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ON ADDITIVE AND MULTIPLICATIVE MAGIC CUBES

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Abstract.
An additive magic cube is a cubical array containing different natural numbers such that the sum of the numbers along every row and every diagonal is the same. A multiplicative magic cube is cubical array containing mutually different natural numbers such that the product of the numbers along each row and diagonal is the same. In this paper we give several ways to construct additive and multiplicative magic cubes.

1. Introduction

Magic squares have fascinated people for centuries. An $n \times n$ *additive magic square* contains the natural numbers $1, 2, \dots, n^2$, such that the sum of every rows, column and diagonal is the same. Figure 1 depicts a 3×3, a 4×4 and a 5×5 table each containing a different set of natural numbers in such a way that the product of the numbers in every row, column and diagonal is the same. (In the first table the product is 6^3, in the second one is 7! and in the third one is 9!.) Such tables are called *multiplicative magic squares* (See [3]).

12	1	18
9	6	4
2	36	3

1	24	14	15
21	10	4	6
20	7	18	2
12	3	5	28

1	15	42	16	36
14	32	9	3	30
27	6	10	28	8
20	7	24	54	2
48	18	4	5	21

Fig. 1. Multiplicative magic squares

A *magic cube* is a natural generalization of a magic square. (In this paper we will call it an *additive magic cube.*) The first magic cube probably appeared about 1640 in a letter of *Pierre de Fermat* (see [1, p. 365]). Information and many interesting results about magic squares and cubes can be found in the references and web-pages.

An *additive magic cube* of order n is a cubical array (3-dimensional matrix of order n)

$$\mathbf{M}_n = |\mathbf{m}_n(i,j,k); \quad 1 \le i,j,k \le n|$$

containing natural numbers $1, 2, 3, \dots, n^3$ such that the sum of the numbers along every row and diagonal is the same, i.e. $\frac{n(n^3+1)}{2}$. By a row of a magic cube we mean an n-tuple of elements having the same coordinates in two places. (Note: We use the same term for a row or a column or a pilar.) Every additive magic cube of order n has exactly $3n^2$ rows and 4 diagonals connecting the eight corners of the cube.

In [4] there are depicted additive magic cubes $\mathbf{M}_3$ and $\mathbf{M}_4$.

A *multiplicative magic cube* of order n is a cubical array

$$\mathbf{Q}_n = |\mathbf{q}_n(i,j,k); \quad 1 \le i,j,k \le n|$$

containing n^3 mutually different natural numbers such that the product of the numbers along each row and every one of its four diagonals is the same. We call this product the *magic constant* and denote $\sigma(\mathbf{Q}_n)$.

Figure 2 shows $\mathbf{Q}_3$ with the magic constant $(2.3.5)^3$. The element $\mathbf{m}_3(1,1,1) = 18$ is contained in the rows $\{18, 20, 75\}$, $\{18, 300, 5\}$, $\{18, 60, 25\}$ and in the diagonal $\{18, 30, 50\}$.

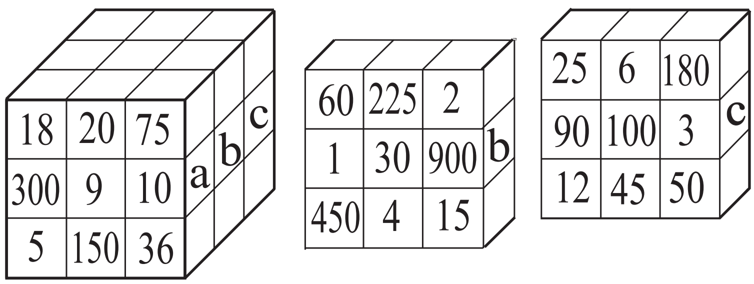

Fig. 2. Multiplicative magic cube $\mathbf{Q}_3$

In [4] it is proved that an additive magic cube $\mathbf{M}_n$ of order n exists for every $n \ne 2$. If we know a construction of $\mathbf{M}_n = |\mathbf{m}_n(i,j,k)|$, then we can easily make a multiplicative magic cube

$$\mathbf{Q}_n = |\mathbf{q}_n(i,j,k) = 2^{\mathbf{m}_n(i,j,k)-1}; \ 1 \le i,j,k \le n|$$

with the magic constant $\sigma(\mathbf{Q}_n) = 2^{\frac{n(n^3-1)}{2}}$. This paper contains formulas for construction of magic cubes $\mathbf{M}_n$ and $\mathbf{Q}_n$ for all $n \neq 2$. Moreover, the constructed cubes $\mathbf{Q}_n$ have a significantly smaller magic constant than cubes constructed using (1).

We construct an additive magic cube $\mathbf{M}_n = |\mathbf{m}_n(i,j,k); 1 \leq i,j,k \leq n|$ of order n and a multiplicative magic cube $\mathbf{Q}_n = |\mathbf{q}_n(i,j,k); 1 \leq i,j,k \leq n|$ of order n for all $n \neq 2$ using the following formulas. We consider three cases (n is an odd integer; if n is an even integer, then we distinguish whether n is or is not divisible by four.) The correctness of formulas for additive magic cubes follow immediately from the proofs in [4, 6].

We use the following notation:

$x \pmod n$ is the remainder in the division of x by n,

$\overline{x} = n + 1 - x$,

$x^* = \min\{x, \overline{x}\}$,

$$\widetilde{x} = \begin{cases} 0 & \text{for} \quad 1 \leq x \leq \frac{n}{2}, \\ 1 & \text{for} \quad \frac{n}{2} < x \leq n. \end{cases}$$

1. If $n \equiv 1 \pmod 2$, then

$$\mathbf{m}_n(i,j,k) = \alpha n^2 + \beta n + \gamma + 1, \tag{1.1}$$

$$\mathbf{q}_n(i,j,k) = 2^\alpha . 3^\beta . 5^\gamma, \tag{1.2}$$

where

$\alpha = (i - j + k - 1) \pmod n$,

$\beta = (i - j - k) \pmod n$,

$\gamma = (i + j + k - 2) \pmod n$.

99	182	300	408	16
1456	75	102	4	792
600	816	1	198	364
204	8	1584	91	150
2	396	728	1200	51

1*st* layer

26	540	952	80	33
135	238	20	264	208
1904	5	66	52	1080
40	528	13	270	476
132	104	2160	119	10

2*nd* layer

180	136	144	77	130
34	36	616	1040	450
9	154	260	360	272
1232	65	90	68	72
520	720	17	18	308

3*rd* layer

680	48	11	234	420
12	88	1872	105	170
22	468	840	1360	3
117	210	340	24	176
1680	85	6	44	936

4*th* layer

112	55	78	60	1224
440	624	15	306	28
156	120	2448	7	110
30	612	56	880	39
153	14	220	312	240

5*th* layer

Fig. 3. Multiplicative magic cube $\mathbf{Q}_5$

If $n \not\equiv 0 \pmod 3$, then in every row and also in every diagonal $\mathbf{Q}_n$ constructed by (1.2) there is exactly one number which is divisible by the zth power but is not divisible by the $(z+1)$th power of the number 2 (3 or 5, respectively). We obtain a multiplicative magic cube $\mathbf{Q}_n$ with a smaller magic constant $\sigma(\mathbf{Q}_n)$ if in formula (1.2) we replace powers of 3 by the numbers $(2\beta+1)$ for $\beta = 1, 2, \dots, n-1$ and the powers of 5 by the numbers $(2n+2\gamma-1)$ for $\gamma = 1, 2, \dots, n-1$. Figure 3 shows five layers of $\mathbf{Q}_5$. (By a different substitution we can obtain a $\mathbf{Q}_5$ with a smaller magic constant).

2. If $n \equiv 0 \pmod 4$, then

$$\mathbf{m}_n(i,j,k) = \begin{cases} (i-1)\,n^2 + (j-1)\,n + k & \text{if } \mathcal{F}(i,j,k) = 1, \\ (\bar{i}-1)\,n^2 + (\bar{j}-1)\,n + \bar{k} & \text{if } \mathcal{F}(i,j,k) = 0; \end{cases} \tag{2.1}$$

$$\mathbf{q}_n(i,j,k) = \begin{cases} 2^{(i-1)}.3^{(j-1)}.5^{(k-1)} & \text{if } \mathcal{F}(i,j,k) = 1, \\ 2^{(\bar{i}-1)}.3^{(\bar{j}-1)}.5^{(\bar{k}-1)} & \text{if } \mathcal{F}(i,j,k) = 0, \end{cases} \tag{2.2}$$

where $\mathcal{F}(i,j,k) = (i + \widetilde{i} + j + \widetilde{j} + k + \widetilde{k}) \pmod 2$.

If $n \equiv 0 \pmod 4$, we construct a cube $\mathbf{Q}_n$ with a smaller magic constant $\sigma(\mathbf{Q}_n)$ using a method which we demonstrate in the following example. In Figure 4 there are depicted the four layers of $\mathbf{M}_4$ (constructed by (2.1)) whose numbers are the binary representation of the numbers $\mathbf{m}_4(i,j,k) - 1$.

000000	111110	111101	000011
111011	000101	000110	111000
110111	001001	001010	110100
001100	110010	110001	001111

1*st* layer

101111	010001	010010	101100
010100	101010	101001	010111
011000	100110	100101	011011
100011	011101	011110	100000

2*nd* layer

011111	100001	100010	011100
100100	011010	011001	100111
101000	010110	010101	101011
010011	101101	101110	010000

3*rd* layer

110000	001110	001101	110011
001011	110101	110110	001000
000111	111001	111010	000100
111100	000010	000001	111111

4*th* layer

Fig. 4

By closely examining Figure 4 you can find out that in every 4-tuple of numbers in any row or diagonal it holds that on the zth position,

$z = 1, 2, \ldots, 6$, there are exactly two ones and two zeroes. We use this fact in the construction. If $b_1 b_2 b_3 \ldots b_6$ is the representation of the number $\mathbf{m}_4(i, j, k)$ lowered by 1 in binary code, then

$$\mathbf{q}_4(i, j, k) = 2^{b_1} 3^{b_2} 4^{b_3} 5^{b_4} 7^{b_5} 9^{b_6}.$$

We have chosen the set $\{2, 3, 4, 5, 7, 9\}$ in such a way that it does not contain two nonempty subsets of numbers whose product is the same. The magic constant of the cube $\mathbf{Q}_4$ (see Figure 5) is $\sigma(\mathbf{Q}_4) = (2.3.4.5.7.9)^2 = 57\ 153\ 600$.

1	840	1080	63
1512	45	35	24
1890	36	28	30
20	42	54	1260

1*st* layer

2520	27	21	40
15	56	72	945
12	70	90	756
126	540	420	2

2*nd* layer

3780	18	14	60
10	84	108	630
8	105	135	504
189	360	280	3

3*rd* layer

6	140	180	378
252	270	210	4
315	216	168	5
120	7	9	7560

4*th* layer

Fig. 5. Multiplicative magic cube $\mathbf{Q}_4$

Because binary representation of elements (lowered by one) of the magic cube $\mathbf{M}_n$ fulfills the condition about the same number of ones and zeroes in the corresponding positions, we can generalize the given construction for all $n \equiv 0 \pmod 4$.

3. If $n \equiv 2 \pmod 4$ (in this case $\frac{n}{2}$ is odd and let $t = \frac{n}{2}$), then

$$\mathbf{m}_n(i, j, k) = \mathbf{d}(u, v)t^3 + \mathbf{m}_t(i^*, j^*, k^*), \tag{3.1}$$

$$\mathbf{q}_n(i, j, k) = 7^{\mathbf{d}(u,v)}.\mathbf{q}_t(i^*, j^*, k^*), \tag{3.2}$$

where

$\mathbf{q}_t(i^*, j^*, k^*)$ is constructed using (1.2),

$u = (i^* - j^* + k^*) \pmod t + 1$,

$v = 4\widetilde{i} + 2\widetilde{j} + \widetilde{k} + 1$,

$\mathbf{d}(u, v)$ for $1 \le u \le t,\ 1 \le v \le 8$ is defined by the table in Fig. 6, ($a = 1, 2, \ldots, \frac{n-6}{4}$).

	$\mathbf{d}(u,1)$	$\mathbf{d}(u,2)$	$\mathbf{d}(u,3)$	$\mathbf{d}(u,4)$	$\mathbf{d}(u,5)$	$\mathbf{d}(u,6)$	$\mathbf{d}(u,7)$	$\mathbf{d}(u,8)$
$\mathbf{d}(1,v)$	7	3	6	2	5	1	4	0
$\mathbf{d}(2,v)$	3	7	2	6	1	5	0	4
$\mathbf{d}(3,v)$	0	1	3	2	5	4	6	7
$\mathbf{d}(2a+2,v)$	0	1	2	3	4	5	6	7
$\mathbf{d}(2a+3,v)$	7	6	5	4	3	2	1	0

Figure 6

Problem 1. Using analogous formulas it is easy to make a computer program which constructs an additive and a multiplicative magic square for every $n \neq 2$.

Problem 2. Find a smaller magic constant for a multiplicative magic cubes of order n.

Remark. By the end of the 19-th century (see [1]) mathematicians began to consider also 4-dimensional magic cubes. But only in 2001 the following result was published:

Theorem. *An additive magic d-dimensional cube of order n exists if and only if $d > 1$ and $n \neq 2$ or $d = 1$.*

Similarly we can consider the existence of multiplicative magic d-dimensional cubes for any natural d. The constructions given in [6] allow us to experiment with magic cubes also in higher dimensional spaces.

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