Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XV, Częstochowa 2010

INDECOMPOSABLE PROJECTIVE REPRESENTATIONS OF DIRECT PRODUCTS OF FINITE GROUPS OVER A RING OF FORMAL POWER SERIES

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Abstract. Let F be a field of characteristic $p > 0$, $S = F[[X]]$ the ring of formal power series in the indeterminate X with coefficients in the field F, F^* the multiplicative group of F, $G = G_p \times B$ a finite group, where G_p is a p-group and B is a p'-group. We give necessary and sufficient conditions for G and F under which there exists a cocycle $\lambda \in Z^2(G, F^*)$ such that every indecomposable projective S-representation of G with the cocycle λ is the outer tensor product of an indecomposable projective S-representation of G_p and an irreducible projective S-representation of B.

1. Introduction

Let F be a field of characteristic $p > 0$ and $G = G_p \times B$, where G_p is a Sylow p-subgroup. Blau [6] and Gudyvok [10, 11] proved that every finitely generated FG-module is the outer tensor product $V\#W$ of an indecomposable FG_p-module V and an irreducible FB-module W if and only if either G_p is cyclic or F is a splitting field for B. Gudyvok [12, 13] also investigated a similar problem for group rings KG, where K is a complete discrete valuation ring. In particular, he proved that if K is of characteristic $p > 0$ and T is the quotient field of K, then every indecomposable KG-module is of the form $V\#W$ if and only if either $|G_p| = 2$ or T is a splitting field for B. In the paper [2], the results of Blau and Gudyvok were generalized to the twisted group rings $S^\lambda G$, where $G = G_p \times B$, $S = F$ or S is a complete discrete valuation ring of characteristic $p > 0$.

In this paper we continue the study of indecomposable projective representations of $G = G_p \times B$ over the ring $S = F[[X]]$ as begun in [2].

Let us present the main results of the paper. We assume that F is a field of characteristic $p > 0$, S^* the unit group of S, $|G_p| \neq 1$, $|B| \neq 1$, and if G_p is non-Abelian, then F contains a primitive q^{th} root of 1 for every prime $q \mid |B|$ such that $p \mid (q-1)$. Given a cocycle $\lambda\colon G \times G \to S^*$ in $Z^2(G, S^*)$, we denote by $S^\lambda G$ the twisted group ring of the group G over the ring S with the 2-cocycle λ. By an $S^\lambda G$-module we mean a finitely generated left $S^\lambda G$-module which is S-free. Given $\mu \in Z^2(G_p, S^*)$, the kernel $\mathrm{Ker}(\mu)$ of μ is the union of all cyclic subgroups $\langle\, g\,\rangle$ of G_p such that the restriction of μ to $\langle\, g\,\rangle \times \langle\, g\,\rangle$ is a coboundary. We recall from [4, p. 268] that $G_p' \subset \mathrm{Ker}(\mu)$, $\mathrm{Ker}(\mu)$ is a normal subgroup of G_p and the restriction of μ to $\mathrm{Ker}(\mu) \times \mathrm{Ker}(\mu)$ is a coboundary (see also [3, p. 197] for a simple proof). Up to cohomology in $Z^2(G_p, S^*)$, we have $\mu_{g,a} = \mu_{a,g} = 1$ for all $g \in G_p$ and $a \in \mathrm{Ker}(\mu)$. In what follows, we assume that every cocycle $\mu \in Z^2(G_p, S^*)$ under consideration satisfies this condition. If H is a subgroup of G, then the restriction of $\lambda \in Z^2(G, S^*)$ to $H \times H$ will also be denoted by λ. In this case, $S^\lambda H$ is a subring of $S^\lambda G$. A group G is of symmetric type if it decomposes into a direct product of two isomorphic groups. Denote

$$i(F) = \begin{cases} t & \text{if } [F : F^p] = p^t, \\ \infty & \text{if } [F : F^p] = \infty. \end{cases}$$

Let $G = G_p \times B$, $\mu \in Z^2(G_p, S^*)$ and $\nu \in Z^2(B, S^*)$. Then the map $\mu \times \nu\colon G \times G \to S^*$ defined by

$$(\mu \times \nu)_{x_1 b_1, x_2 b_2} = \mu_{x_1, x_2} \cdot \nu_{b_1, b_2}$$

for all $x_1, x_2 \in G_p$, $b_1, b_2 \in B$ belongs to $Z^2(G, S^*)$. Every cocycle $\lambda \in Z^2(G, S^*)$ is cohomologous to $\mu \times \nu$, where μ is the restriction of λ to $G_p \times G_p$ and ν is the restriction of λ to $B \times B$. From now on, we suppose that each cocycle $\lambda \in Z^2(G, S^*)$ under consideration satisfies the condition $\lambda = \mu \times \nu$.

For any $\lambda = \mu \times \nu \in Z^2(G, S^*)$, we have $S^\lambda G \cong S^\mu G_p \otimes_S S^\nu B$. If every indecomposable $S^\lambda G$-module is isomorphic to the outer tensor product $V \# W$, where V is an indecomposable $S^\mu G_p$-module and W is an irreducible $S^\nu B$-module, then we will say that the ring $S^\lambda G$ is of OTP representation type.

Let Ω be a subgroup of S^*. We say that a group $G = G_p \times B$ is of OTP projective (S, Ω)-representation type if there exists a cocycle $\lambda \in Z^2(G, \Omega)$ such that the ring $S^\lambda G$ is of OTP representation type. A group $G = G_p \times B$ is defined to be of purely OTP projective (S, Ω)-representation type if $S^\lambda G$ is of OTP representation type for any $\lambda \in Z^2(G, \Omega)$. If $\Omega = S^*$, then instead of "(S, Ω)-representation type" we write "S-representation type".

In Section 3, we characterize twisted group rings of OTP representation type. Let $G = G_p \times B$, $\mu \in Z^2(G_p, S^*)$, $\nu \in Z^2(B, S^*)$, $\lambda = \mu \times \nu$ and $H = \mathrm{Ker}(\mu)$. In Theorem 1, we prove that if $|H| > 2$, then the ring $S^\lambda G$ is of OTP representation type if and only if F is a splitting field for the F-algebra $S^\nu B/XS^\nu B$. Assume that $|G'_p| \neq 2$, $\mu \in Z^2(G_p, F^*)$, $\nu \in Z^2(B, S^*)$ and $\lambda = \mu \times \nu$. In Proposition 3, we show that $S^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:

(i) $F^\mu G_p$ is a field;

(ii) $p = 2$, $|G'_2| = 1$ and $2\dim_F(F^\mu G_2/\operatorname{rad} F^\mu G_2) = |G_2|$;

(iii) F is a splitting field for the F-algebra $S^\nu B/XS^\nu B$.

In Section 4, we study the groups of OTP projective representation type. Let $G = G_p \times B$, $|G'_p| \neq 2$ and s be the number of invariants of G_p/G'_p. In Theorem 2, we prove that G is of OTP projective (S, F^*)-representation type if and only if one of the following conditions is satisfied:

(i) $|G'_p| = 1$ and $s \leq i(F)$;

(ii) $p = 2$, $|G'_2| = 1$, $s = i(F) + 1$ and G_2 has at least one invariant equal to 2;

(iii) F is a splitting field for $F^\sigma B$ for some $\sigma \in Z^2(B, F^*)$.

Let $G = G_p \times B$ be an Abelian group and s the number of invariants of G_p. In Proposition 5, we establish that G is of OTP projective (S, F^*)-representation type if and only if one of the following conditions is satisfied:

(i) $s \leq i(F)$;

(ii) $p = 2$, $s = i(F) + 1$ and G_2 has at least one invariant equal to 2;

(iii) B has a subgroup H such that B/H is of symmetric type and F contains a primitive m^{th} root of 1, where $m = \max\{\exp(B/H), \exp H\}$.

In Section 5, we show in Theorem 3 that $G = G_p \times B$ is of purely OTP projective S-representation type if and only if $|G_p| = 2$ or F is a splitting field for any $F^\nu B$. Corollary to Theorem 3 asserts that if G is a nilpotent group, then G is of purely OTP projective S-representation type if and only if one of the following conditions is satisfied:

(i) $|G_p| = 2$;

(ii) $F = F^q$ and F contains a primitive q^{th} root of 1 for every prime $q \mid |B|$.

2. Preliminaries

Throughout this paper, we use the following notations: $p \geq 2$ is a prime; F is a field of characteristic $p > 0$; $S = F[[X]]$ is the ring of formal power series in the indeterminate X with coefficients in the field F; $P = XS$ is unique maximal ideal of S; F^* is the multiplicative group of F; $F^q = \{\alpha^q : \alpha \in F\}$; S^* is the unit group of S; $G = G_p \times B$ is a finite group, where G_p is a p-group and B is a p'-group; H' is the commutant of a group H, e is the identity element of H, $|h|$ is the order of $h \in H$; $\operatorname{soc} A$ is the socle of an Abelian group A and $\exp A$ is the exponent of A. We suppose that $|G_p| > 1$ and $|B| > 1$. Given a subgroup Ω of S^*, we denote by $Z^2(H, \Omega)$ the group of all Ω-valued normalized 2-cocycles of the group H, where we assume that H acts trivially on Ω. An S-basis $\{u_h \colon h \in H\}$ of $S^\lambda H$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in H$ is called natural (corresponding to $\lambda \in Z^2(H, S^*)$). Given an $S^\lambda H$-module V, we write $\operatorname{End}_{S^\lambda H}(V)$ for the ring of all $S^\lambda H$-endomorphisms of V, $\operatorname{rad}\operatorname{End}_{S^\lambda H}(V)$ for the Jacobson radical of $\operatorname{End}_{S^\lambda H}(V)$ and $\overline{\operatorname{End}_{S^\lambda H}(V)}$ for the quotient ring

$$\operatorname{End}_{S^\lambda H}(V)/\operatorname{rad}\operatorname{End}_{S^\lambda H}(V).$$

Moreover, we denote by $\widetilde{S^\lambda H}$ the F-algebra $S^\lambda H / X S^\lambda H$ and by $\widetilde{V}$ the factor module V/XV. Given $\lambda \in Z^2(H, F^*)$, $F^\lambda H$ denotes the twisted group algebra of H over F and $\overline{F^\lambda H}$ the quotient algebra of $F^\lambda H$ by the radical $\operatorname{rad} F^\lambda H$. We identify an element $a + P$, $a \in F$, of the field $\bar{S} = S/P$ with the element a.

Lemma 1. [8, p.125] *Let H be a finite group, $\lambda \in Z^2(H, S^*)$ and V an $S^\lambda H$-module. Then V is indecomposable if and only if $\overline{\operatorname{End}_{S^\lambda H}(V)}$ is a skewfield.*

Lemma 2. *Let H be a finite p-group, D a subgroup of H, $\lambda \in Z^2(H, S^*)$ and M an indecomposable $S^\lambda D$-module. Assume that $\overline{\operatorname{End}_{S^\lambda D}(M)}$ is isomorphic to a field K, $K \supset F$ and one of the following conditions is satisfied:*

(i) H is Abelian;

(ii) $[s(K) : F]$ is not divisible by p, where $s(K)$ is the separable closure of F in K.

Then $M^H := S^\lambda H \otimes_{S^\lambda D} M$ is an indecomposable $S^\lambda H$-module and

$$\overline{\operatorname{End}_{S^\lambda H}(M^H)}$$

is isomorphic to a field that is a finite purely inseparable extension of the field K.

The proof is similar to that of Lemma 2.2 [2, p.540]. It uses the same idea as in Theorem 8 of [9].

Lemma 3. *Let K be a finite separable extension of the field F and H a finite p-group. If $|H| > 2$, then there exists an indecomposable SH-module V such that $\overline{\operatorname{End}_{SH}(V)}$ is isomorphic to K.*

P r o o f. Let $K = F(\theta)$, $f(t)$ be the monic minimal polynomial of θ over F and Γ the companion matrix of $f(t)$. Assume that either H is cyclic of order

$|H| > 2$ or H is a group of type $(2,2)$. Let $H = \langle\, a \,\rangle$ and V be the underlying SH-module of the representation

$$a \mapsto \begin{pmatrix} E & XE & \Gamma \\ 0 & E & XE \\ 0 & 0 & E \end{pmatrix}$$

of H, where E is the identity matrix of order $n = \deg f(t)$. Then, by [13, pp. 70–71], $\overline{\mathrm{End}_{SH}(V)} \cong K$. If $H = \langle\, a \,\rangle \times \langle\, b \,\rangle$ is a group of type $(2,2)$, then as V we take the underlying SH-module of the representation

$$a \mapsto \begin{pmatrix} E & E \\ 0 & E \end{pmatrix}, \quad b \mapsto \begin{pmatrix} E & \Gamma \\ 0 & E \end{pmatrix}.$$

By [13, p. 71], we have $\overline{\mathrm{End}_{SH}(V)} \cong K$. □

Lemma 4. *Let $p = 2$, $[F : F^2] = 2$, H be a 2-group such that $|H| \neq 8$ and $|H'| = 2$. Assume also that K is a finite separable extension of the field F and $[K : F]$ is not divisible by 2. Then, for any $\lambda \in Z^2(H, F^*)$, there exists an indecomposable $S^\lambda H$-module V such that $\overline{\mathrm{End}_{S^\lambda H}(V)}$ is isomorphic to a field that is a finite purely inseparable extension of the field K.*

P r o o f. Let $H' = \langle c \rangle$, s be the number of invariants of the Abelian group H/H', D the subgroup of H such that $H' \subset D$ and $D/H' = \mathrm{soc}(H/H')$. We have

$$S^\lambda D / S^\lambda D(u_c - u_e) \cong S^{\bar\lambda}\bar D,$$

where $\bar D = D/H'$ and $\bar\lambda_{xH',yH'} = \lambda_{x,y}$ for all $x, y \in D$. Assume $s > 2$. Since $i(F) = 1$,

$$F^{\bar\lambda}\bar D \cong F^{\bar\lambda}\bar D_1 \otimes_F F\bar D_2,$$

where $\bar D = \bar D_1 \times \bar D_2$ and $|\bar D_2| \geq 4$. It follows that $S^{\bar\lambda}\bar D \cong S^{\bar\lambda}\bar D_1 \otimes_S S\bar D_2$. By Lemmas 2 and 3, there exists an indecomposable $S^{\bar\lambda}\bar D$-module V such that

$$\overline{\mathrm{End}_{S^{\bar\lambda}\bar D}(V)}$$

is a finite purely inseparable extension of the field K. The module V is also an $S^\lambda D$-module. In view of Lemma 2, V^H is an indecomposable $S^\lambda H$-module and

$$\overline{\mathrm{End}_{S^\lambda H}(V^H)}$$

is a finite purely inseparable extension of K.

Now we consider the case $s = 2$. Since $|H| > 8$, then D is Abelian. Let $D = \langle\, a \,\rangle \times \langle\, b \,\rangle$, where $a^2 = c$ and $b^2 = e$. Then

$$S^\lambda D = \bigoplus_{i,j,k} S u_a^i u_b^j u_c^k,$$

where

$$u_a^2 = \alpha u_c, \quad u_b^2 = \beta u_e, \quad u_c^2 = u_e$$

and $\alpha, \beta \in F^*$. If $\alpha \in F^2$, then $S[u_a]$ is the group ring of the group $\langle\, a \,\rangle$ over the ring S. If $\beta \in F^2$ then $S^\lambda D$ contains the group ring SQ, where $Q = \langle\, c \,\rangle \times \langle\, b \,\rangle$. Assume that $\alpha \notin F^2$ and $\beta \notin F^2$. Since $i(F) = 1$, $\alpha^{-1} = \delta_0^2 + \delta_1^2\beta$ for some $\delta_0, \delta_1 \in F$. Let $v = u_a(\delta_0 u_e + \delta_1 u_b)$. Then $v^2 = \alpha u_c \cdot \alpha^{-1} u_e = u_c$.

If $D = \langle\, a \,\rangle \times \langle\, b \,\rangle \times \langle\, c \,\rangle$ is of type $(2, 2, 2)$, then $S^\lambda D$ contains SQ, where Q is a group of type $(2, 2)$.

Applying Lemmas 2 and 3, we finish the proof. □

Lemma 5. *Let $G = G_p \times B$ and $\lambda \in Z^2(G, S^*)$. The ring $S^\lambda G$ is of OTP representation type if and only if the outer tensor product of any indecomposable $S^\lambda G_p$-module and any irreducible $S^\lambda B$-module is an indecomposable $S^\lambda G$-module.*

The proof is similar to that of the corresponding fact for a group ring (see [6, p. 41], [13, p. 68]).

Let B be a finite p'-group and $\lambda \in Z^2(B, S^*)$. We denote by $\widetilde{S^\lambda B}$ the F-algebra $S^\lambda B/XS^\lambda B$. For $y \in S^\lambda B$, let $\tilde{y}$ denote $y + XS^\lambda B$. The F-algebra $\widetilde{S^\lambda B}$ is separable. By Theorem 6.8 [8, p. 124], if

$$\widetilde{S^\lambda B} = \widetilde{S^\lambda B}\varepsilon_1 \oplus \ldots \oplus \widetilde{S^\lambda B}\varepsilon_n$$

is a decomposition into minimal left ideals, then there exists a decomposition

$$S^\lambda B = S^\lambda B e_1 \oplus \ldots \oplus S^\lambda B e_n,$$

where ε_i is an idempotent of $\widetilde{S^\lambda B}$, e_i is an idempotent of $S^\lambda B$ and $\tilde{e_i} = \varepsilon_i$ for every $i \in \{1, \ldots, n\}$. Each ideal $S^\lambda B e_i$ is an irreducible $S^\lambda B$-module. By Theorem 76.8 [7, p. 532] and Corollary 76.15 [7, p. 536], any irreducible $S^\lambda B$-module is isomorphic to $S^\lambda B e_j$ for some $j \in \{1, \ldots, n\}$. Moreover, by Proposition 5.22 [8, p. 112] and Theorem 76.8 [7, p. 532],

$$\overline{\mathrm{End}_{S^\lambda B}\, S^\lambda B e_j} \cong \mathrm{End}_{S^\lambda B}\, S^\lambda B e_j / X\, \mathrm{End}_{S^\lambda B}\, S^\lambda B e_j \cong \mathrm{End}_{\widetilde{S^\lambda B}}\, \widetilde{S^\lambda B}\varepsilon_j.$$

Lemma 6. *Let $G = G_p \times B$ and $\lambda \in Z^2(G, S^*)$. If V is an indecomposable $S^\lambda G_p$-module and W is an irreducible $S^\lambda B$-module, then*

$$\overline{\mathrm{End}_{S^\lambda G}(V \# W)} \cong \overline{\mathrm{End}_{S^\lambda G_p}(V)} \otimes_F \overline{\mathrm{End}_{S^\lambda B}(W)}.$$

P r o o f. By Proposition 7.6 [14, p. 652],

$$\mathrm{End}_{S^\lambda G}(V\#W) \cong \mathrm{End}_{S^\lambda G_p}(V) \otimes_S \mathrm{End}_{S^\lambda B}(W).$$

Applying Proposition 2 [6, p. 39], we obtain

$$\overline{\mathrm{End}_{S^\lambda G}(V\#W)} \cong \left(\overline{\mathrm{End}_{S^\lambda G_p}(V)} \otimes_F \overline{\mathrm{End}_{S^\lambda B}(W)}\right)/R,$$

where $R := \mathrm{rad}\left(\overline{\mathrm{End}_{S^\lambda G}(V)} \otimes_F \overline{\mathrm{End}_{S^\lambda B}(W)}\right)$. Since $\overline{\mathrm{End}_{S^\lambda B}(W)}$ is a separable F-algebra, then

$$\overline{\mathrm{End}_{S^\lambda G_p}(V)} \otimes_F \overline{\mathrm{End}_{S^\lambda B}(W)}$$

is a semisimple algebra. Hence $R = 0$ and the result follows. □

Lemma 7. *Let $G = G_p \times B$ and $\lambda \in Z^2(G, S^*)$. If F is a splitting field for the algebra $\widetilde{S^\lambda B}$, then $S^\lambda G$ is of OTP representation type.*

P r o o f. Let W be an irreducible $S^\lambda B$-module. Then

$$\overline{\mathrm{End}_{S^\lambda B}\, W} \cong \mathrm{End}_{\widetilde{S^\lambda B}}\, \widetilde{W} \cong F,$$

where $\widetilde{W} = W/XW$. By Lemmas 1 and 6, $V\#W$ is an indecomposable $S^\lambda G$-module for every indecomposable $S^\lambda G_p$-module V. By Lemma 5, $S^\lambda G$ is of OTP representation type. □

Lemma 8. *Let B be a finite p'-group. Assume that F contains a primitive q^{th} root of 1 for every prime $q \mid |B|$ such that $p \mid (q-1)$. Then, for any F-algebra $\widetilde{S^\lambda B}$, there exists a splitting field K such that $[K : F]$ is not divisible by p.*

P r o o f. See [2, p. 548]. □

Proposition 1. *Let $S = F[[X]]$, T be the quotient field of S, B a finite p'-group and $\lambda \in Z^2(B, S^*)$. The field T is a splitting field for the algebra $T^\lambda B$ if and only if F is a splitting field for the F-algebra $\widetilde{S^\lambda B}$.*

P r o o f. Assume that T is a splitting field for $T^\lambda B$. Denote by W an irreducible $S^\lambda B$-module. Since $T \otimes_S W$ is an absolutely irreducible $T^\lambda B$-module, by Schur's Lemma, $\mathrm{End}_{S^\lambda B}(W) \cong S$. It follows that

$$\mathrm{End}_{\widetilde{S^\lambda B}}(\widetilde{W}) \cong F. \tag{1}$$

Hence F is a splitting field for $\widetilde{S^\lambda B}$.

Now suppose that F is a splitting field for $\widetilde{S^\lambda B} = S^\lambda B/XS^\lambda B$. Then there exists an isomorphism (1) for any irreducible $S^\lambda B$-module W. It follows, by Theorem 76.8 [7, p. 532] and Corollary 76.16 [7, p. 536], that $\mathrm{End}_{S^\lambda B}(W) \cong S$, therefore $\mathrm{End}_{T^\lambda B}(T \otimes_S W) \cong T$. Hence T is a splitting field for $T^\lambda B$. □

3. Twisted group rings of OTP representation type

In this Section, $S = F[[X]]$ and $G = G_p \times B$, where G_p is a Sylow p-subgroup of G, $|G_p| \neq 1$ and $|B| \neq 1$. We assume that if G_p is non-Abelian, then F contains a primitive q^{th} root of 1 for every prime $q \mid |B|$ such that $p \mid (q-1)$.

Theorem 1. *Let $G = G_p \times B$, $\mu \in Z^2(G_p, S^*)$, $\nu \in Z^2(B, S^*)$, $\lambda = \mu \times \nu$ and $H = \mathrm{Ker}(\mu)$. Assume that $|H| > 2$. The ring $S^\lambda G$ is of OTP representation type if and only if F is a splitting field for $\widetilde{S^\nu B}$.*

P r o o f. If F is a splitting field for $\widetilde{S^\nu B}$, then, by Lemma 7, the ring $S^\lambda G$ is of OTP representation type.

Assume now that F is not a splitting field for $\widetilde{S^\nu B}$. There exists an irreducible $S^\nu B$-module W such that $D := \overline{\mathrm{End}_{S^\lambda B}(W)}$ is a division F-algebra of dimension greater than one. By [4, p. 268], the restriction of μ to $H \times H$ is a coboundary and $G_p' \subset H$. Suppose that G_p is non-Abelian. Then, by Lemma 8, there exists a splitting field K for $\widetilde{S^\nu B}$, which is a finite separable extension of the field F and satisfies $[K : F] \not\equiv 0 (\mathrm{mod}\, p)$. In view of Lemma 3, there is an indecomposable SH-module M such that $\overline{\mathrm{End}_{SH}(M)}$ is isomorphic to K. According to Lemma 2, we conclude that M^{G_p} is an indecomposable $S^\mu G_p$-module and

$$\overline{\mathrm{End}_{S^\mu G_p}(M^{G_p})}$$

is isomorphic to a field L that is a finite purely inseparable extension of the field K. Since L is a splitting field for D, $L \otimes_F D$ is not a skewfield. Hence, by Lemmas 1 and 6, $M^{G_p} \# W$ is not an indecomposable $S^\lambda G$-module. In view of Lemma 5, $S^\lambda G$ is not of OTP representation type.

The case, when G_p is Abelian, is treated similarly. □

Corollary. [2, p. 553] *Let $G = G_p \times B$, $|G_p'| > 2$ and $\lambda \in Z^2(G, S^*)$. The ring $S^\lambda G$ is of OTP representation type if and only if F is a splitting field for $\widetilde{S^\lambda B}$.*

P r o o f. Let μ be the restriction of λ to $G_p \times G_p$. Since $G_p' \subset \mathrm{Ker}(\mu)$, we have $|\mathrm{Ker}(\mu)| > 2$. Next apply Theorem 1. □

Proposition 2. *Let B be a nilpotent p'-group.*

(i) If the field F does not contain a primitive q^{th} root of 1 for some prime $q \mid |B|$, then F is not a splitting field for each algebra $F^\lambda B$.

(ii) The field F is a splitting field for all twisted group algebras $F^\lambda B$ if and only if $F = F^q$ and F contains a primitive q^{th} root of 1 for every prime $q \mid |B|$.

P r o o f. (i) Assume that F does not contain a primitive q^{th} root of 1 for some prime $q \mid |B|$. The center of a Sylow q-subgroup B_q of B contains an element b of order q. If $\{u_g : g \in B\}$ is a natural F-basis of the algebra $F^\lambda B$, then u_b lies in the center of $F^\lambda B$. Let $u_b^q = \gamma u_e$, $\gamma \in F^*$, and let F be a splitting field for the algebra $F^\lambda B$. Denote by $f_1, \ldots, f_m$ a complete system of minimal pairwise orthogonal central idempotents of $F^\lambda B$. We have $u_b = \beta_1 f_1 + \ldots + \beta_m f_m$, where $\beta_j \in F$ for any $j \in \{1, \ldots, m\}$. Then $\gamma = \beta_j^q$ for every j. It follows that $\beta_1 = \ldots = \beta_m$, hence $u_b = \beta_1 u_e$. This contradiction proves that F is not a splitting field for the algebra $F^\lambda B$.

(ii) Suppose that F is a splitting field for $F^\lambda B$ for each $\lambda \in Z^2(B, F^*)$. Then every irreducible projective F-representation of the group B is absolutely irreducible. Let q be a prime divisor of $|B|$. There exists a normal subgroup D of B such that $|B/D| = q$. Denote by $\pi : B \to B/D$ the canonical group homomorphism and by V a finite-dimensional vector space over F. If $\bar{\Gamma} : B/D \to \mathrm{GL}(V)$ is an irreducible projective F-representation of B/D on V, then $\Gamma := \bar{\Gamma} \circ \pi$ is an irreducible projective F-representation of B on the space V and $D \subset \mathrm{Ker}(\Gamma)$. Assume that $B/D = \langle bD \rangle$ and $\bar{\Gamma}(bD)^q = \gamma\, \mathrm{id}_V$, $\gamma \in F^*$. Since every $\bar{\Gamma}$ is absolutely irreducible, $\gamma \in F^q$ and F contains a primitive q^{th} root of 1.

Assume now that the field F contains a primitive q^{th} root of 1 and $F = F^q$ for each prime $q \mid |B|$. Let $\lambda \in Z^2(B, F^*)$. Then $F^\lambda B = F^\mu B$, where $\mu_{x,y}^{|B|} = 1$ for all $x, y \in B$. There exists an F-algebra homomorphism of FH onto $F^\mu B$, where H is a central extension of a cyclic group of order $|B|$ by the group B. Since F contains a primitive $|H|^{\text{th}}$ root of 1, by Corollary 70.24 [7, p. 475], F is a splitting field for FH. Hence, F is a splitting field for $F^\lambda B$ for each $\lambda \in Z^2(B, F^*)$. □

Proposition 3. *Let $G = G_p \times B$, $|G_p'| \neq 2$, $\mu \in Z^2(G_p, F^*)$, $\nu \in Z^2(B, S^*)$ and $\lambda = \mu \times \nu$. The ring $S^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:*

(i) $F^\mu G_p$ is a field;

(ii) $p = 2$, $|G_2'| = 1$ and $2 \dim_F \overline{F^\mu G_2} = |G_2|$;

(iii) F is a splitting field for the F-algebra $\widetilde{S^\nu B}$.

P r o o f. If $|G'_p| > 2$ then, by Corollary to Theorem 1, the ring $S^\lambda G$ is of OTP representation type if and only if F is a splitting field for $\widetilde{S^\nu B}$. Let $|G'_p| = 1$ and $K = F^\mu G_p$. If K is a field, then $S^\mu G_p = K[[X]]$ is a principal ideal ring. Every indecomposable $S^\mu G_p$-module is isomorphic to $S^\mu G_p$. We have

$$\overline{\mathrm{End}_{S^\mu G_p}(S^\mu G_p)} \cong S^\mu G_p / X S^\mu G_p \cong K.$$

The field K is a finite purely inseparable extension of F. Let W be an irreducible $S^\nu B$-module and $D := \overline{\mathrm{End}_{S^\nu B}(W)}$. Then $D \cong \mathrm{End}_{\widetilde{S^\nu B}}(\widetilde{W})$. Since $\widetilde{S^\nu B}$ is a separable algebra, the center of the division F-algebra D is a separable extension of F [7, p. 485]. The index of D is not divisible by p [16]. It follows that $K \otimes_F D$ is a skewfield. Applying Lemmas 1 and 6, we conclude that $S^\mu G_p \# W$ is an indecomposable $S^\lambda G$-module. Hence, by Lemma 5, $S^\lambda G$ is of OTP representation type.

Assume that $p > 2$ and K is not a field. Let H be the socle of G_p. We have $F^\mu H \cong F^\mu H_1 \otimes_F FH_2$, where $|H_2| \geq p$. It follows that $S^\mu H \cong S^\mu H_1 \otimes_S SH_2$. By Lemmas 2 and 3, for any finite separable extension L of the field F, there exists an indecomposable $S^\mu G_p$-module V such that $\overline{\mathrm{End}_{S^\mu G_p}(V)}$ is a finite purely inseparable extension of L. Arguing as in the proof of Theorem 1, we conclude that $S^\lambda G$ is of OTP representation type if and only if F is a splitting field for the algebra $\widetilde{S^\nu B}$.

Suppose that $p = 2$ and K is not a field. If $4 \dim_F \overline{F^\mu G_2} \leq |G_2|$ then, as in the case $p > 2$, we prove that $S^\lambda G$ is of OTP representation type if and only if F is a splitting field for the algebra $\widetilde{S^\nu B}$. If $2 \dim_F \overline{F^\mu G_2} = |G_2|$ then, by Theorem 4.2 [2, p. 552], the ring $S^\lambda G$ is of OTP representation type. □

Corollary. *Let G_p be an Abelian p-group, B a nilpotent p'-group, $G = G_p \times B$, $\mu \in Z^2(G_p, F^*)$, $\nu \in Z^2(B, S^*)$ and $\lambda = \mu \times \nu$. Assume that the field F does not contain a primitive q^{th} root of 1 for some prime $q \mid |B|$. The ring $S^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:*

(i) $F^\mu G_p$ is a field;

(ii) $p = 2$ and $2 \dim_F \overline{F^\mu G_2} = |G_2|$.

P r o o f. Apply Propositions 2 and 3. □

Proposition 4. *Let $p = 2$, $G = G_2 \times B$, $\mu \in Z^2(G_2, F^*)$, $\nu \in Z^2(B, S^*)$ and $\lambda = \mu \times \nu$. Assume that $|G_2| \neq 8$, $|G'_2| = 2$ and $[F : F^2] \leq 2$. Then $S^\lambda G$ is of OTP representation type if and only if F is a splitting field for $\widetilde{S^\nu B}$.*

P r o o f. If F is a perfect field, then μ is a coboundary [15, p. 43]. In this case $S^\mu G_2$ is the group ring SG_2. Since $|G_2| > 8$, by Theorem 1, $S^\lambda G$ is of OTP representation type if and only if F is a splitting field for $\widetilde{S^\nu B}$. Assume now that $[F : F^2] = 2$. Arguing as in the proof of Theorem 1, we deduce, by Lemmas 1, 4, 5, 6 and 7, that $\overline{S^\lambda G}$ is of OTP representation type if and only if F is a splitting field for $\widetilde{S^\nu B}$. □

4. Groups of OTP projective representation type

We recall from [3, p. 200] that $i(F)$ is the supremum of the set that consists of 0 and all positive integers m such that an F-algebra of the form

$$F[t]/(t^p - \alpha_1) \otimes_F \ldots \otimes_F F[t]/(t^p - \alpha_m)$$

is a field for some $\alpha_1, \ldots, \alpha_m \in K$.

Theorem 2. *Let $G = G_p \times B$, $|G'_p| \neq 2$ and s be the number of invariants of G_p/G'_p. The group G is of OTP projective (S, F^*)-representation type if and only if one of the following conditions is satisfied:*

(i) $|G'_p| = 1$ and $s \leq i(F)$;

(ii) $p = 2$, $|G'_2| = 1$, $s = i(F) + 1$ and G_2 has at least one invariant equal to 2;

(iii) F is a splitting field for $F^\sigma B$ for some $\sigma \in Z^2(B, F^*)$.

P r o o f. Let $p = 2$ and G_2 be Abelian. If $s \geq i(F) + 2$, then $4 \dim_F \overline{F^\lambda G_2} \leq |G_2|$ for any $\lambda \in Z^2(G_2, F^*)$. In this case, by Proposition 3, G is of OTP projective (S, F^*)-representation type if and only if the condition (iii) is satisfied. Assume that $s = i(F) + 1$. If G_2 has at least one invariant equal to 2, then there exists a cocycle $\lambda \in Z^2(G_2, F^*)$ such that $2 \dim_F \overline{F^\lambda G_2} = |G_2|$. Hence, by Proposition 3, G is of OTP projective (S, F^*)-representation type. Suppose that every invariant of G_2 is greater than 2. Then $4 \dim_F \overline{F^\lambda G_2} \leq |G_2|$ for each $\lambda \in Z^2(G_2, F^*)$. By Proposition 3, G is of OTP projective (S, F^*)-representation type if and only if the condition (iii) is satisfied.

Let $p \geq 2$ and G_p be Abelian. There exists a cocycle $\mu \in Z^2(G_p, F^*)$ such that $F^\mu G_p$ is a field if and only if $s \leq i(F)$. For any $\nu \in Z^2(B, F^*)$, we have $\widetilde{S^\nu B} \cong F^\nu B$. Applying Proposition 3, we finish the proof. □

Corollary. *Let G_p be an Abelian p-group, s the number of invariants of G_p, B a nilpotent p'-group and $G = G_p \times B$. Assume that the field F does not contain a primitive q^{th} root of 1 for some prime $q \mid |B|$. The group G is of OTP projective (S, F^*)-representation type if and only if one of the following conditions is satisfied:*

(i) $s \leq i(F)$;

(ii) $p = 2$, $s = i(F) + 1$ and G_2 has at least one invariant equal to 2.

P r o o f. Apply Proposition 2 and Theorem 2. □

Lemma 9. *Let B be an Abelian p'-group. The field F is a splitting field for some algebra $F^{\lambda}B$ if and only if B has a subgroup H such that B/H is of symmetric type and F contains a primitive m^{th} root of 1, where* $m = \max\{\exp(B/H), \exp H\}$.

P r o o f. Let $\lambda \in Z^2(B, F^*)$, $\{u_b : b \in B\}$ be a natural F-basis of the algebra $F^{\lambda}B$, Z the center of $F^{\lambda}B$ and $H = \{g \in B : u_g \in Z\}$. Then H is a subgroup of B and $Z = F^{\lambda}H$. The algebra $F^{\lambda}B$ may be viewed as a twisted group ring of the group $\bar{B} := B/H$ over the ring Z. By Lemma 3 [1, p. 785],

$$F^{\lambda}B = Z^{\bar{\lambda}}\bar{B} \cong Z^{\bar{\lambda}}N_1 \otimes_Z \ldots \otimes_Z Z^{\bar{\lambda}}N_r,$$

where N_i is a group of type $(q_i^{n_i}, q_i^{n_i})$, q_i is a prime divisor of $|\bar{B}|$ and $Z^{\bar{\lambda}}N_i$ is a central Z-algebra, moreover

$$\gamma_{x,y} := \bar{\lambda}_{x,y} \cdot \bar{\lambda}_{y,x}^{-1} \in F$$

and

$$\gamma_{x,y}^{q_i^{n_i}} = 1$$

for all $x, y \in N_i$. It follows that F contains a primitive $(\exp \bar{B})^{\text{th}}$ root of 1.

If F is a splitting field for $F^{\lambda}B$, then F is a splitting field for the commutative F-algebra $Z = F^{\lambda}H$. Therefore F contains a primitive $(\exp H)^{\text{th}}$ root of 1. The group $\bar{B} = N_1 \times \ldots \times N_r$ is of symmetric type. This proves the necessity.

Let us prove the sufficiency. Denote by K a finite subfield of the field F which contains a primitive m^{th} root of 1, where $m = \max\{\exp(B/H), \exp H\}$. We may assume that B is an Abelian q-group, where $q \neq p$. Let

$$\bar{B} := B/H = \langle\, x_1H \,\rangle \times \langle\, y_1H \,\rangle \times \ldots \times \langle\, x_rH \,\rangle \times \langle\, y_rH \,\rangle,$$

where $|x_iH| = |y_iH| = q^{n_i}$ for each $i \in \{1, \ldots, r\}$. We have

$$x_i^{q^{n_i}} = h_i, \quad y_i^{q^{n_i}} = h_i^*,$$

where $h_i, h_i^* \in H$. Let $Z = KH$ with K-basis $\{u_h : h \in H\}$ and let $A = Z^{\mu}\bar{B}$ be the twisted group ring of $\bar{B}$ over Z with Z-basis $\{v_{bH} : b \in B\}$ satisfying the following conditions:

1) if $bH = (x_1H)^{i_1}(y_1H)^{j_1} \ldots (x_rH)^{i_r}(y_rH)^{j_r}$, where $0 \leq i_s, j_s < q^{n_s}$, then

$$v_{bH} = v_{x_1H}^{i_1} v_{y_1H}^{j_1} \cdots v_{x_rH}^{i_r} v_{y_rH}^{j_r};$$

2) $v_{x_sH}^{q^{n_s}} = u_{h_s}$, $v_{y_sH}^{q^{n_s}} = u_{h_s^*}$ for all $s \in \{1, \ldots, r\}$;

3) $v_{bH} \cdot v_{\bar{b}H} = \xi_1^{j_1\bar{i}_1} \ldots \xi_r^{j_r\bar{i}_r} v_{x_1H}^{i_1+\bar{i}_1} v_{y_1H}^{j_1+\bar{j}_1} \ldots v_{x_rH}^{i_r+\bar{i}_r} v_{y_rH}^{j_r+\bar{j}_r}$,

where ξ_s is a primitive $(q^{n_s})^{\rm th}$ root of 1 for every $s \in \{1, \ldots, r\}$. Then

$$A \cong Z^\mu N_1 \otimes_Z \ldots \otimes_Z Z^\mu N_r,$$

where $Z^\mu N_s$ is a central twisted group ring of the group $N_s = \langle x_s H \rangle \times \langle y_s H \rangle$ over the ring Z.

Let g be an element of the group B. Then

$$g = x_1^{d_1} y_1^{t_1} \ldots x_r^{d_r} y_r^{t_r} h,$$

where $0 \leq d_s$, $t_s < q^{n_s}$ for every $s \in \{1, \ldots, r\}$ and $h \in H$. We set

$$w_g = v_{x_1 H}^{d_1} v_{y_1 H}^{t_1} \ldots v_{x_r H}^{d_r} v_{y_r H}^{t_r} u_h.$$

Then $\{w_g \colon g \in B\}$ is a K-basis of the algebra A and $w_{g_1} w_{g_2} = \lambda_{g_1,g_2} w_{g_1 g_2}$, where $\lambda_{g_1,g_2} \in K^*$ for all $g_1, g_2 \in B$. Hence $A = K^\lambda B$ and K is a splitting field for the algebra $K^\lambda B$. It follows that F is a splitting field for the algebra $F^\lambda B = F \otimes_K K^\lambda B$. □

Lemma 10. *Let B be an Abelian p'-group of symmetric type and $\exp B = q_1^{m_1} \ldots q_t^{m_t}$, where $q_1, \ldots, q_t$ are pairwise distinct prime numbers. The field F is a splitting field for certain algebra $F^\lambda B$ if and only if F contains a primitive $n^{\rm th}$ root of 1, where $n = q_1^{k_1} \ldots q_t^{k_t}$ and $2k_j \geq m_j$ for every $j \in \{1, \ldots, t\}$.*

P r o o f. Without loss of generality, we may assume that B is an Abelian q-group of exponent q^m. Let F contain a primitive $(q^l)^{\rm th}$ root of 1 and F does not contain a primitive $(q^{l+1})^{\rm th}$ root of 1. If $l \geq m$ then F is a splitting field for the group algebra FB. Let $\frac{m}{2} \leq l < m$. The group B has a subgroup H of exponent q^{m-l} such that B/H is of symmetric type and $\exp(B/H) = q^l$. Since $m - l \leq l$, by Lemma 9, F is a splitting field for certain algebra $F^\nu B$. Suppose now that $l < \frac{m}{2}$. Let $\lambda \in Z^2(B, F^*)$, Z be the center of $F^\lambda B$ and H a subgroup of B such that $Z = F^\lambda H$. Then $\exp H \geq q^{m-l}$. If F is a splitting field for $F^\lambda B$, then $\exp H \leq q^l$. We have $q^{m-l} \leq q^l$, whence $m - l \leq l$. Hence $l \geq \frac{m}{2}$. This contradiction shows that F is not a splitting field for every algebra $F^\lambda B$. □

Proposition 5. *Let $G = G_p \times B$ be an Abelian group and s the number of invariants of G_p. The group G is of OTP projective (S, F^*)-representation type if and only if one of the following conditions is satisfied:*

(i) $s \leq i(F)$;

(ii) $p = 2$, $s = i(F) + 1$ and G_2 has at least one invariant equal to 2;

(iii) B has a subgroup H such that B/H is of symmetric type and F contains a primitive $m^{\rm th}$ root of 1, where $m = \max\{\exp(B/H), \exp H\}$.

P r o o f. Apply Theorem 2 and Lemma 9. □

Proposition 6. *Let $G = G_p \times B$ be an Abelian group and s the number of invariants of G_p. Assume that B is of symmetric type and $\exp B = q_1^{m_1} \dots q_t^{m_t}$, where $q_1, \dots, q_t$ are pairwise distinct prime numbers. The group G is of OTP projective (S, F^*)-representation type if and only if one of the following conditions is satisfied:*

(i) $s \leq i(F)$;

(ii) $p = 2$, $s = i(F) + 1$ and G_2 has at least one invariant equal to 2;

(iii) F contains a primitive n^{th} root of 1, where $n = q_1^{k_1} \dots q_t^{k_t}$ and $2k_j \geq m_j$ for each $j \in \{1, \dots, t\}$.

P r o o f. Apply Theorem 2 and Lemma 10. □

5. Groups of purely OTP projective representation type

Lemma 11. [5, p. 322] *Let R be a Noetherian integral domain whose integral closure is a finitely generated R-module. Then every finitely generated torsion free R-module is a direct sum of ideals in R if and only if each ideal in R is generated by one or two elements.*

Theorem 3. *Let $G = G_p \times B$. The group G is of purely OTP projective S-representation type if and only if $|G_p| = 2$ or F is a splitting field for $F^\nu B$ for any $\nu \in Z^2(B, F^*)$.*

P r o o f. Assume that $|G_p| > 2$ and $\sigma \in Z^2(B, S^*)$. By Theorem 1, the ring $S^\lambda G = SG_p \otimes_S S^\sigma B$ is of OTP representation type if and only if F is a splitting field for $\widetilde{S^\sigma B}$. Hence, by Lemma 7, if $|G_p| > 2$ then G is of purely OTP projective S-representation type if and only if F is a splitting field for every algebra $F^\nu B$.

Let $p = 2$ and $G_2 = \langle\, a \,\rangle$ be the group of order 2. If V is an indecomposable SG_2-module then, by [13, p. 70], $\overline{\operatorname{End}_{SG_2}(V)} \cong F$. Hence, by Lemmas 1, 5 and 6, the ring $SG_2 \otimes_S S^\nu B$ is of OTP representation type for any $\nu \in Z^2(B, S^*)$. Suppose now that $\lambda \in Z^2(G, S^*)$ and $S^\lambda G_2$ is not a group ring. Then $S^\lambda G_2 = Su_e + Su_a$, where $u_a^2 = f(X)u_e$, $f(X) \in S^*$ and $f(X) \notin S^2$. Let $f(X) = a_0 + a_1X + a_2X^2 + \dots$, where $a_j \in F$ for every $j \in \{0, 1, 2, \dots\}$, θ be a root of the polynomial $t^2 - f(X)$ and $K = T(\theta)$, where T is the quotient field of S. We have $S^\lambda G_2 \cong S[\theta]$. Denote by L the integral closure of $S[\theta]$ in the field K. Then $L = S[\omega]$, where $\omega = \theta$ or $\omega = X^{-n}(b_0 + b_1X + \dots + b_{n-1}X^{n-1} + \theta)$, moreover in the second case

$$f(X) = b_0^2 + b_1^2X^2 + \dots + b_{n-1}^2X^{2(n-1)} + \sum_{j \geq 2n} a_jX^j,$$

$n \geq 1$, $a_{2n} \notin F^2$ or $a_{2n+1} \neq 0$.

Every ideal of the ring $S[\theta]$ is generated by one or two elements. Let V be an indecomposable $S[\theta]$-module. If $z \in S[\theta]$, $v \in V$ and $zv = 0$, then $z^2 v = 0$. Since $z^2 \in S$ and V is a free S-module, $z^2 = 0$ or $v = 0$. Hence $z = 0$ or $v = 0$. This means that V is a torsion-free $S[\theta]$-module. By Lemma 11, V is isomorphic to an ideal J of the ring $S[\theta]$. The ideal J is a free S-module of rank 2. It follows that $T \otimes_S J$ is an indecomposable $T^\lambda G_2$-module. By Theorem 3.1 [2, p. 549], the algebra $T^\lambda G$ is of OTP representation type. Therefore, $(T \otimes_S J)\#(T \otimes_S W)$ is an indecomposable $T^\lambda G$-module for any irreducible $S^\lambda B$-module W. It follows that $J\#W$ is an indecomposable $S^\lambda G$-module. By Lemma 5, the ring $S^\lambda G$ is of OTP representation type. Hence, G is of purely OTP projective S-representation type. □

Corollary. *Let $G = G_p \times B$ be a nilpotent group. The group G is of purely OTP projective S-representation type if and only if one of the following conditions is satisfied:*

(i) $|G_p| = 2$;

(ii) $F = F^q$ *and F contains a primitive* q^{th} *root of* 1 *for each prime* $q \mid |B|$.

P r o o f. Apply Proposition 2 and Theorem 3. □

Proposition 7. *Let $G = G_p \times B$. Assume that $F = F^q$ and F contains a primitive q^{th} root of 1 for each prime $q \mid |B|$. Then G is of purely OTP projective S-representation type.*

P r o o f. The field F is a splitting field for any algebra $F^\nu B$. Hence, by Lemma 7, $S^\lambda G$ is of OTP representation type for every $\lambda \in Z^2(G, S^*)$. □

Corollary. *If F is a separably closed field, then every group $G = G_p \times B$ is of purely OTP projective S-representation type.*

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