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CONSTRUCTION OF TRIGONOMETRY USING THE SCALAR PRODUCT

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Abstract
At school, trigonometry is constructed traditionally starting from the definition of trigonometric functions in right triangle and generalizing those definitions for arbitrary angles in rectangular coordinates without using vector tools.

The method of construction of trigonometry using the scalar product was presented in the article [1] published in 1961 in Göttingen. This work is difficult of access. We intend to popularize this method, because it is general, brief and clear.

1. Scalar product of vectors

In linear space V over field of real numbers, the scalar product $\circ$ is defined as a function $\circ : V \times V \to \mathbb{R}$ which for arbitrary $\vec{a}, \vec{b} \in V$ and arbitrary $k, l \in \mathbb{R}$ fulfils the following conditions:

(1) $$(k\vec{a} + l\vec{b}) \circ \vec{c} = k(\vec{a} \circ \vec{b}) + l(\vec{b} \circ \vec{c}),$$

(2) $$\vec{a} \circ \vec{b} = \vec{b} \circ \vec{a},$$

(3) $\vec{a} \neq \vec{0} \;\Rightarrow\; \vec{a} \circ \vec{a} > 0.$

The norm (the length) of vector $\vec{a}$ is defined as:

(4) $|\vec{a}| = \sqrt{\vec{a} \circ \vec{a}},$

whereas the distance between the vectors $\vec{a}$ and $\vec{b}$ is described by

(5) $\rho(\vec{a}, \vec{b}) = |\vec{a} - \vec{b}|$

The unit vector for the vector $\vec{a}$ is enterpreted as

(6) $\vec{a}_0 = \dfrac{1}{|\vec{a}|} \cdot \vec{a}.$

Then $|\vec{a}_0| = \left|\dfrac{1}{|\vec{a}|} \cdot \vec{a}\right| = \left|\dfrac{1}{|\vec{a}|}\right| \cdot |\vec{a}| = \dfrac{1}{|\vec{a}|} \cdot |\vec{a}| = 1.$

The following formulae are fulfilled:

(7) $|\vec{a}|^2 = \vec{a} \circ \vec{a}, \qquad \vec{a} = |\vec{a}| \cdot \vec{a}_0.$

Traditionally, the scalar product of two vector is defined as

(8) $\vec{a} \circ \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\langle \vec{a}, \vec{b} \rangle$

For teaching at school, the scalar product in rectangular coordinates on a plane can be defined as follows:

$$\text{If } \vec{a} = (a_x, a_y), \quad \vec{b} = (b_x, b_y), \quad \text{then} \quad \vec{a} \circ \vec{b} = a_x b_x + a_y b_y.$$

The csalar product defined in such a manner fulfills the conditions (1)–(3).

In a linear space V with the scalar product „ $\circ$”, the following rules assert:

– the parallelogram rule:

$$|\vec{a} + \vec{b}| + |\vec{a} - \vec{b}| = 2(|\vec{a}|^2 + |\vec{b}|^2);$$

– the Pythagorean theorem:

$$\text{if } \vec{a} \circ \vec{b} = 0, \quad \text{then} \quad |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2.$$

The above-mentioned rules follow immediately from (1), (2) and (4).

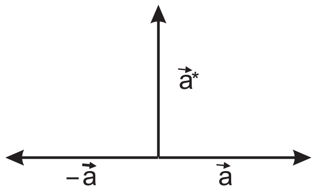

Fig. 1

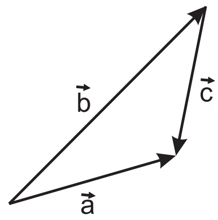

Fig. 2

Let us introduce the following notation (in rectangular coordinates on a plane) (Fig.1):

$-\vec{a}$ is the vector opposite to the vector $\vec{a}$;

$\vec{a}^*$ is the vector perpendicular to the vector $\vec{a}$ such that $|\vec{a}^*| = |\vec{a}|$.

Let $\vec{c} = \vec{a} - \vec{b}$ (Fig. 2). Then, accoding to the properties (1), (2) and (7), we have:

$$(\vec{a} - \vec{b})^2 = (\vec{a} - \vec{b}) \circ (\vec{a} - \vec{b}) = \vec{a}^2 - 2(\vec{a} \circ \vec{b}) + \vec{b}^2 =$$
$$= |\vec{a}|^2 - 2(\vec{a} \circ \vec{b}) + |\vec{b}|^2 = \vec{c}^2 = |\vec{c}|^2,$$

therefore,

(9) $\vec{a} \circ \vec{b} = \frac{1}{2}(|\vec{a}|^2 + |\vec{b}|^2 - |\vec{c}|^2).$

For non-vanishing vectors $\vec{a}$ and $\vec{b}$, we get from (7)

(10) $$\vec{a}_0 \circ \vec{b}_0 = \frac{1}{2|\vec{a}| \cdot |\vec{b}|} \cdot \left(|\vec{a}|^2 + |\vec{b}|^2 - |\vec{c}|^2\right).$$

Moreover, we have

(11) $$\vec{a}_0 \circ \vec{b}_0 = \vec{a}_0^* \circ \vec{b}_0^*, \qquad (\vec{a}_0^*)^* = -\vec{a}_0.$$

On the basis of (10), we obtain for the unit vector:

(12) $$\vec{e} \circ \vec{e} = 1, \quad \text{if} \quad \vec{c} = \vec{0},$$

(13) $$\vec{e} \circ \vec{e}^* = 0, \quad \text{if} \quad |\vec{c}| = \sqrt{2}|\vec{e}|.$$

2. Definition and basic properties of trigonometric functions

The number $\vec{a}_0 \circ \vec{b}_0$ can be interpreted as the cosine function of the angle γ between the vectors $\vec{a}_0$ and $\vec{b}_0$ (Fig. 3), i.e.

$$\cos\gamma = \cos\langle\vec{a}_0, \vec{b}_0\rangle = \vec{a}_0 \circ \vec{b}_0. \tag{14}$$

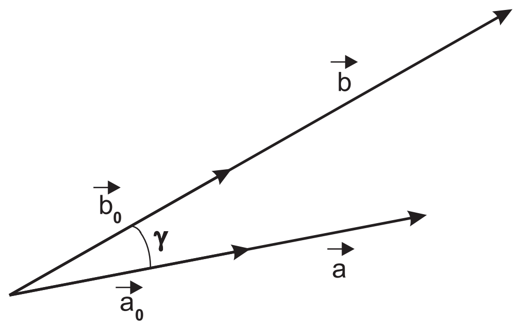

Fig. 3

It can be seen from (10) that

$$\cos(-\gamma) = \cos\langle\vec{b}_0, \vec{a}_0\rangle = \cos\langle\vec{a}_0, \vec{b}_0\rangle = \cos\gamma. \tag{15}$$

Hence, the cosine function is an even function. It is evident that

$$\vec{a} \circ \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot (\vec{a}_0 \circ \vec{b}_0) = |\vec{a}| \cdot |\vec{b}| \cdot \cos\gamma. \tag{16}$$

The sine function of the angle γ between the vectors $\vec{a}_0$ and $\vec{b}_0$ is defined as follows:

$$\sin\gamma = \sin\langle\vec{a}_0, \vec{b}_0\rangle = \vec{a}_0^{\,*} \circ \vec{b}_0. \tag{17}$$

From (11) we get:

$$\vec{a}_0^{\,*} \circ \vec{b}_0 = (\vec{a}_0^{\,*})^* \circ \vec{b}_0^{\,*} = (-\vec{a}_0) \circ \vec{b}_0^{\,*} = -(\vec{b}_0^{\,*} \circ \vec{a}_0),$$

hence (see Fig. 4), $\sin\langle\vec{b}_0^{\,*}, \vec{a}_0\rangle = -\sin\langle\vec{a}_0^{\,*}, \vec{b}_0\rangle$ or

$$\sin(-\gamma) = -\sin\gamma. \tag{18}$$

Therefore, the sine function is an odd function.

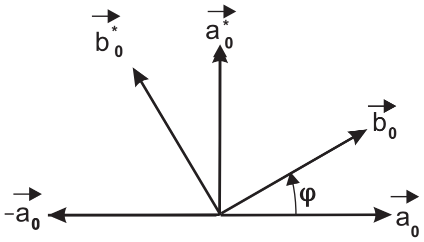

Fig. 4

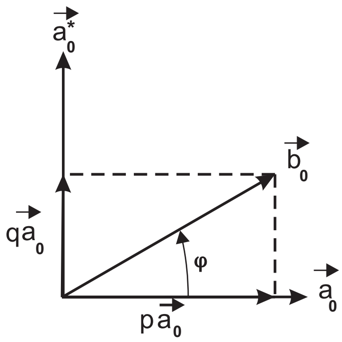

Fig. 5

For the unit vectors $\vec{e}_1, \vec{e}_2$, we have:

(19) $$\text{If } \varphi = \langle \vec{e}_1, \vec{e}_2 \rangle, \text{ then } \begin{cases} \cos\varphi = \cos\langle \vec{e}_1, \vec{e}_2 \rangle = \vec{e}_1 \circ \vec{e}_2, \\ \sin\varphi = \sin\langle \vec{e}_1, \vec{e}_2 \rangle = \vec{e}_1^* \circ \vec{e}_2. \end{cases}$$

Now, we prove that

(20) $$\sin^2\varphi + \cos^2\varphi = 1.$$

Proof:
Let $\vec{b}_0 = p\vec{a}_0 + q\vec{a}_0^*$ (Fig. 5). Then $\vec{a}_0 \circ \vec{b}_0 = \vec{a}_0 \circ (p\vec{a}_0 + q\vec{a}_0^*) = p$, as

$\vec{a}_0 \circ \vec{a}_0 = 1,\ \vec{a}_0 \circ \vec{a}_0^* = 0;$

$\vec{a}_0^* \circ \vec{b}_0^* \circ (p\vec{a}_0 + q\vec{a}_0^*) = q$, gdyż $\vec{a}_0^* \circ \vec{a}_0 = 0,\ \vec{a}_0^* \circ \vec{a}_0^* = 1.$

According to definitions (14) and (17) we obtain

$$\sin^2\varphi + \cos^2\varphi = (\vec{a}_0^* \circ \vec{b}_0)^2 + (\vec{a}_0 \circ \vec{b}_0)^2 = q^2 + p^2 = |\vec{b}_0| = 1.$$

3. Trigonometric functions in right-angled triangle

In a right-angled triangle based on vectors $\vec{a}$ i $\vec{b}$ (Fig. 6.), we have

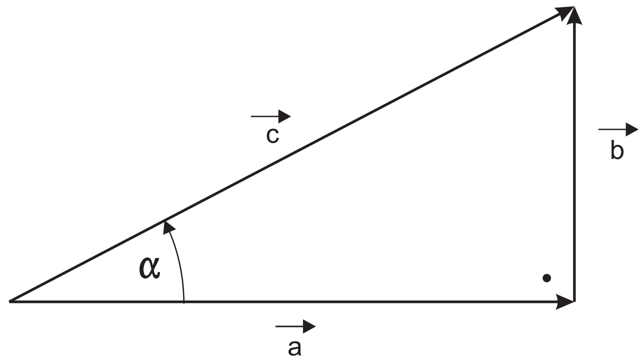

Fig. 6

$$\vec{c} = \vec{a} + \vec{b}, \qquad |\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2,$$

$$\vec{a} \circ \vec{c} = \tfrac{1}{2}(|\vec{a}|^2 + |\vec{c}|^2| - |\vec{b}|^2) = \tfrac{1}{2}(|\vec{a}|^2 + |\vec{a}|^2) = |\vec{a}|^2$$

or

$$\vec{a} \circ \vec{c} = (|\vec{a}| \cdot \vec{a}_0) \circ (|\vec{c}| \cdot \vec{c}_0) = |\vec{a}| \cdot |\vec{c}| \cdot (\vec{a}_0 \circ \vec{c}_0) = |\vec{a}|^2,$$

therefore

$$(21) \qquad \vec{a}_0 \circ \vec{c}_0 = \frac{|\vec{a}|}{|\vec{c}|}.$$

Similarly,

$$\vec{b} \circ \vec{c} = \tfrac{1}{2}(|\vec{b}|^2 + |\vec{c}|^2 - |\vec{a}|^2) = \tfrac{1}{2}(|\vec{b}|^2 + |\vec{b}|^2) = |\vec{b}|^2$$

or

$$\vec{b} \circ \vec{c} = (|\vec{b}| \cdot \vec{b}_0) \circ (|\vec{c}| \cdot \vec{c}_0) = |\vec{a}| \cdot |\vec{c}| \cdot (\vec{b}_0 \circ \vec{c}_0) = |\vec{b}| \cdot |\vec{c}| \cdot (\vec{a}_0^* \circ \vec{c}_0) = |\vec{b}|^2.$$

Hence,

$$(22) \qquad \vec{a}_0^* \circ \vec{c}_0 = \frac{|\vec{e}|}{|\vec{c}|}.$$

Taking into account formulae (21), (22) and defifnitions (14), (17), we get

$$\cos\alpha = \frac{|\vec{a}|}{|\vec{c}|}, \qquad \sin\alpha = \frac{|\vec{b}|}{|\vec{c}|}. \tag{23}$$

The tangent and cotangent functions are defined as follows:

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \qquad \operatorname{cotan}\alpha = \frac{\cos\alpha}{\sin\alpha}.$$

Then on the basis of (23) in a right-angled triangle we obtain

$$\tan\alpha = \frac{|\vec{b}|}{|\vec{a}|}, \qquad \operatorname{cotan}\alpha = \frac{|\vec{a}|}{|\vec{b}|}. \tag{24}$$

Definitions (14) and (17) allow us to obtain easily the reduction formulae and formulae describing trigonometric functions of the sum and difference of angles and to ground the sine and cosine theorems.

4. Reduction formulae

Consider the unit vectors shown in Fig. 7:

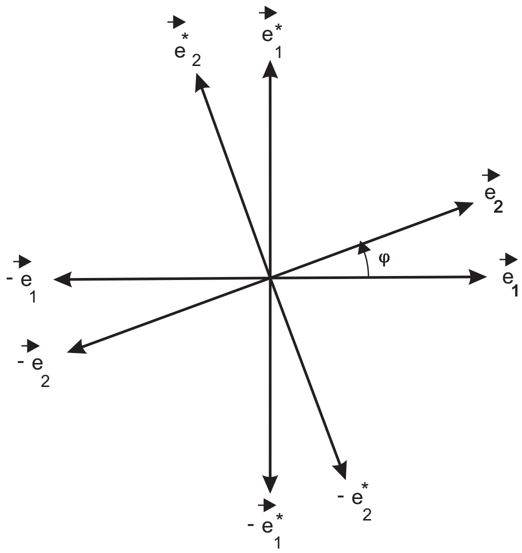

Fig. 7

From definitions (14) and (17) we have:

$$\sin\left(\frac{\pi}{2} - \varphi\right) = \sin\langle \vec{e}_2, \vec{e}_1^{\,*}\rangle = \vec{e}_2^{\,*} \circ \vec{e}_1^{\,*} = \vec{e}_1 \circ \vec{e}_2 = \cos\varphi,$$

$$\sin\left(\frac{\pi}{2}+\varphi\right)=\sin\langle\vec{e}_1,\vec{e}_2^{\,*}\rangle=\vec{e}_1^{\,*}\circ\vec{e}_2^{\,*}=\vec{e}_1\circ\vec{e}_2=\cos\varphi,$$

$$\sin(\pi-\varphi)=\sin\langle\vec{e}_2-\vec{e}_1\rangle=\vec{e}_2^{\,*}\circ(-\vec{e}_1)=-(\vec{e}_2^{\,*}\circ\vec{e}_1)=$$
$$=-((\vec{e}_2^{\,*})^*\circ\vec{e}_1^{\,*})=-((-\vec{e}_2)\circ\vec{e}_1^{\,*})=\vec{e}_1^{\,*}\circ\vec{e}_2=\sin\varphi,$$

$$\sin(\pi+\varphi)=\sin\langle\vec{e}_1,-\vec{e}_2\rangle=\vec{e}_1^{\,*}\circ(-\vec{e}_2)=-(\vec{e}_1^{\,*}\circ\vec{e}_2)=-\sin\varphi,$$

$$\sin\left(\frac{3}{2}\pi-\varphi\right)=\sin\langle\vec{e}_2,-\vec{e}_1^{\,*}\rangle=\vec{e}_2^{\,*}\circ(-\vec{e}_1^{\,*})=-(\vec{e}_1\circ\vec{e}_2)=-\cos\varphi,$$

$$\sin\left(\frac{3}{2}\pi+\varphi\right)=\sin\langle\vec{e}_1,-\vec{e}_2^{\,*}\rangle=\vec{e}_1^{\,*}\circ(-\vec{e}_2^{\,*})=-(\vec{e}_1\circ\vec{e}_2)=-\cos\varphi,$$

$$\sin(2\pi-\varphi)=\sin\langle\vec{e}_2,\vec{e}_1\rangle=\vec{e}_2^{\,*}\circ\vec{e}_1=\vec{e}_1^{\,*}\circ\vec{e}_2=\sin\varphi,$$

$$\cos\left(\frac{\pi}{2}-\varphi\right)=\cos\langle\vec{e}_2,\vec{e}_1^{\,*}\rangle=\vec{e}_2\circ\vec{e}_1^{\,*}=\vec{e}_1^{\,*}\circ\vec{e}_2=\sin\varphi$$

$$\cos\left(\frac{\pi}{2}+\varphi\right)=\cos\langle\vec{e}_1,\vec{e}_2^{\,*}\rangle=\vec{e}_1\circ\vec{e}_2^{\,*}=\vec{e}_1^{\,*}\circ(\vec{e}_2^{\,*})^*=$$
$$=\vec{e}_1^{\,*}\circ(-\vec{e}_2)=-(\vec{e}_1^{\,*}\circ\vec{e}_2)=-\sin\varphi,$$

$$\cos(\pi-\varphi)=\cos\langle\vec{e}_2,-\vec{e}_1\rangle=\vec{e}_2\circ(-\vec{e}_1)=-(\vec{e}_1\circ\vec{e}_2)=-\cos\varphi,$$

$$\cos(\pi+\varphi)=\cos\langle\vec{e}_1,-\vec{e}_2\rangle=\vec{e}_1\circ(-\vec{e}_2)=-(\vec{e}_1\circ\vec{e}_2)=-\cos\varphi,$$

$$\cos\left(\frac{3}{2}\pi-\varphi\right)=\cos\langle\vec{e}_2,-\vec{e}_1^{\,*}\rangle=\vec{e}_2\circ(-\vec{e}_1^{\,*})=-(\vec{e}_1^{\,*}\circ\vec{e}_2)=-\sin\varphi,$$

$$\cos\left(\frac{3}{2}\pi+\varphi\right)=\cos\langle\vec{e}_1,-\vec{e}_2^{\,*}\rangle=\vec{e}_1\circ(-\vec{e}_2^{\,*})=-(\vec{e}_1\circ\vec{e}_2^{\,*})=$$
$$=-(\vec{e}_1^{\,*}\circ(\vec{e}_2^{\,*})^*)=-(\vec{e}_1^{\,*}\circ(-\vec{e}_2))=\vec{e}_1^{\,*}\circ\vec{e}_2=\sin\varphi,$$

$$\cos(2\pi-\varphi)=\cos\langle\vec{e}_2,\vec{e}_1\rangle=\vec{e}_2\circ\vec{e}_1=\vec{e}_1\circ\vec{e}_2=\cos\varphi.$$

5. The trigonometric functions of sum and difference of angles

Let $\vec{e}_1 = p\vec{e}_3 + q\vec{e}_3^*$ (Fig. 8).

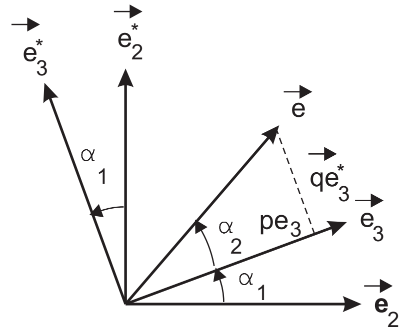

Fig. 8

Then

$$\vec{e}_3 \circ \vec{e}_1 = p = \cos\alpha_2, \quad \vec{e}_3^* \circ \vec{e}_1 = q = \sin\alpha_2.$$

Hence,

$$\vec{e}_1 = \vec{e}_3 \cdot \cos\alpha_2 + \vec{e}_3^* \cdot \sin\alpha_2. \tag{25}$$

Multiplying the above equality scalarwise by $\vec{e}_2$, we obtain

$$\vec{e}_1 \circ \vec{e}_2 = (\vec{e}_2 \circ \vec{e}_2)\cos\alpha_2 + (\vec{e}_2 \circ \vec{e}_3^*)\sin\alpha_2. \tag{26}$$

As $\vec{e}_1 \circ \vec{e}_2 = \cos(\alpha_1 + \alpha_2)$, $(\vec{e}_2 \circ \vec{e}_3) = \cos\alpha_1$, then

$$\vec{e}_2 \circ \vec{e}_3^* = \vec{e}_2^* \circ (\vec{e}_3^*)^* = \vec{e}_2^* \circ (-\vec{e}_3) = -(\vec{e}_2^* \circ \vec{e}_3) = -\sin\alpha_1$$

and, hence, Eq. (26) takes the form

$$\cos(\alpha_1 + \alpha_2) = \cos\alpha_1 \cdot \cos\alpha_2 - \sin\alpha_1 \cdot \sin\alpha_2. \tag{27}$$

For the expression $\sin(\alpha_1 + \alpha_2)$, we obtain on the basis of (25)
$\sin(\alpha_1 + \alpha_2) = \sin\langle\vec{e}_2, \vec{e}_1\rangle = \vec{e}_2^* \circ \vec{e}_1 = \vec{e}_2^* \circ (\vec{e}_3 \cdot \cos\alpha_2 + \vec{e}_3^* \cdot \sin\alpha_2) =$
$= (\vec{e}_2^* \circ \vec{e}_3) \cdot \cos\alpha_2 + (\vec{e}_2^* \circ \vec{e}_3^*) \cdot \sin\alpha_2.$

As $\vec{e}_2^* \circ \vec{e}_3 = \sin\alpha_1$, $\vec{e}_2^* \circ \vec{e}_3^* = \vec{e}_2 \circ \vec{e}_3 = \cos\alpha_1$, then

(28) $\sin(\alpha_1 + \alpha_2) = \sin\alpha_1 \cdot \cos\alpha_2 + \cos\alpha_1 \cdot \sin\alpha_2.$

Using Eqs. (27) and (28) and taking into account the even property of cosine function and odd property of sine function, we obtain

$$\sin(\alpha_2 - \alpha_1) = \sin[\alpha_2 + (-\alpha_1)] = \sin\alpha_2 \cdot \cos(-\alpha_1) + \cos\alpha_2 \cdot \sin(-\alpha_1)$$

or

(29) $\sin(\alpha_2 - \alpha_1) = \sin\alpha_2 \cdot \sin\alpha_1 - \cos\alpha_2 \cdot \sin\alpha_1$

$$\cos(\alpha_2 - \alpha_1) = \cos[\alpha_2 + (-\alpha_1)] = \cos\alpha_2 \cdot \cos(-\alpha_1) - \sin\alpha_2 \cdot \sin(-\alpha_1)$$

or

(30) $\cos(\alpha_2 - \alpha_1) = \cos\alpha_2 \cdot \cos\alpha_2 \cdot \cos\alpha_1 + \sin\alpha_2 \cdot \sin\alpha_1.$

6. The cosine theorem and the sine theorem

Consider the following triangle (Fig. 9)

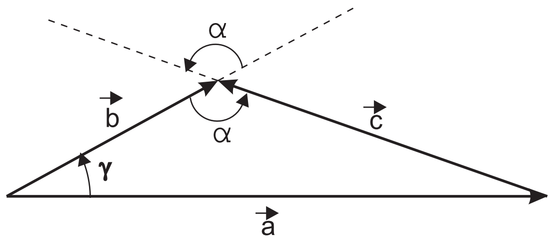

Fig. 9

Since $\vec{c} = \vec{b} - \vec{a}$, then

$|\vec{c}|^2 = \vec{c}^2 = (\vec{b} - \vec{a})^2 = \vec{b}^2 - 2(\vec{a} \circ \vec{b}) + \vec{c}^2 = |\vec{b}|^2 - 2(\vec{a} \circ \vec{b}) + |\vec{c}|^2.$

Hence, on the basis of Eq. (8), we obtain the cosine theorem

(31) $|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}| \cdot |\vec{b}| \cdot \cos\gamma.$

The sine theorem can be obtained as follows.

As $\vec{a} = \vec{b} - \vec{c}$ (Fig. 9), then we can multiply both sides of this equality scalarly by the vector $-\vec{b}_0^*$:

$\vec{a} \circ (-\vec{b}_0^*) = (\vec{b} - \vec{c}) \circ (-\vec{b}_0^*) = -(\vec{b} \circ \vec{b}_0^*) + \vec{c} \circ \vec{b}_0^*.$

Therefore,

$$-\vec{a} \circ \vec{b}_0^* = \vec{c} \circ \vec{b}_0^*, \quad \text{as} \quad \vec{b} \circ \vec{b}_0^* = 0.$$

Introducing the unit vectors, we get

(32) $$-|\vec{a}|\cdot(\vec{a}_0\circ\vec{b}_0^*)=|\vec{c}|\cdot(\vec{c}_0\circ\vec{b}_0^*).$$

Since

$\vec{a}_0\circ\vec{b}_0 = \vec{a}_0^*\circ(\vec{b}_0^*)^* = \vec{a}_0^*\circ(-\vec{b}_0) = -\vec{a}_0^*\circ\vec{b}_0 = -\sin\langle\vec{a}_0,\vec{b}_0\rangle = -\sin\gamma$,

then

$$\vec{c}_0\circ\vec{b}_0^*=\vec{b}_0^*\circ\vec{c}_0=\sin\langle\vec{b}_0,\vec{c}_0\rangle=\sin\alpha.$$

Hence, on the basis of (32), we have:

$$-|\vec{a}|\cdot(-\sin\gamma)=|\vec{c}|\cdot\sin\alpha.$$

From this equation, the sine theorem follows:

(33) $$\frac{|\vec{a}|}{\sin\alpha}=\frac{|\vec{c}|}{\sin\gamma}.$$

Remark.

The functions sin and cos can also be introduced using the Schwarz theorem:

(34) $$|\vec{a}\circ\vec{b}|\le|\vec{a}|\cdot|\vec{b}|.$$

Proof:

For an arbitrary number $\lambda\in R$ we have:

$0\le(\lambda\vec{a}-\vec{b})^2=\lambda\vec{a}^2-2\lambda(\vec{a}\circ\vec{b})+\vec{b}^2$

or

$$|\vec{a}|^2\cdot\lambda^2-2(\vec{a}\circ\vec{b})\lambda+|\vec{b}|^2\ge 0.$$

Because this quadratic polynomial (in λ) is always non-negative, then

$$\Delta=4(\vec{a}\circ\vec{b})-4|\vec{a}|^2\cdot|\vec{b}|^2\le 0,$$

hence,

$$(\vec{a}\circ\vec{b})^2\le|\vec{a}|^2\cdot|\vec{b}|^2 \quad\text{or}\quad \sqrt{(\vec{a}\circ\vec{b})^2}\le|\vec{a}|\cdot|\vec{b}|.$$

Finally, we get $|\vec{a}\circ\vec{b}|\le|\vec{a}|\cdot\vec{b}|$.

Expressing the non-vanishing vectors $\vec{a}$ and $\vec{b}$ in terms of unit vectors, we obtain

$$|(|\vec{a}|\cdot\vec{a}_0)\circ(|\vec{b}|\cdot\vec{b}_0)|\le|\vec{a}|\cdot|\vec{b}|,$$

whence

$$|\vec{a}_0 \circ \vec{b}_0| \leq 1, \quad -1 \leq \vec{a}_0 \circ \vec{b}_0 \leq 1.$$

The number $\vec{a}_0 \circ \vec{b}_0$ is interpreted as the cosine of the angle between the vectors $\vec{a}_0$ and $\vec{b}_0$:

$$\cos\langle \vec{a}_0, \vec{b}_0 \rangle = \vec{a}_0 \circ \vec{b}_0.$$

The Schwarz theorem for the vectors $\vec{a}^*, \vec{b}$ has the following form:

$$|\vec{a}^* \circ \vec{b}| \leq |\vec{a}^*| \cdot |\vec{b}|,$$

whence, similarly to the preceding consideration, we get

$$-1 \leq \vec{a}_0^* \circ \vec{b}_0 \leq 1.$$

The number $\vec{a}_0^* \circ \vec{b}_0$ is interpreted as the sine of the angle between the vectors $\vec{a}_0$ and $\vec{b}_0$, i.e.

$$\sin\langle \vec{a}_0, \vec{b}_0 \rangle = \vec{a}_0^* \circ \vec{b}_0.$$

It should be mentioned that the proposed vector approach to trigonometry can be also extended on spherical trigonometry [1].

References

[1] H. Athen. Vectorielle Begründung der Trigonometrie. *Mathematisch-Physikalische Semesterberichte*, Ed. Vandenhoeck & Ruprecht, Göttingen, 8 (1), 83-94, 1961.