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STOCHASTIC GRAPHS AND THEIR APPLICATIONS

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Abstract. The article deals with one example of so called stochastic graph. The paper demonstrates some of possible applications of stochastic graphs in practice using the well known example about seven brigdes of the town of Königsberg.

In Figure 1 is a schematic city plan of Königsberg with seven bridges over the river of Pregel. In 1736 L. Euler showed that it is not possible to cross all seven bridges without crossing one of the bridges twice. Since then has been this task used as a motivation and introductory problem to graph theory. We use this problem differently.

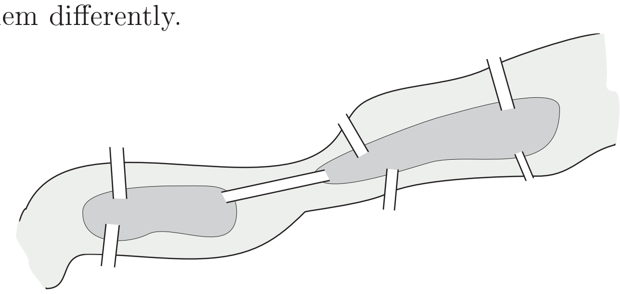

Fig. 1.

Let's suppose that spring flooding can destroy each of the bridges (independent on the others) with probability $0 < p < 1$. What is the probability that it is possible to cross the river of Pregel during the spring flooding?

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We can approach the problem in various ways.

1. In total, there are seven bridges. Each of them will either śurviveór will be destroyed. (i.e. there are only two possibilities). Taking into account the independence, we obtain $2^7 = 128$ situations in total. We can determine for each of the situation whether it is possible to cross the river and also the probability of its happening. The resulting probability is equal to the sum of partial probabilities. This method is more time-consuming with the increasing number of bridges.

2. The situation in the city can be illustrated by the following schema:

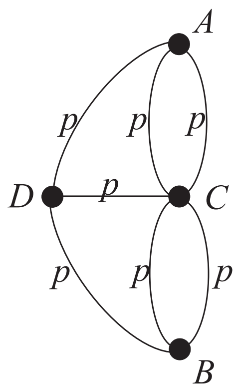


Fig. 2.

Let's denote
AB - event of crossing the river (i.e. from point A to point B).
$\overline{AB}$ - event when it is not possible to cross the river.

We will take advantage of the following characteristics of dependent probability

$$P(AB) = P(AB \mid Y) \cdot P(Y) + P(AB \mid \overline{Y}) \cdot P(\overline{Y}). \qquad (0.1.0.1)$$

As event Y we choose the event of destruction of bridge 7.
We obtain two disjunct cases illustrated by Figures 3 and 4:

destruction of bridge 7

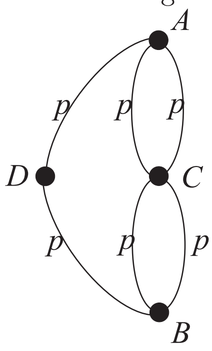


Fig. 3.

survival of bridge 7

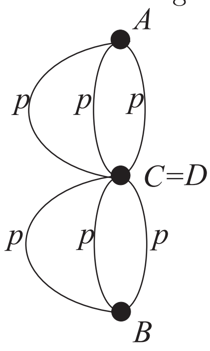


Fig. 4.

Set
ACB - event of going from point A through point C to point B
ADB - event of going from point A to point B through D .

In case of Figure 3 for $P(AB \mid Y)$ it is true that:

$$P(AB \mid Y) = P(ACB) + P(ADB) - P(ACB \cap ADB).$$

As $P(ACB) = P(AC) \cdot P(CB)$ using 1a), b) we obtain $P(AC) = P(CB) = 1 - p^2$ and $P(ADB) = (1-p)^2$. from which flows that

$$P(AB \mid Y) = (1-p^2)^2 + (1-p)^2 - (1-p^2)^2 \cdot (1-p)^2 =$$

$$= 1 - 4p^3 + 2p^4 + 2p^5 - p^6.$$

Considering situation in Figure 4, it is true for $P(AB \mid \overline{Y})$ that:

$$P(AB \mid \overline{Y}) = P(AC) \cdot P(CB).$$

Using 1a) we obtain $P(AC) = P(CB) = 1 - p^3$. This results in

$$P(AB \mid \overline{Y}) = (1-p^3)^2.$$

As $P(Y) = p,\ P(\overline{Y}) = (1-p)$ we can substitute into (1) and then we obtain

$$P(AB) = \left(1 - 4p^3 + 2p^4 + 2p^5 - p^6\right) \cdot p + (1-p^3)^2 \cdot (1-p),$$

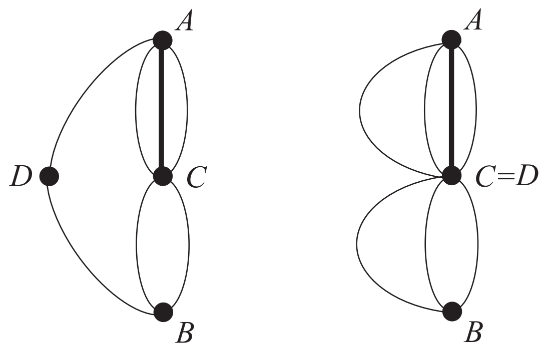
resulting in polynom

$$P(AB) = -2p^7 + 3p^6 + 2p^5 - 2p^4 - 2p^3 + 1.$$

For concrete values of parameter p, it is easy to determine the probability of the possibility to crass the river.

Let's talk about a new situation when we can built a new bridge connecting either points AC, AD, or CD. It is obvious that if we build the new bridge at any of the places, it will be possible to cross all the bridges just once. Now, the task is where to build the bridge to increase the possibility of crossing the river during the flooding to maximum.

1. The new bridge connects points A and C
   Like in previous case, we use (1) to obtain two disjunct cases (first one with probability p, the second one with probability $(1-p)$)



$$P_1(AB \mid Y) = (1-p^3)\cdot(1-p^2) + (1-p)^2 - (1-p^3)\cdot(1-p^2)\cdot(1-p)^2,$$

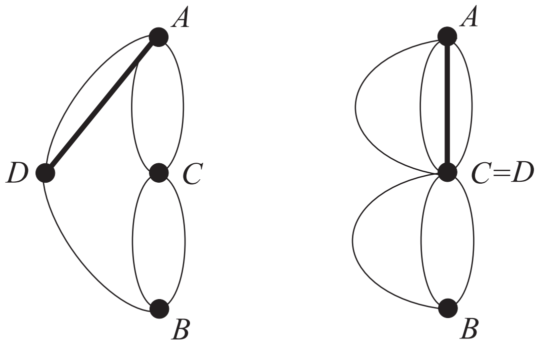
$$P_1(AB \mid \overline{Y}) = (1-p^4)\cdot(1-p^3)$$

resulting in

$$P_1(AB) = \Big((1-p^3)\cdot(1-p^2) + (1-p)^2 - (1-p^3)\cdot(1-p^2)\cdot(1-p)^2\Big)\cdot p +$$

$$+\Big((1-p^4)\cdot(1-p^3)\Big)\cdot(1-p) = -2p^8 + 3p^7 + p^6 - 2p^4 - p^3 + 1.$$

2. The new bridge connects points A and D
   Like in previous case, we use (1) to obtain two disjunct cases (first one with probability p, the second one with probability $(1-p)$)



$$P_2(AB \mid Y) = (1-p^2)^2 + (1-p^2)\cdot(1-p) - (1-p^2)^2\cdot(1-p^2)\cdot(1-p),$$

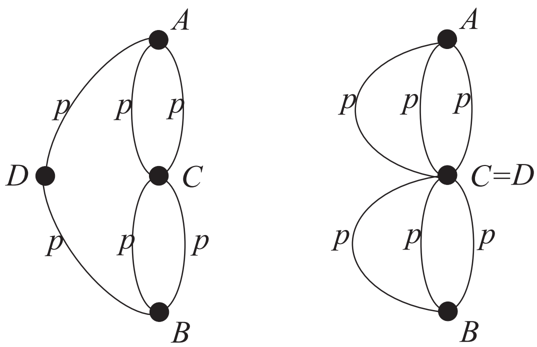
$$P_2(AB \mid \overline{Y}) = (1-p^4)\cdot(1-p^3)$$

resulting in

$$P_2(AB) = \Big((1-p^2)^2 + (1-p^2)\cdot(1-p) - (1-p^2)^2\cdot(1-p^2)\cdot(1-p)\Big)\cdot p +$$

$$+\Big((1-p^4)\cdot(1-p^3)\Big)\cdot(1-p) = -2p^8 + 2p^7 + 3p^6 - p^5 - 2p^4 - p^3 + 1.$$

3. The new bridge connects points C and D
   In this case, the first situation happens with probability of p^2, the second with probability of $1-p^2$



$$P_3(AB \mid Y) = (1-p^2)^2 + (1-p)^2 - (1-p^2)^2\cdot(1-p)^2,$$

$$P_3(AB \mid \overline{Y}) = (1-p^3)^2$$

resulting in

$$P_3(AB) = \Big((1-p^2)^2 + (1-p)^2 - (1-p^2)^2 \cdot (1-p)^2\Big) \cdot p^2 +$$
$$+\Big((1-p^3)^2\Big) \cdot (1-p^2) = -2p^8 + 2p^7 + 3p^6 - 2p^5 - 2p^3 + 1.$$

Let's compare first $P_1(AB)$ and $P_2(AB)$:

$$P_1(AB) - P_2(AB) = p^5 \cdot (1-p)^2 \geq 0.$$

Now, we compare $P_2(AB)$ and $P_3(AB)$:

$$P_2(AB) - P_3(AB) = p^3 \cdot (1-p)^2 \geq 0.$$

Now (for completeness), we compare $P_1(AB)$ and $P_3(AB)$:

$$P_1(AB) - P_3(AB) = p^3 \cdot (p^2+1) \cdot (1-p)^2 \geq 0.$$

It is obvious that it is most effective to build the bridge between points A and C.

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