\Leftrightarrow$

$$\Leftrightarrow \underset{T\neq 0}{\exists} \underset{x\in X}{\forall} [x+T \in X \wedge f(x+T) = f(x)] \qquad [3].$$

Definition 4.
A function f is progressive periodic ("forward"-periodic) in a domain $X \Leftrightarrow$

$$\Leftrightarrow \underset{T>0}{\exists} \underset{x\in X}{\forall} [x+T \in X \wedge f(x+T) = f(x)] \qquad [5].$$

Definition 5.
A function f is regressive periodic ("backward"-periodic) in a domain $X \Leftrightarrow$

$$\Leftrightarrow \underset{T<0}{\exists} \underset{x\in X}{\forall} [x+T \in X \wedge f(x+T) = f(x)].$$

It is easy to show that

Theorem 3.
The following implications are true for particular types of periodicity:

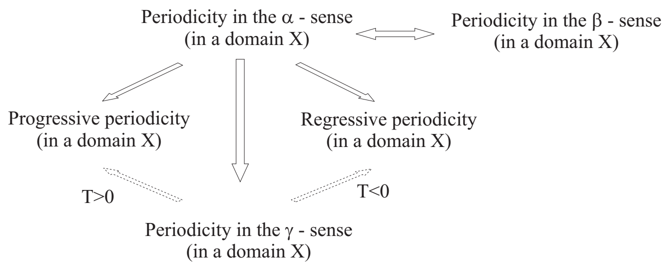

Fig. 1.

There are progressive periodic functions and regressive periodic functions which are not periodic in the α-sense. For example, the function $f(x) = \sin x$ (with the primitive period $T = 2\pi$) in a domain $X = \langle \frac{\pi}{4}, +\infty\rangle$ is a progressive periodic one, whereas the function $g(x) = \cos x$ (with the primitive period $T = -2\pi$) in a domain $X = (-\infty, \frac{\pi}{3}\rangle$ is a regressive periodic one (Fig. 2) [10].

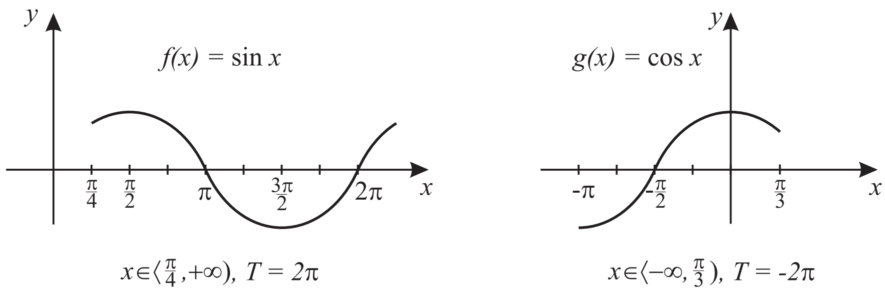

Fig. 2.

2. Periodicity of a function $f : X \to \mathbb{R}$ in a domain $\mathbb{R}$

For a function $f : \mathbb{R} \to \mathbb{R}$ the following definitions of periodicity are considered:

Definition 1*.
The function f is periodic (in the α-sense) $\Leftrightarrow$

$$\Leftrightarrow \underset{T\neq 0}{\exists} \underset{x\in\mathbb{R}}{\forall} [f(x+T) = f(x)] \qquad [2,\ 6,\ 8,\ 9].$$

Definition 2*.
A function f is periodic (in the β-sense) [*] $\Leftrightarrow$

$$\Leftrightarrow \underset{T>0}{\exists} \underset{x\in\mathbb{R}}{\forall} [f(x+T) = f(x)] \qquad [5].$$

Theorem 4.
A function f in a real domain is periodic in the α-sense if and only if it is periodic in the β-sense in this domain.

The proof of this theorem is similar to the proof of Theorem 1. On the basis of the abovementioned theorem periodicity of a function in the domain $\mathbb{R}$ can be characterized by both the definitions 1* and 2*.

[*]In the textbook [7] periodicity of a function in a domain $\mathbb{R}$ is characterized as follows: A function f is periodic (in the β-sense) $\Leftrightarrow \underset{T>0}{\exists} \underset{x\in\mathbb{R}}{\forall} [f(x) = f(x \pm T) = f(x \pm 2T) = \cdots = f(x \pm kT)]$, where k is an arbitrary integer.

3. The notion of a primitive period

The notion of a primitive period for a periodic function is usually defined as follows: if there exists the least positive number T satisfying the condition $f(x+T) = f(x)$ (for arbitrary $x \in X$), then it is called the primitive period of a function $f : X \to \mathbb{R}$.

In the paper [10] it was noted that "if there exists the least positive period or the largest negative period, then the larger of these two numbers which exists (when two numbers do not exist at the same time) is called a primitive period".

With this definition of a primitive period, one can also consider a primitive period for progressive and regressive periodic functions.

It should be emphasized that there are periodic functions which do not have a primitive period. As an example we can consider the following functions (compare [1]):

$$f(x) = c \qquad (c = \text{const}), \qquad x \in \mathbb{R};$$

$$g(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

where $\mathbb{Q}$ is a set of all rational numbers.

An arbitrary nonzero real number constitutes a period of a function f, while an arbitrary nonzero rational number constitutes a period of a function g.

A periodic function in a domain X ($X \subseteq \mathbb{R}$) which does not have a primitive period can have a domain bounded above as well as bounded below.

A progressive periodic function $f : X \to \mathbb{R}$ having a primitive period is not a function with a domain bounded above, whereas a regressive periodic function having a primitive period is not a function with a domain bounded below.

By the way, it should be mentioned that apart from periodic functions (having one period) there are also functions having two and more periods. For example, a two-periodic function $f : \mathbb{R} \to \mathbb{R}$ having two periods 1 and $\sqrt{2}$ is defined as follows:

$$f(x) = \begin{cases} 0 & \text{for numbers } x \text{ of the form } m + n\sqrt{2}, \\ & \text{where } m \text{ and } n \text{ are integers,} \\ 1 & \text{otherwise.} \end{cases} \qquad [8]$$

Functions with many periods were studied by Polish mathematician A. Łomnicki (1881–1941). Extensive information about functions with many periods can be found in the paper [8].

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