Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XV, Częstochowa 2010

RIEMANN INTEGRABILITY AND QUASI-UNIFORM CONVERGENCE

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Abstract. We consider quasi-uniform convergence of sequences of functions in a context of Riemann integrability of its limit. Some generalizations are discussed as well.

Arzelá considered an additional regularity condition for a pointwise convergent sequence of functions. Its role is particularly interesting while dealing with convergence in the space of continuous functions. The precise definition reads as follows.

Definition 1. (Arzelá [1]. *A pointwise convergent sequence* $(f_n)_{n=1}^{\infty}$ *of real functions defined in a topological space* X *is called quasi-uniformly convergent to a function* $f : X \longrightarrow \mathbb{R}$ *if*

$$\forall_{\varepsilon>0} \forall_{n\in\mathbb{N}} \exists_{k_n} \exists_{p_1,\dots,p_{k_n}\geq n} \forall_{t\in X} \left(\min\{|f_{p_i}(t) - f(t)| \,:\, i \in \{1,\dots,k_n\}\} < \varepsilon\right). \qquad (1)$$

The following two facts concerning this convergence are well known.

Fact 1. (Szökefalvi-Nagy [2]). *Assume that a sequence of continuous real functions defined in a topological space* X *is quasi-uniformly convergent to a function* $f : X \longrightarrow \mathbb{R}$*. Then* f *itself is continuous.*

Fact 2. *Let X be a compact topological space and let $(f_n)_{n=1}^{\infty}$ be a pointwise convergent sequence of continuous real functions defined in X. If the limit function $f : X \longrightarrow \mathbb{R}$ is continuous as well, then the sequence $(f_n)_{n=1}^{\infty}$ is quasi-uniformly convergent to f.*

In what follows, we shall consider a locally compact topological group $(G, +)$ with Haar measure h. It turns out that, in such circumstances, Fact 1 carries over to functions that are merely h-almost everywhere continuous. Namely, the following theorem holds true.

Theorem 1. *Let A stand for a Haar measurable subset of G with $h(A) > 0$. Assume that a sequence $(f_n)_{n=1}^{\infty}$ of h-almost everywhere continuous real functions defined in A is quasi-uniformly convergent to a function $f : A \longrightarrow \mathbb{R}$. Then f itself is h-almost everywhere continuous.*

Proof. Let E_n denote the set of all continuity points of the function f_n, $n \in \mathbb{N}$. Moreover, let

$$E = \bigcap_{n=1}^{\infty} E_n.$$

Since $h(E_n) = h(A)$ for all $n \in \mathbb{N}$, we also have $h(E) = h(A)$.

Fix arbitrarily x_0 from E. For any positive ε there exists a positive integer n_0 such that

$$|f_n(x_0) - f(x_0)| < \frac{\varepsilon}{3} \quad \text{provided that} \quad n \geq n_0.$$

In view of condition (1), we infer that there exist $n_1, \ldots, n_k$ such that

$$n_1 \geq n_0, \ldots, n_k > n_0 \quad \text{and} \quad |f_{n_1}(t) - f(t)| < \frac{\varepsilon}{3} \vee \ldots \vee |f_{n_k}(t) - f(t)| < \frac{\varepsilon}{3}$$

for all $t \in A$.

Each of the functions f_{n_i} is continuous at x_0; therefore there exists a neighborhood U_0 of x_0 such that

$$|f_{n_i}(t) - f_{n_i}(x_0)| < \frac{\varepsilon}{3}$$

for all $t \in U_0 \cap A$ and $i \in \{1, \ldots, k\}$.

Let $x \in U_0 \cap A$, and let n_{i_o} be such that

$$\left| f_{n_{i_0}}(x) - f(x) \right| < \frac{\varepsilon}{3}.$$

Then

$$|f(x) - f(x_0)| \leq$$

$$\leq \left|f(x) - f_{n_{i_0}}(x)\right| + \left|f_{n_{i_0}}(x) - f_{n_{i_0}}(x_0)\right| + \left|f_{n_{i_0}}(x_0) - f(x_0)\right| < \varepsilon,$$

which proves that f is continuous at every point x_0 from the set E, i.e. h-almost everywhere in A. Thus the proof has been completed.

Recall that a function $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable if and only if it is almost everywhere continuous with respect to the one-dimensional Lebesgue measure. Therefore, applying Theorem 1 for the group $(\mathbb{R}, +)$ and A being a compact interval in $\mathbb{R}$, we obtain immediately the following result.

Theorem 2. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of Riemann integrable functions defined in $[a, b]$. If $f : [a, b] \longrightarrow \mathbb{R}$ is a quasi-uniform limit of the sequence $(f_n)_{n=1}^{\infty}$, then f is Riemann integrable as well.*

Plainly, in general, the Riemann integrability of f does not imply that its Riemann integral is the limit of the sequence of integrals of functions f_n, $n \in \mathbb{N}$. Nevertheless, we have the following *dominated convergence* result.

Theorem 3. *Let $f_n : [a, b] \longrightarrow \mathbb{R}$ be Riemann integrable functions. If $\lim\limits_{n \longrightarrow \infty} f_n = f$ quasi-uniformly and there exists a Riemann integrable function $g : [a, b] \longrightarrow \mathbb{R}$ such that for every positive integer n one has*

$$|f_n| \leq g,$$

then f is Riemann integrable and

$$\lim_{n \longrightarrow \infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n \longrightarrow \infty} f_n(x)\, dx.$$

For the proof it suffices to apply Theorem 2 jointly with the classical Lebesgue theorem on majorized convergence.

A careful inspection of the proof of Theorem 1 shows that the group structure as well as the translation invariance of the measure in question are inessential. As a matter of fact, the following abstract setting will allow us to reproduce this proof with no essential changes. Namely, given a topological space X and a proper σ-ideal $\mathscr{I}$ of subsets of X, we say that a function f is $\mathscr{I}$-almost everywhere continuous in X whenever the set of all discontinuity points of the function f yields a member of $\mathscr{I}$. So, we terminate this paper with the following

Theorem 4. *Assume that a sequence* $(f_n)_{n=1}^{\infty}$ *of* $\mathscr{I}$*-almost everywhere continuous real functions defined on* X *is quasi-uniformly convergent to a function* $f : X \longrightarrow \mathbb{R}$*. Then* f *itself is* $\mathscr{I}$*-almost everywhere continuous.*

Now, Theorem 1 becomes a special case of the latter result on setting $X := A$ and $\mathscr{I} := \{F \subset X : h(F) = 0\}$ which, obviously, forms a proper σ-ideal $\mathscr{I}$ of subsets of X.

References

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