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ON CONNECTED FUNCTIONS IN ORDERED SPACES

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Abstract. We consider some properties of functions defined in a topological space X with values in a topological space Y. The definitions 1, 2 and 3 define the same class of functions when X and Y are equal to $\mathbb{R}$ with natural topology. In this article we discuss some properties of those classes and give some sufficient conditions for the space X in which real functions defined in X form the same class.

1. Classes of connected functions

We shall consider some properties of functions defined in a topological space X with values in a topological space Y.

Definition 1. [6] *We shall say that a function $f : X \longrightarrow Y$ is connected if its graph is a connected set in $X \times Y$.*

The set of all functions which are connected will be denoted by $\mathscr{C}$.

Definition 2. [6] *We shall say that a function $f : X \longrightarrow Y$ is strongly connected if $f|E$ is a connected set for each connected subset E of X.*

The set of all functions which are strongly connected will be denoted by $\mathscr{C}_s$.

Definition 3. [4] *We shall say that a function $f : X \longrightarrow Y$ is locally strongly connected if for each x in X and its open neighbourhood U there exists open and connected neighbourhood E of x, $E \subset U$, such that $f|E$ is a connected set in the space $X \times Y$.*

The set of all functions which are locally strongly connected will be denoted by $\mathscr{C}_{ls}$.

The definitions 1, 2 and 3 define the same class of functions when X and Y are equal to $\mathbb{R}$ with natural topology.

In the article we shall discuss some properties of those classes and give some sufficient conditions for the space X in which real functions defined in X form the same class.

The terminology and properties concerned with ordered spaces are taken from the articles [1] and [2]. All other topological notions and properties are taken from [3] and [5].

The next properties follow immediately from the above definitions.

Property 1. *Each continuous function is strongly connected.*

Property 2. *Each continuous function defined in a connected space is connected.*

Property 3. *Each continuous function defined in a locally connected space is locally strongly connected.*

Theorem 1. *If a topological spaces X is connected and Y is an arbitrary topological space, then*

$$\mathscr{C}_s \subset \mathscr{C}.$$

This theorem can be completed to get a sufficient condition for a space X to be connected.

Theorem 2. *If a topological space Y has at least two elements and for topological space X*

$$\mathscr{C}_s \subset \mathscr{C},$$

then X is connected.

Theorem 3. *If a topological spaces X is locally connected and Y is an arbitrary topological space, then*

$$\mathscr{C}_s \subset \mathscr{C}_{ls}.$$

Theorem 4. *If a topological spaces X is connected and locally connected and Y is an arbitrary topological space, then*

$$\mathscr{C}_{ls} \subset \mathscr{C}.$$

Proof. For each point x from X there is an open and connected set U_x such that $f|U_x$ is a connected set in $X \times Y$. Of course,

$$\bigcup_{x \in X} U_x = X.$$

Let $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$ be arbitrary points of the graph of the function f. The class of sets

$$\{U_x : x \in X\}$$

forms an open cover; then (see [3]) there exists a finite sequence of points $(t_1, \dots, t_n)$ of the set X such that

$$x_1 \in U_{t_1}, \quad x_2 \in U_{t_2} \quad \text{and} \quad U_{t_i} \cap U_{t_j} \neq \emptyset$$

if and only if $|i-j| \le 1$.

The sets $f|U_{t_i}$ are connected, $f|U_{t_i}$ and $f|U_{t_{i+1}}$ are not disjoint. Hence the set $f|\bigcup_{i=1}^{n} U_{t_i}$ is connected and contains points $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$.

We have proved that each two points of the graph of f can be joined by a connected set, therefore the graph of f is connected. □

Example 1. *Let us define a function* $f_1\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ *in the following way*

$$f_1(x,y) = \begin{cases} x & \text{if} \qquad x>0,\ y>0, \\ 0 & \text{otherwise.} \end{cases}$$

The graph of this function is connected but has no other property.

Example 2. *Let* L_1 *be the union of segments* P_n *connecting points* $\left(\frac{1}{2^n}, 0\right)$, $\left(\frac{3}{2^{n+2}}, \frac{8}{10}\right)$ *and points* $\left(\frac{3}{2^{n+2}}, \frac{8}{10}\right)$, $\left(\frac{1}{2^{n+2}}, 0\right)$ *and a half-straight-line from* $(1,0)$ *towards* $(2,0)$.

Let L_2 *be the union of segments* Q_n *connecting points* $\left(\frac{1}{2^n}, \frac{1}{10}\right)$, $\left(\frac{3}{2^{n+2}}, \frac{9}{10}\right)$ *and points* $\left(\frac{3}{2^{n+2}}, \frac{9}{10}\right)$, $\left(\frac{1}{2^{n+2}}, \frac{1}{10}\right)$ *and a half-straight-line from* $\left(1, \frac{1}{10}\right)$ *towards* $\left(2, \frac{1}{10}\right)$.

Let L_3 *be the union of segments* S_n *connecting points* $\left(\frac{1}{2^n}, \frac{2}{10}\right)$, $\left(\frac{3}{2^{n+2}}, 1\right)$ *and points* $\left(\frac{3}{2^{n+2}}, 1\right)$, $\left(\frac{1}{2^{n+2}}, \frac{2}{10}\right)$ *and a half-straight-line from* $\left(1, \frac{2}{10}\right)$ *towards* $\left(2, \frac{2}{10}\right)$.

Let us define a function $f_2 \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ *in the following way*

$$f_2(x,y) = \begin{cases} 0 & \textit{if} & (x,y) \in L_1 \cup L_3, \\ 1 & \textit{if} & (x,y) \in L_2, \\ 0 & \textit{if} & x > 0,\ y > 1, \\ 0 & \textit{if} & x > 0,\ y < 0, \\ 0 & \textit{if} & x \le 0, \\ \begin{matrix}\textit{continuous} \\ \textit{and linear}\end{matrix} & \textit{in each vertical segment} & \textit{between lines } L_1,\ L_2, \\ \begin{matrix}\textit{continuous} \\ \textit{and linear}\end{matrix} & \textit{in each vertical segment} & \textit{between lines } L_2,\ L_3, \\ 0 & \textit{otherwise.} & \end{cases}$$

The above-defined function f_2 is connected and locally strongly connected and, of course, has the local Darboux property, but it has no other, considered in the article, properties.

Example 3. *Let us define a function* $f_3 \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ *in the following way*

$$f_3(x,y) = \begin{cases} f_2(x,y) & \textit{if} \quad x > 0,\ y \in \mathbb{R}, \\ 1 & \textit{if} \quad x \le 0,\ y \in \left(\frac{1}{10}, \frac{9}{10}\right), \\ 10y & \textit{if} \quad x \le 0,\ y \in \left(0, \frac{1}{10}\right), \\ -10y + 10 & \textit{if} \quad x \le 0,\ y \in \left(\frac{9}{10}, 1\right), \\ 0 & \textit{if} \quad x \le 0,\ y \in (-\infty, 0) \cup (1, \infty). \end{cases}$$

The function f_3 is connected, strongly connected, locally strongly connected and has the local Darboux property, but it has no other properties.

The above examples complete all the relations among the considered properties.

2. Spaces in which classes of considered connected functions coincide

If we consider real functions defined in an interval of real numbers, i.e. functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ or $f\colon (a,b) \longrightarrow \mathbb{R}$, then the four above-defined classes are equal. When we want to compare those classes, it is necessary to consider the space X to be connected (connected functions), locally connected (locally strongly connected functions). Nevertheless, those classes of functions are different if the domain of the functions is $\mathbb{R}^2$ as we have seen in the first part of the article.

If we assume that the space X has dimension 1, the situation is not better: the function f defined in the unit circle of the complex plane by the formula

$$f\left(e^{it}\right) = t \quad \text{if} \quad t \in [0, 2\pi]$$

is connected but it is not strongly connected.

It seems to be very useful the idea of cut points of a connected space, which means that a point x_0 is a cut point of a connected space X if the set $X \setminus \{x_0\}$ is not connected. A point x_0 is a strong cut point of a connected space X if the set $X \setminus \{x_0\}$ has (exactly) 2 connected components.

Similarly, if we assume that every point is a cut point of the space X, the situation is not sufficiently good. The function $f\colon X \longrightarrow Y$ defined by:

$$f(x,y) = \begin{cases} 1 + \sin\frac{1}{x} & \text{if} \quad x \in [-1, 0),\ y = 0, \\ 2 + \sin\frac{1}{y} & \text{if} \quad x = 0,\ y \in (0, 1], \\ 0 & \text{if} \quad x \in [0, 1],\ y = 0, \end{cases}$$

where $X = [-1,1] \times \{0\} \cup \{0\} \times [0,1]$, is connected but it is not strongly connected.

In this way we come to the conclusion that comparing our classes of connected functions we should bound our considerations to functions which have connected or locally connected spaces for which each point is a strong cut point. However, such properties of topological spaces imply that they are linearly ordered spaces. That is the reason for assuming that the spaces X and Y are connected, locally connected and linearly ordered. In such a case it is possible to consider the fourth class of functions, i.e. functions which cut continuum.

If $\prec$ is an order relation in a topological space X, then this space is called ordered if the sets

$$\{x \in X \,:\, x \prec a\} \quad \text{and} \quad \{x \in X \,:\, a \prec x\}$$

form a subbase of the topology in X.

The sets

$$\{x \in X : a \prec x \wedge x \prec b\} \quad \text{and} \quad \{x \in X : a \prec x \wedge x \prec b\} \cup \{a, b\}$$

are called open and closed intervals in X. These sets are denoted by (a, b) and $[a, b]$, respectively.

The sets

$$\{x \in X : a \prec x\} \quad \text{and} \quad \{x \in X : x \prec b\}$$

are denoted by $(a, \rightarrow)$ and $(\leftarrow, b)$, respectively.

Of course, the class of all open intervals in a linearly ordered space form a base of this topology.

Lemma 1. *In a linearly ordered, connected and locally connected topological space each closed interval is compact.*

Proof. Let X be a linearly ordered and connected topological space, moreover let $[a, b]$ be an arbitrary closed interval in the space X. Suppose that $\{U_s : s \in S\}$ is an arbitrary open cover of the set $[a, b]$. Since X is a linearly ordered space, then each open set can be represented as a union of open intervals:

$$U_s = \bigcup_{t \in T_s} I_{s,t},$$

where $I_{s,t}$ are intervals in X and T_s are some sets of indexes. Then

$$[a, b] \subset \bigcup_{s \in S} \bigcup_{t \in T_s} I_{s,t}.$$

Since $[a, b]$ is a connected subset of the space X, then (see [3]) there exists a finite sequence $(s_1, t_1), \ldots, (s_n, t_n)$ of indexes such that

$$a \in I_{s_1,t_1}, \quad b \in I_{s_n,t_n}$$

and

$$I_{s_i,t_i} \cap I_{s_j,t_j} \neq \emptyset \iff |i - j| \leq 1. \tag{1}$$

Suppose now that some element x_0 from $[a, b]$ does not belong to the set $\bigcup_{i=1}^{n} I_{s_i,t_i}$. Let us assume for shortening of notation that $I_{s_i,t_i} = (a_i, b_i)$. Let

$$i_0 = \max\{i \in \{1, \ldots, n\} : a_i \prec x_0\}.$$

Hence (1) implies that

$$x_0 \in (a_{i_0}, b_{i_0}),$$

what contradicts to the assumption. Thus

$$[a,b] \subset \bigcup_{i=1}^{n} I_{s_i,t_i}$$

and consequently

$$[a,b] \subset \bigcup_{i=1}^{n} U_{s_i}.$$

It proves that the interval $[a,b]$ is compact. □

Theorem 5. *If topological spaces X and Y are linearly ordered, connected and locally connected topological spaces, then each connected function $f\colon X \longrightarrow Y$ is strongly connected.*

Proof. Suppose that there exists a connected function $f\colon X \longrightarrow Y$ which is not strongly connected. Then there exists a connected subset K of X such that $f|K$ is not a connected subset of the space Y. The set K can be one of the following sets:

$$[a,b], \quad (a,b), \quad [a,b), \quad (a,b], \quad (\leftarrow,b), \quad \text{or} \quad (a,\rightarrow).$$

Assume first that $K=[a,b]$. Since the set $f|K$ is not connected, then there are two nonempty separated sets A and B in $X \times Y$ such that

$$f|K = A \cup B.$$

Suppose that $(a,f(a)) \in A$. There are two possibilities:

1. $(b,f(b)) \in A$,
2. $(b,f(b)) \in B$.

In the first case, let

$$A_1 = A \cup f|(\leftarrow,a) \cup f|(b,\rightarrow), \quad B_1 = B.$$

Then

$$f = A_1 \cup B_1, \quad A_1 \neq \emptyset \neq B_1,$$

and the sets A_1 and B_1 are separated, which contradicts to the assumption.

If $(b,f(b)) \in B$, let

$$A_1 = A \cup f|(\leftarrow,a), \quad B_1 = B \cup f|(b,\rightarrow).$$

Then the sets A_1 and B_1 are nonempty and separated, which is impossible in view of connectivity of the function (set) f.

Let now $K = (a, b)$. Then there are nonempty and separated sets A and B such that $f|K = A \cup B$. There exist elements c and d in X such that $a \prec c \prec d \prec b$ and the sets A_1 and B_1 are nonempty and separated, where

$$A \cap ([c, d] \times Y)\,, \quad B \cap ([c, d] \times Y)\,.$$

It is impossible in view of connectivity of the function f.

Similar arguments can be applied in all remained cases for the set K. □

The next theorem is a simple corollary of theorem 4.

Theorem 6. *If topological spaces X and Y are linearly ordered, connected and locally connected, then each locally strongly connected function $f\colon X \longrightarrow Y$ is strongly connected.*

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