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THE LOGIC DUAL TO SOBOCIŃSKI'S n–VALUED LOGIC

Anetta Górnicka

Institute of Mathematics and Computer Science
Jan Długosz University of Częstochowa
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
e-mail: a.gornicka@ajd.czest.pl

Abstract. In this paper, we describe the logic dual to n–valued Sobociński logic. According to the idea presented by Malinowski and Spasowski [1], we introduce the consequence dual to the consequence of n–valued Sobociński logic in two ways: by a logical matrix and by a set of rules of inference. Then we prove that both approaches are equivalent and the consequence is dual in Wójcicki sense (see [3]).

1. Introduction

By a *language of a propositional logic* (*propositional calculus*) we mean an absolutely free algebra $J = (S, \mathbb{F})$, where S is the set of all formulas built in the standard way on a countable set of propositional variables $p_1, p_2, \ldots$ using functors from the set $\mathbb{F}$.

Let $\mathbf{C}$ denote the family of all consequences in S and let $Cn \in \mathbf{C}$. The consequence dCn dual to the consequence Cn is defined as follows:

Definition 1.

$$\alpha \in dCn(X) \Leftrightarrow \exists_Y \Big(Y \subseteq X \wedge card(Y) < \aleph_0 \wedge \bigcap_{\beta \in Y} Cn(\{\beta\}) \subseteq Cn(\{\alpha\}) \Big)$$

for all formulas $\alpha, \beta \in S$ *and every* $X \subseteq S$.

The definition of a dual consequence applied here was given by Wójcicki [3].

Let $J = (S, \{\Rightarrow, \neg\})$ be the language of Sobociński's n–valued logic described in [2].

Definition 2. *n–valued implicational-negational Sobociński propositional calculus is determined by the following matrix:*

$$\mathfrak{M}_{Sob} = (\{0, 1, 2, \ldots, n-1\}, \{1, 2, \ldots, n-1\}, \{\Rightarrow, \neg\}), \quad n \geq 3.$$

Here the only nondesignated value is 0.
Functions $\Rightarrow, \neg$ are defined as follows:

$$x \Rightarrow y = \begin{cases} y & \text{if} \quad x \neq y, \\ n-1 & \text{if} \quad x = y, \end{cases}$$

$$\neg x = \begin{cases} x+1 & \text{if} \quad x < n-1, \\ 0 & \text{if} \quad x = n-1, \end{cases}$$

for any $x, y \in \{0, 1, \ldots, n-1\}$.

Let us consider the following matrix, which will be called dual to the matrix $\mathfrak{M}_{Sob}$:

$$\mathfrak{M}^d_{Sob} = (\{0, 1, 2, \ldots, n-1\}, \{0\}, \{\Rightarrow, \neg\}), \quad n \geq 3,$$

where functions $\Rightarrow$ and $\neg$ are defined in the same way as in the matrix $\mathfrak{M}_{Sob}$.

Definition 3.

1. $\neg^* \alpha \overset{df}{=} (\alpha \Rightarrow \neg(\alpha \Rightarrow \alpha))$.
2. $\alpha \vee^* \beta \overset{df}{=} (\neg^* \alpha \Rightarrow \beta)$.

We call the functors $\neg^*$ and $\vee^*$ the strong negation and the strong disjunction, respectively.

It is easy to observe that a function $\neg^*$ defined by

$$\neg^*(x) = \begin{cases} n-1 & \text{if} \quad x = 0, \\ 0, & \text{otherwise}, \end{cases}$$

corresponds in the matrix $\mathfrak{M}_{Sob}$ to the functor $\neg^*$.
Similarly, a function $\vee^*$ defined by

$$x \vee^* y = \begin{cases} y & \text{if} \quad y \geq 1, \\ 0 & \text{if} \quad x = 0 \text{ and } y = 0, \\ n-1 & \text{if} \quad x \geq 1 \text{ and } y = 0, \end{cases}$$

corresponds in the matrix $\mathfrak{M}_{Sob}$ to the functor $\vee^*$.

Lemma 1. *For arbitrary formulas* $\alpha, \beta \in S$ *and for any homomorphism* $h : J \rightarrow (\{0, 1, 2, \dots, n-1\}, \{\Rightarrow, \neg^*, \vee^*\})$ *the following statements are true:*

1. *if* $h(\alpha \Rightarrow \beta), h(\alpha) \in \{1, 2, \dots, n-1\}$, *then* $h(\beta) \in \{1, 2, \dots, n-1\}$,
2. $h(\alpha \Rightarrow \beta) = 0$ *iff* $h(\alpha) \in \{1, 2, \dots, n-1\}$ *and* $h(\beta) = 0$,
3. $h(\alpha) \in \{1, 2, \dots, n-1\}$ *iff* $h(\neg^* \alpha) = 0$,
4. $h(\alpha \vee^* \beta) \in \{1, 2, \dots, n-1\}$ *iff* $h(\alpha) \in \{1, 2 \dots, n-1\}$ *or* $h(\beta) \in \{1, 2, \dots, n-1\}$.

Let us consider two inference rules:

$$r_{mp} : \frac{\alpha \Rightarrow \beta, \alpha}{\beta}, \qquad r^d_{mp} : \frac{\neg^*(\alpha \Rightarrow \beta), \beta}{\alpha}.$$

Let $R = \{r_{mp}\}, R^d = \{r^d_{mp}\}$.

Denote by Hom the set of all homomorphisms from $(S, \{\Rightarrow, \neg\})$ into $(\{0, 1, \dots, n-1\}, \{\Rightarrow, \neg\})$ and let $X \subseteq S$. We define the matrix consequence $C_{\mathfrak{M}}(X)$, the content $E(\mathfrak{M})$ of the matrix $\mathfrak{M}$ and the consequence $C_R(X)$ based on inference rules from the set X in the standard way:

Definition 4.

1. $C_{\mathfrak{M}_{Sob}}(X) =$
$= \{\alpha \in S : \forall_{h \in Hom}(h(X) \subseteq \{1, \dots, n-1\} \Rightarrow h(\alpha) \in \{1, \dots, n-1\})\}$.
2. $C_{\mathfrak{M}^d_{Sob}}(X) = \{\alpha \in S : \forall_{h \in Hom}(h(X) \subseteq \{0\} \Rightarrow h(\alpha) = 0)\}$.
3. $E(\mathfrak{M}_{Sob}) = \{\alpha \in S : \forall_{h \in Hom} h(\alpha) \in \{1, 2, \dots, n-1\}\}$.
4. $E(\mathfrak{M}^d_{Sob}) = \{\alpha \in S : \forall_{h \in Hom} h(\alpha) = 0\}$.
5. $C_R(X)$ *is the least set* Y, *which is closed under the rule* r_{mp} *and which satisfies* $E(\mathfrak{M}_{Sob}) \cup X \subseteq Y$.
6. $C_{R^d}(X)$ *is the least set* Y, *which is closed under the rule* r^d_{mp} *and which satisfies* $E(\mathfrak{M}^d_{Sob}) \cup X \subseteq Y$.

2. Some properties of $C_{\mathfrak{M}_{Sob}}, C_{\mathfrak{M}^d_{Sob}}, C_R$ and C_{R^d}

Since modus ponens is the primitive rule of $C_R(X)$ and, as can be easily seen, $\alpha \Rightarrow \alpha, \alpha \Rightarrow (\beta \Rightarrow \alpha), (\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma)) \in E(\mathfrak{M}_{Sob})$, then the classical deduction theorem holds:

Lemma 2. *For arbitrary $\alpha, \beta \in S$ and $X \subseteq S$*

$$\beta \in C_R(X \cup \{\alpha\}) \text{ iff } \alpha \Rightarrow \beta \in C_R(X).$$

Proof. Let us assume that the sequence $\alpha_1, \dots, \alpha_n$ is the proof based on the set $X \cup \{\alpha\}$ of a formula β. We prove, by induction, that for any $1 \leqslant k \leqslant n$ it holds

$$\alpha \Rightarrow \alpha_k \in C_R(X).$$

Let $k = 1$. Then $\alpha_1 = \alpha$ or $\alpha_1 \in X$.

If $\alpha_1 = \alpha$, then since $\alpha \Rightarrow \alpha \in E(\mathfrak{M}_{Sob})$, we get $\alpha \Rightarrow \alpha_1 \in C_R(X)$.

If $\alpha_1 \in X$, then noticing that $\alpha_1 \Rightarrow (\alpha \Rightarrow \alpha_1) \in E(\mathfrak{M}_{Sob})$, we can see that the sequence $\alpha_1 \Rightarrow (\alpha \Rightarrow \alpha_1), \alpha_1, \alpha \Rightarrow \alpha_1$ is the proof based on X of the formula $\alpha \Rightarrow \alpha_1$.

Assume now that $k > 1$ and for any $i < k, \alpha \Rightarrow \alpha_i \in C_R(X)$.

If $\alpha_k \in X \cup \{\alpha\}$, then the proof is analogous as in the case $k = 1$.

Thus, let α_k results by r_{mp} from α_i, α_j for some $i, j < k$.
Therefore $\alpha_j = \alpha_i \Rightarrow \alpha_k$ and $\alpha \Rightarrow \alpha_i$, $\alpha \Rightarrow (\alpha_i \Rightarrow \alpha_k) \in C_R(X)$. Suppose $\beta_0, \dots, \beta_{n-1}, \alpha \Rightarrow \alpha_i$ and $\gamma_0, \dots, \gamma_{m-1}, \alpha \Rightarrow (\alpha_i \Rightarrow \alpha_k)$ are proofs of $\alpha \Rightarrow \alpha_i$ and $\alpha \Rightarrow \alpha_j$, respectively. Then the sequence
$\beta_0, \dots, \beta_{n-1}, \gamma_0, \dots, \gamma_{m-1}, (\alpha \Rightarrow (\alpha_i \Rightarrow \alpha_k)) \Rightarrow ((\alpha \Rightarrow \alpha_i) \Rightarrow (\alpha \Rightarrow \alpha_k))$,
$\alpha \Rightarrow (\alpha_i \Rightarrow \alpha_k), (\alpha \Rightarrow \alpha_i) \Rightarrow (\alpha \Rightarrow \alpha_k), \alpha \Rightarrow \alpha_i, \alpha \Rightarrow \alpha_k$ is a proof of $\alpha \Rightarrow \alpha_k$, because $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma)) \in E(\mathfrak{M}_{Sob})$.

In the end, let us assume that the sequence $\alpha_1, \dots, \alpha_n$ is the proof based on X of the formula $\alpha \Rightarrow \beta$. Then $\alpha_n = \alpha \Rightarrow \beta$. It is easy to observe that the sequence $\alpha_1, \dots, \alpha_n, \alpha, \beta$ is the proof based on $X \cup \{\alpha\}$ of the formula β. □

The next Lemma follows directly from definitions and Lemma 1.

Lemma 3. *For arbitrary $\alpha, \beta \in S$ and $X \subseteq S$*

1. $\beta \in C_{\mathfrak{M}_{Sob}}(X \cup \{\alpha\})$ *iff* $\alpha \Rightarrow \beta \in C_{\mathfrak{M}_{Sob}}(X)$.
2. $\alpha \in C_{\mathfrak{M}_{Sob}}(\{\beta\})$ *iff* $\beta \in C_{\mathfrak{M}^d_{Sob}}(\{\alpha\})$.
3. $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\beta\})$ *iff* $\neg^*(\alpha \Rightarrow \beta) \in C_{\mathfrak{M}^d_{Sob}}(\emptyset)$.
4. *The consequences $C_{\mathfrak{M}_{Sob}}, C_{\mathfrak{M}^d_{Sob}}, C_R$ and C_{R^d} are finitary.*

Lemma 4.

1. *The rule r_{mp} is an admissible rule of the consequence $C_{\mathfrak{M}_{Sob}}$.*
2. *The rule r^d_{mp} is an admissible rule of the consequence $C_{\mathfrak{M}^d_{Sob}}$.*

Proof.

1. By Lemma 1, for any homomorphism $h \in Hom$ such that $h(\alpha \Rightarrow \beta), h(\alpha) \in \{1, \ldots, n-1\}$ we have $h(\beta) \in \{1, \ldots, n-1\}$. This means that $\beta \in C_{\mathfrak{M}_{Sob}}(\{\alpha \Rightarrow \beta, \alpha\})$ and then modus ponens is an admissible rule in $C_{\mathfrak{M}_{Sob}}$.

2. The proof can be carried out on the basis of Definition 4 and Lemma 1.

□

Lemma 5.

1. $C_{\mathfrak{M}^d_{Sob}}(\emptyset) = C_{R^d}(\emptyset) = E(\mathfrak{M}^d_{Sob})$.

2. $C_{\mathfrak{M}_{Sob}}(\emptyset) = C_R(\emptyset) = E(\mathfrak{M}_{Sob})$.

3. $C_{\mathfrak{M}_{Sob}} = C_R$.

Proof. Equalities *1.* and *2.* follow directly from definitions. The proof of the equality *3.* runs as follows:

Let $X \subseteq S$. To prove the inclusion $C_{\mathfrak{M}_{Sob}}(X) \subseteq C_R(X)$ assume that $\alpha \in C_{\mathfrak{M}_{Sob}}(X)$. Due to the finitariness of the matrix consequence $C_{\mathfrak{M}_{Sob}}$ there exists a finite set $X_0 \subseteq X$ such that $\alpha \in C_{\mathfrak{M}_{Sob}}(X_0)$.

If $X_0 = \emptyset$, then using equality *2.*, we infer that $\alpha \in C_R(X_0)$ and therefore $\alpha \in C_R(X)$.

Let $X_0 = \{\alpha_1, \ldots, \alpha_m\}$.
By Lemma 3, we get $\alpha_1 \Rightarrow (\ldots \Rightarrow (\alpha_m \Rightarrow \alpha)\ldots) \in C_{\mathfrak{M}_{Sob}}(\emptyset)$. Then, by equality *2.* and Lemma 2, we have that $\alpha \in C_R(\{\alpha_1, \ldots, \alpha_m\})$. As $X_0 \subseteq X$, we see that $\alpha \in C_R(X)$.

To prove the inclusion $C_R(X) \subseteq C_{\mathfrak{M}_{Sob}}(X)$, we apply Lemma 2, Lemma 3 and the fact that C_R is finitary. □

Let us define recursively a generalized strong disjunction by

Definition 5.

1. $\vee^*(\alpha) = \alpha$,

2. $\vee^*(\alpha, \beta) = \alpha \vee^* \beta$,

3. $\vee^*(\alpha_1, \ldots, \alpha_{n+1}) = \vee^*(\vee^*(\alpha_1, \ldots, \alpha_n), \alpha_{n+1}), \quad n \geqslant 2$.

Lemma 6. *For any natural number $m \geqslant 1$:*

$$C_{\mathfrak{M}^d_{Sob}}(\{\vee^*(\alpha_1, \ldots, \alpha_m)\}) = C_{\mathfrak{M}^d_{Sob}}(\{\alpha_1, \ldots, \alpha_m\}).$$

Proof. We are going to show that for any formula $\alpha \in S$,

$$\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\vee^*(\alpha_1, \dots, \alpha_m)\}) \text{ iff } \alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\alpha_1, \dots, \alpha_m\}).$$

By Lemma 3, we have the following chain of equivalent statements:

$$\begin{aligned}\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\vee^*(\alpha_1, \dots, \alpha_m)\}) &\text{ iff } \vee^*(\alpha_1, \dots, \alpha_m) \in C_{\mathfrak{M}_{Sob}}(\{\alpha\}) \\ &\text{ iff } \alpha \Rightarrow \vee^*(\alpha_1, \dots, \alpha_m) \in C_{\mathfrak{M}_{Sob}}(\emptyset).\end{aligned}$$

The equivalence $\alpha \Rightarrow \vee^*(\alpha_1, \dots, \alpha_m) \in C_{\mathfrak{M}_{Sob}}(\emptyset)$ iff $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\alpha_1, \dots, \alpha_m\})$ can be justified in the following way:

„$\Rightarrow$”. Suppose that there exists a homomorphism $h_0 \in Hom$ such that $h_0(\{\alpha_1, \dots, \alpha_m\}) \subseteq \{0\}$ and $h_0(\alpha) \in \{1, \dots, n-1\}$. Then, by Lemma 1, we get $h_0(\alpha \Rightarrow \vee^*(\alpha_1, \dots, \alpha_m)) = 0$.

„$\Leftarrow$”. Let $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\alpha_1, \dots, \alpha_m\})$ and let us suppose that there exists a homomorphism h_1 such that $h_1(\alpha \Rightarrow \vee^*(\alpha_1, \dots, \alpha_m)) = 0$. By Lemma 1, we have $h_1(\alpha) \in \{1, \dots, n-1\}$ and $h_1(\vee^*(\alpha_1, \dots, \alpha_m)) = 0$. According to Lemma 1, we obtain $h_1(\{\alpha_1, \dots, \alpha_m\}) \subseteq \{0\}$, so $h_1(\alpha) = 0$. This contradicts our assumption. □

Lemma 7. *For any natural number $m \geqslant 1$:*

$$C_{R^d}(\{\vee^*(\alpha_1, \dots, \alpha_m)\}) \subseteq C_{R^d}(\{\alpha_1, \dots, \alpha_m\}).$$

Proof. The proof is inductive on m.

Let us observe that $\neg^*(\neg^*(\alpha_1 \vee^* \alpha_2 \Rightarrow \alpha_1) \Rightarrow \alpha_2) \in E(\mathfrak{M}^d_{Sob})$. By Lemma 5 and Definition 4, we have $\alpha_1 \vee^* \alpha_2 \in C_{R^d}(\{\alpha_1, \alpha_2\})$.
Thus $C_{R^d}(\{\alpha_1 \vee^* \alpha_2\}) \subseteq C_{R^d}(\{\alpha_1, \alpha_2\})$.

Assume that $C_{R^d}(\{\vee^*(\alpha_1, \dots, \alpha_k)\}) \subseteq C_{R^d}(\{\alpha_1, \dots, \alpha_k\})$ for some $k \geq 2$. We show that

$$C_{R^d}(\vee^*(\alpha_1, \dots, \alpha_{k+1})\}) \subseteq C_{R^d}(\{\alpha_1, \dots, \alpha_{k+1}\}).$$

Indeed, $C_{R^d}(\{\vee^*(\alpha_1, \dots, \alpha_{k+1})\}) = C_{R^d}(\{\vee^*(\vee^*(\alpha_1, \dots, \alpha_k), \alpha_{k+1})\}) \subseteq$

$\subseteq C_{R^d}(\{\vee^*(\alpha_1, \dots, \alpha_k), \alpha_{k+1}\}) = C_{R^d}(\{\vee^*(\alpha_1, \dots, \alpha_k)\} \cup \{\alpha_{k+1}\}) =$

$= C_{R^d}(C_{R^d}(\{\vee^*(\alpha_1, \dots, \alpha_k)\}) \cup \{\alpha_{k+1}\}) \subseteq C_{R^d}(C_{R^d}(\{\alpha_1, \dots, \alpha_k\}) \cup \{\alpha_{k+1}\}) =$

$= C_{R^d}(\{\alpha_1, \dots, \alpha_k\} \cup \{\alpha_{k+1}\}) = C_{R^d}(\{\alpha_1, \dots, \alpha_{k+1}\}).$ □

Lemma 8. *For arbitrary formulas $\alpha, \alpha_1, \dots, \alpha_m \in S$*
$\alpha \in C_{\mathfrak{M}_{Sob}}(\{\vee^*(\alpha_1, \dots, \alpha_m)\})$ iff $\alpha \in C_{\mathfrak{M}_{Sob}}(\{\alpha_1\}) \cap \dots \cap C_{\mathfrak{M}_{Sob}}(\{\alpha_m\})$.

Proof. It is a direct consequence of Lemma 1 and Definition 4. □

Lemma 9.

1. $C_{\mathfrak{M}_{Sob}}(\{\alpha\}) = S \Leftrightarrow \alpha \in C_{\mathfrak{M}^d_{Sob}}(\emptyset)$.
2. $C_{\mathfrak{M}^d_{Sob}}(\{\alpha\}) = S \Leftrightarrow \alpha \in C_{\mathfrak{M}_{Sob}}(\emptyset)$.

Proof.

1. „$\Rightarrow$". Let us assume that $C_{\mathfrak{M}_{Sob}}(\{\alpha\}) = S$. Since $\neg^*(p \Rightarrow p) \in C_{\mathfrak{M}_{Sob}}(\{\alpha\})$, then applying Lemma 3, we get $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\neg^*(p \Rightarrow p)\})$.
   But $\neg^*(p \Rightarrow p) \in C_{\mathfrak{M}^d_{Sob}}(\emptyset)$, so $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\emptyset)$.

   „$\Leftarrow$". Let us assume that $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\emptyset)$. By Lemma 1 and Definition 4, we get $h(\alpha \Rightarrow \gamma) \in \{1, \ldots, n-1\}$ for every homomorphism h and any formula $\gamma \in S$. By Definition 4 and Lemma 3, we obtain that $\gamma \in C_{\mathfrak{M}_{Sob}}(\{\alpha\})$ for any formula $\gamma \in S$, so $S \subseteq C_{\mathfrak{M}_{Sob}}(\{\alpha\})$. As the opposite inclusion trivially holds, we obtain $C_{\mathfrak{M}_{Sob}}(\{\alpha\}) = S$.
2. The proof is analogous as above. □

3. Main result

Now, we consider the consequences dual in the sense of Definition 1 to the consequences C_R and $C_{\mathfrak{M}_{Sob}}$ and their relation to $C_{\mathfrak{M}^d_{Sob}}$ and C_{R^d}.

Theorem 1.

$$C_{R^d} = C_{\mathfrak{M}^d_{Sob}} = dC_{\mathfrak{M}_{Sob}} = dC_R.$$

Proof. 1° $C_{R^d} = C_{\mathfrak{M}^d_{Sob}}$.

By Lemma 5, we know that $C_{R^d}(\emptyset) = C_{\mathfrak{M}^d_{Sob}}(\emptyset)$ and since, by Lemma 4, the rule r^d_{mp} is an admissible rule of the consequence $C_{\mathfrak{M}^d_{Sob}}$, we get $C_{R^d}(X) \subseteq C_{\mathfrak{M}^d_{Sob}}(X)$ for every $X \subseteq S$, which means that $C_{R^d} \leq C_{\mathfrak{M}^d_{Sob}}$.

Now, let $\alpha \in C_{\mathfrak{M}^d_{Sob}}(X)$. Since the consequence $C_{\mathfrak{M}^d_{Sob}}$ is finitary, there exists a finite set X_0 such that $X_0 \subseteq X$ and $\alpha \in C_{\mathfrak{M}^d_{Sob}}(X_0)$.

If $X_0 = \emptyset$, then by Lemma 5 we get $\alpha \in C_{R^d}(X)$.

Assume then that $X_0 = \{\alpha_1, \ldots, \alpha_m\}$.
Applying Lemma 6, we have $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\vee^*(\alpha_1, \ldots, \alpha_m)\})$. In turn, Lemma 3 yields that $\neg^*(\alpha \Rightarrow \vee^*(\alpha_1, \ldots, \alpha_m)) \in C_{\mathfrak{M}^d_{Sob}}(\emptyset)$. Therefore, by Lemma 5, we obtain that $\neg^*(\alpha \Rightarrow \vee^*(\alpha_1, \ldots, \alpha_m)) \in C_{R^d}(\emptyset)$.

Hence, $\alpha \in C_{R^d}(\{\vee^*(\alpha_1, \ldots, \alpha_m)\}) \subseteq C_{R^d}(\{\alpha_1, \ldots, \alpha_m\})$ and then $\alpha \in C_{R^d}(X)$.

Thus we have shown that $C_{\mathfrak{M}^d_{Sob}} \leq C_{R^d}$.

2° $C_{\mathfrak{M}^d_{Sob}} = dC_{\mathfrak{M}_{Sob}}$.

Let $\alpha \in C_{\mathfrak{M}^d_{Sob}}(X)$. Then, by finitariness of $C_{\mathfrak{M}^d_{Sob}}$, we deduce that $\alpha \in C_{\mathfrak{M}^d_{Sob}}(X_1)$ for a finite set $X_1 \subseteq X$.

If $X_1 = \emptyset$, then by Lemma 9
$C_{\mathfrak{M}_{Sob}}(\{\alpha\}) = S$. Hence $\bigcap_{\beta\in\emptyset} C_{\mathfrak{M}_{Sob}}(\{\beta\}) \subseteq C_{\mathfrak{M}_{Sob}}(\{\alpha\})$, i.e. $\alpha \in dC_{\mathfrak{M}_{Sob}}(X)$.

If $X_1 = \{\alpha_1, \dots, \alpha_m\}$, then $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\alpha_1, \dots, \alpha_m\})$. Applying Lemmas 6 and 3, we obtain that $\vee^*(\alpha_1, \dots, \alpha_m) \in C_{\mathfrak{M}_{Sob}}(\{\alpha\})$. From this and Lemma 8, we have $C_{\mathfrak{M}_{Sob}}(\{\alpha_1\}) \cap \dots \cap C_{\mathfrak{M}_{Sob}}(\{\alpha_m\}) \subseteq C_{\mathfrak{M}_{Sob}}(\{\alpha\})$. Thus $\alpha \in dC_{\mathfrak{M}_{Sob}}(X)$ by Definition 1. We have just shown that $C_{\mathfrak{M}^d_{Sob}} \leq dC_{\mathfrak{M}_{Sob}}$.

Suppose now that $\alpha \in dC_{\mathfrak{M}_{Sob}}(X)$. By Definition 1, there exists a finite set $Y \subseteq X$ such that $\bigcap_{\beta\in Y} C_{\mathfrak{M}_{Sob}}(\{\beta\}) \subseteq C_{\mathfrak{M}_{Sob}}(\{\alpha\})$.

If $Y = \emptyset$, then from the fact that $\bigcap_{\beta\in\emptyset} C_{\mathfrak{M}_{Sob}}(\{\beta\}) = S$ and Lemma 9, we obtain that $\alpha \in C_{\mathfrak{M}^d_{Sob}}(X)$.

Therefore, let us assume that $Y = \{\beta_1, \dots, \beta_m\}$.
Thus $C_{\mathfrak{M}_{Sob}}(\{\beta_1\}) \cap \dots \cap C_{\mathfrak{M}_{Sob}}(\{\beta_m\}) \subseteq C_{\mathfrak{M}_{Sob}}(\{\alpha\})$. By Lemma 8, we have that $C_{\mathfrak{M}_{Sob}}(\{\vee^*(\beta_1, \dots, \beta_m)\}) \subseteq C_{\mathfrak{M}_{Sob}}(\{\alpha\})$, i.e.,
$\vee^*(\beta_1, \dots, \beta_m) \in C_{\mathfrak{M}_{Sob}}(\{\alpha\})$.

Applying Lemma 3, we conclude that $\alpha \in C_{{\mathfrak{M}_{Sob}}^d}(\{\vee^*(\beta_1, \dots, \beta_m)\})$. Then, according to Lemma 6, we obtain that $\alpha \in C_{\mathfrak{M}^d_{Sob}}(\{\beta_1, \dots, \beta_m\})$. Hence $\alpha \in C_{\mathfrak{M}^d_{Sob}}(X)$ because $Y = \{\beta_1, \dots, \beta_m\} \subseteq X$. This proves that $dC_{\mathfrak{M}_{Sob}}(X) \subseteq C_{\mathfrak{M}^d_{Sob}}(X)$, so $dC_{\mathfrak{M}_{Sob}} \leq C_{\mathfrak{M}^d_{Sob}}$.

3° The equality $dC_{\mathfrak{M}_{Sob}} = dC_R$ follows directly from Lemma 5. □

Therefore, the sentential logic (S, C_{R^d}) can be regarded as a logic dual to the Sobociński's n–valued logic (S, C_R). Moreover, it is characterized by the matrix ${\mathfrak{M}_{Sob}}^d$.

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