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THE STABILITY OF THE SECOND GENERALIZATION OF D'ALEMBERT'S FUNCTIONAL EQUATION

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Abstract
In the present paper we study solutions of the second generalized d'Alembert's functional equation and its stability.

1. Introduction

We shall start from solutions of the classical d'Alembert functional equation (the cosine equation):

Theorem 1 (Pl. Kannappan [5], see also [2]) *Let* $(G,+)$ *be an Abelian group. Then a function* $f:\ G\to\ \mathbb{C}$ *satisfies the equation*

$$f(x+y)+f(x-y)=2f(x)f(y)\quad \text{for all}\quad x,y\in G \tag{1}$$

if and only if there exists a homomorphism $m:G\to\mathbb{C}$*, i.e.* m *satisfies the exponential functional equation*

$$m(x+y)=m(x)m(y)\quad \text{for all}\quad x,y\in G \tag{2}$$

such that

$$f(x)=\frac{1}{2}(m(x)+m(-x))\quad \text{for all}\quad x\in G.$$

It is known that equation (1) for complex functions defined on an Abelian group is stable in the sense of Hyers-Ulam [4]. Generalizations of this result appeared in various directions. It turned out that equation (1) for complex functions defined on an Abelian group is superstable in the sense of Ger, too. Namely, the following theorems hold true:

Theorem 2 (R. Badora, R. Ger [3]) *Let $(G,+)$ be an Abelian group and let $f : G \to \mathbb{C}$ and $\varphi : G \to \mathbb{R}$ satisfy the inequality*

$$|f(x+y)+f(x-y)-2f(x)f(y)| \le \varphi(x) \quad \textit{for all} \quad x,y \in G.$$

Then either f is bounded or f satisfies d'Alembert's equation (1).

Theorem 3 (R. Badora, R. Ger [3]) *Let $(G,+)$ be an Abelian group and let $f : G \to \mathbb{C}$ and $\varphi : G \to \mathbb{R}$ satisfy the inequality*

$$|f(x+y)+f(x-y)-2f(x)f(y)| \le \varphi(y) \quad \textit{for all} \quad x,y \in G.$$

Then either f is bounded or f satisfies d'Alembert's equation (1).

In monograph of J. Aczél [1] we can find a form of solutions of *Wilson's first generalization of d'Alembert's functional equation*

$$f(x+y)+f(x-y)=2f(x)g(y) \quad \text{for all} \quad x,y \in G, \tag{3}$$

and *Wilson's second generalization of d'Alembert's functional equation*

$$f(x+y)+g(x-y)=h(x)k(y) \quad \text{for all} \quad x,y \in G, \tag{4}$$

where $(G,+)$ is a uniquely 2-divisible Abelian group and $f,g,h,k : G \to \mathbb{C}$.

We deal with the special case of equation (4).

Definition 1 *Let $(G,+)$ be an Abelian group. We say that functions $f,g : G \to \mathbb{C}$ satisfy the second generalization of d'Alembert's functional equation iff f and g satisfy the following functional equation*

$$f(x+y)+f(x-y)=2g(x)f(y) \quad \textit{for all} \quad x,y \in G. \tag{5}$$

Solutions and the stability problem for equation (3) was consider in [6]. In this paper we find a form of solutions of equation (5) and study its stability.

2. Main results

Instead of specification of Aczél's Theorem about solutions of equation (4) [1, p. 175], we formulate and directly prove the following fact concerning solutions of equation (5).

Theorem 4 *Let* $(G, +)$ *be an Abelian group. Then functions* $f, g : G \to \mathbb{C}$ *satisfy equation* (5) *if and only if*

(i) $f = 0$ *and* g *is arbitrary,*

or

(ii) $f \neq 0$ *and* $f(x) = \alpha g(x)$ *for all* $x \in G$*, where* $\alpha \in \mathbb{C}\backslash\{0\}$ *and* g *satisfies the d'Alembert functional equation, i.e.*

$$g(x + y) + g(x - y) = 2g(x)g(y)$$

for all $x, y \in G$*.*

Rroof. Assume that $f \neq 0$. Setting $y = 0$ in (5), we have

$$f(x) = g(x)f(0)$$

for all $x \in G$. From above, we conclude that $f(0) \neq 0$. Putting $\alpha := f(0)$, we get

$$f(x) = \alpha g(x) \quad \text{for all} \quad x \in G.$$

So, from (5) we obtain

$$\alpha g(x + y) + \alpha g(x - y) = 2g(x)\alpha g(y)$$

for all $x, y \in G$, and the theorem follows. □

Let $(G, +)$ be an Abelian group. Take an arbitrary function $\varphi : G \to \mathbb{R}$ (not necessarily constant nor bounded) and let $f, g : G \to \mathbb{C}$. Now, we consider the following inequality

$$|f(x + y) + f(x - y) - 2g(x)f(y)| \leq \varphi(x) \quad \textit{for all} \quad x, y \in G. \qquad (6)$$

Definition 2 *We say that equation* (5) *is stable if and only if for every pair* (f, g) *satisfying inequality* (6) *there exist* $\widetilde{f}, \widetilde{g} : G \to \mathbb{C}$ *and* $\alpha, \beta : G \to \mathbb{R}$ *such that the pair* $(\widetilde{f}, \widetilde{g})$ *is a solution of equation* (5) *and we get the following estimations*

$$|f - \widetilde{f}| \leq \alpha \quad and \quad |g - \widetilde{g}| \leq \beta.$$

Definition 3 *We say that equation* (5) *is superstable if and only if for all pairs* (f, g) *which are solutions of inequality* (6) *the following alternative holds: either at least one from the functions* f, g *is bounded or the pair* (f, g) *satisfies equation* (5).

Consider the following

Example 1 *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *have a form*

$$f(x) = \frac{e^x + e^{-x}}{2} + 1, \quad g(x) = \frac{e^x + e^{-x}}{2} \quad \textit{for all} \quad x \in \mathbb{R}.$$

Then

$$|f(x+y)+f(x-y)-2g(x)f(y)| = |2-e^x-e^{-x}| =: \varphi(x) \quad \textit{for all} \quad x, y \in \mathbb{R}.$$

This example shows that the second generalization of d'Alembert's equation is stable but not superstable (as the d'Alembert equation). Now we check: is this also true for arbitrary complex function f and g defined on an Abelian group?

Theorem 5 *Let* $(G, +)$ *be an Abelian group and let* $f, g : G \to \mathbb{C}$, $\varphi : G \to \mathbb{R}$ *satisfy the inequality* (6). *We have the following possibilities:*

(i) *if* $f = 0$, *then* g *is arbitrary,*

(ii) *if* $g = 0$, *then* $|f(x)| \leq \frac{\varphi(x)}{2}$ *for all* $x \in G$,

(iii) *if* $f \neq 0 \neq g$ *and* f *is bounded, then* $|g(x)| \leq \frac{\varphi(x)+2M}{2M}$ *for all* $x \in G$, *where* $M := \sup\{|f(x)| : x \in G\}$,

(iv) *if* $g \neq 0$ *and* f *is unbounded, then*

$$g(x+y)+g(x-y)=2g(x)g(y) \quad \textit{for all} \quad x,y\in G$$

and

$$|f(x)-g(x)f(0)| \leq \frac{\varphi(x)}{2} \quad \textit{for all} \quad x\in G.$$

Proof. Ad (ii). If $g=0$, then the inequality (6) has a form

$$|f(x+y)+f(x-y)| \leq \varphi(x), \quad x,y\in G.$$

Put $y=0$, for fixed $x\in G$, we get

$$|f(x)| \leq \frac{\varphi(x)}{2}.$$

Ad (iii). Assume that $f\neq 0$, $g\neq 0$ and f is bounded. Then

$$M := \sup\{|f(x)| : x\in G\} < \infty.$$

By (6), we deduce that

$$|2g(x)f(y)| \leq \varphi(x)+2M \quad \text{for all} \quad x,y\in G.$$

Because $f\neq 0$, we get (iii).

Ad (iv). Assume that $g\neq 0$ and f is unbounded. Thus, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ of elements of G such that

$$0\neq |f(t_n)| \longrightarrow \infty \quad \text{as} \quad n\to\infty. \tag{7}$$

Putting $y=t_n$ in (6), we have for all $x\in G$ and $n\in\mathbb{N}$:

$$|f(x+t_n)+f(x-t_n)-2g(x)f(t_n)| \leq \varphi(x).$$

Therefore,

$$\left|\frac{f(x+t_n)+f(x-t_n)}{2f(t_n)}-g(x)\right| \leq \frac{\varphi(x)}{2|f(t_n)|}$$

for all $x\in G$ and $n\in\mathbb{N}$. Passing to the limit as $n\longrightarrow\infty$ and taking (7) into account, we infer that

$$g(x)=\lim_{n\to\infty}\frac{f(x+t_n)+f(x-t_n)}{2f(t_n)} \quad \text{for all} \quad x\in G. \tag{8}$$

Set $y = y + t_n$ in (6), then

$$|f(x+y+t_n) + f(x-y-t_n) - 2g(x)f(y+t_n)| \leq \varphi(x) \tag{9}$$

for all $x, y \in G$ and $n \in \mathbb{N}$. Replacing y by $y - t_n$ in (6), we get

$$|f(x+y-t_n) + f(x-y+t_n) - 2g(x)f(y-t_n)| \leq \varphi(x) \tag{10}$$

for all $x, y \in G$ and $n \in \mathbb{N}$. Now, using (9) and (10) for all $x, y \in G$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \Bigg| \frac{f(x+y+t_n) + f(x+y-t_n)}{2f(t_n)} + \frac{f(x-y+t_n) + f(x-y-t_n)}{2(t_n)} - \\ -2g(x)\frac{f(y+t_n) + f(y-t_n)}{f(t_n)} \Bigg| \leq \frac{2\varphi(x)}{2|f(t_n)|}. \end{aligned} \tag{11}$$

Passing here to the limit as $n \longrightarrow \infty$, by (8), we get

$$g(x+y) + g(x-y) - 2g(x)g(y) = 0 \quad \text{for all} \quad x, y \in G.$$

Moreover, $g \neq 0$ and, thus, $g(0) = 1$. Putting $y = 0$ in (6), we have

$$|f(x) - g(x)f(0)| \leq \frac{\varphi(x)}{2} \quad \text{for all} \quad x \in G,$$

which completes the proof. □

For unbounded functions $f, g : G \to \mathbb{C}$ we can define

$$\widetilde{f}(x) := g(x)f(0), \quad \widetilde{g}(x) := g(x) \quad \text{for all} \quad x \in G.$$

From Theorem 5, we get

$$|f(x) - \widetilde{f}(x)| \leq \frac{\varphi(x)}{2}, \quad |g(x) - \widetilde{g}(x)| = 0 \quad \text{for all} \quad x \in G.$$

Moreover, $\widetilde{f}$ and $\widetilde{g}$ satisfy equation (5). This means the stability of the second generalization of d'Alembert's functional equation.

For $g = f$ from Theorem 5 we obtain the following

Corollary 1 (Theorem 2) *Let* $(G, +)$ *be an Abelian group and let* $f : G \to \mathbb{C}$ *and* $\varphi : G \to \mathbb{R}$ *satisfy the inequality*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varphi(x) \quad \textit{for all} \quad x, y \in G.$$

Then either f *is bounded or* f *satisfies d'Alembert's equation* (1).

Because inequality (6) is not symmetrical with respect to x and y, now we consider the following inequality

$$|f(x+y) + f(x-y) - 2g(x)f(y)| \leq \varphi(y) \tag{12}$$

for all $x, y \in G$, where $(G, +)$ is an Abelian group, $\varphi : G \to \mathbb{R}$ (not necessarily constant nor bounded) and $f, g : G \to \mathbb{C}$.

Theorem 6 *Let* $(G, +)$ *be an Abelian group and let* $f, g : G \to \mathbb{C}$, $\varphi : G \to \mathbb{R}$ *satisfy the inequality* (12). *There are the following possibilities:*

(i) *if* $f = 0$, *then* g *is arbitrary,*

(ii) *if* $f \neq 0$ *and* g *is bounded, then* f *is bounded, too,*

(iii) *if* $f \neq 0$ *and* g *is unbounded, then there exists a function* $h : G \to \mathbb{C}$ *such that*

$$f(x+y) + f(x-y) = 2h(x)f(y) \quad \textit{for all} \quad x, y \in G$$

and

$$|g(x) - h(x)| \leq C \quad \textit{for all} \quad x \in G,$$

where $C := \inf \left\{ \frac{\varphi(x)}{2|f(x)|} : x \in G,\ f(x) \neq 0 \right\}$.

Proof. Assume that $f \neq 0$.

Ad (ii). If g is bounded, then there exists a constant $M \geq 0$ such that

$$|g(x)| \leq M \quad \text{for all} \quad x \in G.$$

Now, putting $y = 0$ in (12), we get

$$|f(x)| \leq \frac{\varphi(0) + 2M|f(0)|}{2} \quad \text{for all} \quad x \in G.$$

Ad (iii). If g is unbounded, then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ of elements from G such that

$$0 \neq |g(t_n)| \longrightarrow \infty \quad \text{as} \quad n \to \infty. \tag{13}$$

Setting $x = t_n$ in (12) we have

$$\left|\frac{f(t_n + y) + f(t_n - y)}{2g(t_n)} - f(y)\right| \leq \frac{\varphi(y)}{2|g(t_n)|} \quad \text{for all} \quad y \in G,\ n \in \mathbb{N}.$$

Passing here to the limit as $n \longrightarrow \infty$ and using (13), we see that for every $y \in G$

$$f(y) = \lim_{n\to\infty} \frac{f(t_n + y) + f(t_n - y)}{2g(t_n)}. \tag{14}$$

On setting $x = t_n + x$ in (12), we get

$$|f(t_n + x + y) + f(t_n + x - y) - 2g(t_n + x)f(y)| \leq \varphi(y) \tag{15}$$

for all $x, y \in G$ and $n \in \mathbb{N}$. Similarly, putting $x = t_n - x$ in (12), we obtain

$$|f(t_n - x + y) + f(t_n - x - y) - 2g(t_n - x)f(y)| \leq \varphi(y) \tag{16}$$

for all $x, y \in G$ and $n \in \mathbb{N}$. Now, using (15) and (16) for all $x, y \in G$ and $n \in \mathbb{N}$ we conclude that

$$\left|\frac{f(t_n + x + y) + f(t_n - (x + y))}{2g(t_n)} + \frac{f(t_n + x - y) + f(t_n - (x - y))}{2g(t_n)} - \right.$$

$$\left. -2\frac{g(t_n + x) + g(t_n - x)}{2g(t_n)} f(y)\right| \leq \frac{2\varphi(y)}{2|g(t_n)|}.$$

Passing to the limit as $n \longrightarrow \infty$ with the use of (14), we infer that there exists the limit

$$h(x) := \lim_{n\to\infty} \frac{g(t_n + x) + g(t_n - x)}{2g(t_n)} \quad \text{for all} \quad x \in G. \tag{17}$$

Moreover, the function $h : G \to \mathbb{C}$ obtained in that way has to satisfy the eqation

$$f(x+y) + f(x-y) = 2h(x)f(y) \quad \text{for all} \quad x, y \in G. \tag{18}$$

Putting (18) into (12), we get

$$2|f(y)||h(x) - g(x)| \leq \varphi(y), \quad x, y \in G. \tag{19}$$

From above and by the assumption that $f \neq 0$, we can define a real constant C as

$$C := \inf \left\{ \frac{\varphi(y)}{2|f(y)|} : x \in G,\ f(y) \neq 0 \right\}.$$

Inequality (19) yields

$$|h(x) - g(x)| \leq C \quad \text{for all} \quad x \in G.$$

□

We introduce definitions analogous to Definition 2 and Definition 3.

Definition 4 *We say that equation* (5) *is stable if and only if for every pair* (f, g) *which is a solution of inequality* (12), *there exist* $\widetilde{f}, \widetilde{g} : G \to \mathbb{C}$ *and* $\alpha, \beta : G \to \mathbb{R}$ *such that the pair* $(\widetilde{f}, \widetilde{g})$ *is a solution of equation* (5) *and*

$$|f - \widetilde{f}| \leq \alpha \quad \text{and} \quad |g - \widetilde{g}| \leq \beta.$$

Definition 5 *We say that equation* (5) *is superstable if and only if for all pairs* (f, g) *which are solutions of inequality* (12), *the following alternative holds: either at least one from functions* f, g *is bounded or a pair* (f, g) *satisfies equation* (5).

From Theorem 6 it follows that if f and g are unbounded, then we can define

$$\widetilde{f}(x) := f(x), \quad \widetilde{g}(x) := h(x) \quad \text{for all} \quad x \in G$$

and

$$|\widetilde{f}(x) - f(x)| = 0, \quad |\widetilde{g}(x) - g(x)| \leq C$$

for all $x \in G$ and some constant C. Functions $\widetilde{f}$, $\widetilde{g}$ satisfy equation (5). This means the stability of the second generalization of d'Alembert's functional equation.

Example 2 *For all* $x \in \mathbb{R}$ *we put*

$$f(x) = \frac{e^x + e^{-x}}{2}, \quad g(x) = \frac{e^x + e^{-x}}{2} + 1.$$

Then functions f *and* g *satisfy*

$$|f(x+y)+f(x-y)-2g(x)f(y)| = |-e^y - e^{-y}| =: \varphi(y) \quad \textit{for all} \quad x, y \in \mathbb{R}.$$

Notice that from Theorem 6 we can get Theorem 3.

Corollary 2 *Let* $(G, +)$ *be an Abelian group and let* $f : G \to \mathbb{C}$, $\varphi : G \to \mathbb{R}$ *satisfy the inequality*

$$|f(x + y) + f(x - y) - 2f(x)f(y)| \leq \varphi(y), \quad x, y \in G.$$

Then either f *is bounded or* f *satisfies d'Alembert's equation* (1).

Proof. Assume that f is unbounded. Define $g = f$ in the case (iii) of Theorem 6. Hence, there exists $h : G \to \mathbb{C}$ such that

$$f(x + y) + f(x - y) = 2h(x)f(y) \quad \text{for all} \quad x, y \in G$$

and

$$|h(x) - f(x)| \leq C \quad \text{for all} \quad x \in G,$$

where $C := \inf \left\{ \frac{\varphi(y)}{2|f(y)|} : x \in G,\ f(y) \neq 0 \right\}$. Applying $g = f$ in (14) and (17) we get $h = f$ which proves the corollary. □

From above theorems we get also

Corollary 3 *Let $(G,+)$ be an Abelian group, $\varepsilon \geq 0$ be a given number, and let $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x-y) - 2g(x)f(y)| \leq \varepsilon \quad \text{for all} \quad x, y \in G.$$

Then there are the following possibilities:

(i) *if $f = 0$, then g is arbitrary,*

(ii) *if $g = 0$, then f is bounded,*

(iii) *if $f \neq 0 \neq g$ and f is bounded, then g is bounded, too,*

(iv) *if $g \neq 0$ and f is unbounded, then g is unbounded, too. Moreover*

$$f(x+y) + f(x-y) = 2g(x)f(y) \quad \text{for all} \quad x, y \in G.$$

Proof. Ad (i). This is the case (i) of Theorem 5 (or the case (i) of Theorem 6).

Ad (ii). If $g = 0$, then for $\varphi(x) \equiv \varepsilon$ from (ii) of Theorem 5 we have

$$|f(x)| \leq \frac{\varepsilon}{2} \quad \text{for all} \quad x \in G.$$

Thus, f is bounded.

Ad (iii). Assume that $f \neq 0 \neq g$ and f is bounded, then by (iii) of Theorem 5 for $\varphi(x) = \varepsilon$ for all $x \in G$ we get

$$|g(x)| \leq \frac{\varepsilon + 2M}{2M} \quad \text{for all} \quad x \in G.$$

Hence, g is bounded.

Ad (iv). If g is non-zero function and f is unbounded, then from (iv) of Theorem 5 for all $x \in G$ we obtain

$$|f(x) - g(x)f(0)| \leq \frac{\varepsilon}{2}.$$

Now, by assumption that f is unbounded, we infer that $f(0) \neq 0$ and g is unbounded, too. From (iii) of Theorem 6, we conclude that there exists function $h : G \to \mathbb{C}$ such that

$$f(x+y) + f(x-y) = 2h(x)f(y) \quad \text{for all} \quad x, y \in G$$

and

$$|h(x) - g(x)| \leq C$$

for all $x \in G$, where $C := \inf \left\{ \frac{\varphi(y)}{2|f(y)|} : x \in G,\ f(y) \neq 0 \right\}$. In this case $\varphi = \varepsilon$ and f is unbounded, thus $C = 0$. Hence, for all $x \in G$ we have

$$h(x) = g(x)$$

and consequently

$$f(x+y) + f(x-y) = 2g(x)f(y) \quad \text{for all} \quad x, y \in G.$$

□

The above corollary, in the light of Definition 3 (or Definition 5), yields superstability of the second generalization of d'Alembert's functional equation in the special case $\varphi(x) \equiv \varepsilon = \text{const}$.

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