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A NOTE ON SI-SPACES AND MI-SPACES

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Abstract. We show that if there exists a second κ-category (or κ-Baire) SI-space, then there exists a second κ-category (resp. κ-Baire) MI-space. Next we discuss some properties of real functions on such spaces.

1. Preliminaries and basic definitions

The topic of our research stems from the ω-problem formulated below (see also [1]), which initially and formally had nothing in common with the spaces under discussion. The connections appeares in the way of analyzing the problem for non-metrizable spaces.

Although we retain all the definitions and notation from [1], we recall some of them for convenience of the reader.

Let $X = (X, \tau)$ be a topological space. To each function $F : X \to \mathbb{R}$ we associate the upper and lower Baire functions

$$M(F, \cdot) : X \to \overline{\mathbb{R}}, \;\; m(F, \cdot) : X \to \overline{\mathbb{R}}$$

defined in a usual way (see [1]). It is well known that $M(F, \cdot)$ is upper semicontinuous (USC), while $m(F, \cdot)$ is lower semicontinuous (LSC) on X.

The value

$$\omega(F, x) = M(F, x) - m(F, x) \in [0, \infty]$$

is called the oscillation of F at a point x.

We can also give an equivalent definition:

$$\omega(F, x) = \inf_{U} \sup_{x', x'' \in U} (F(x') - F(x'')),$$

where the infimum is taken over all elements U of a neighborhood base τ_x of τ at x.

Let $X = (X, \tau)$ be a topological space and a USC function $f : X \to [0, \infty]$ be given. If there exists a function $F : X \to \mathbb{R}$ such that

$$\forall x \in X : \ \omega(F, x) = f(x),$$

then we call F an ω-primitive for f.

By the "ω-problem" on a topological space X we mean the problem of the existence of an ω-primitive for a given USC function $f : X \to [0, \infty]$.[1]

In what follows, we consider only dense-in-themselves topological spaces and finite USC functions f.

In [2] it was shown that the ω-problem is solvable for each metric space. For a non-metrizable space the ω-problem need not be solvable what was shown in the case of an irresolvable space (see, e.g. [1], Theorem 4).

The notion of a resolvable (irresolvable) space was introduced in [3], where the basic properties of such spaces were given. Further, we will discuss the following two special classes of irresolvable spaces introduced in [3].

A dense-in-itself topological space $X = (X, \tau)$ is called an MI-space (or simply, MI) if every dense subset of (X, τ) is open.

A dense-in-itself topological space $X = (X, \tau)$ is called an SI-space (or simply, SI) if X has no resolvable subsets. Each MI-space is an SI-space [3].

We often write X instead of (X, τ). Closure of E is denoted by $\overline{E}$. The phrase "$E \subset X$ is τ-open (or τ-closed, τ-dense, etc.)" means that E is so with respect to the topology τ on X. Similarly, by $\mathrm{Int}_\tau E$ we denote the interior of E with respect to the topology τ. The symbol τ is omitted when no confusion could arise.

2. On second category MI-spaces and Baire MI-spaces

The notions of a first category (second category) set and of a Baire space will be considered in some generalized sense. Namely, we adopt the following definitions (see [4], [5]). Let κ be a cardinal, $\kappa > \aleph_0$.

Definition 1. *A set $E \subset X = (X, \tau)$ is of the first κ-category if it can be written in the form*

$$E = \bigcup_{\alpha \in A} E_\alpha,$$

where $\mathrm{card}\, A < \kappa$ and each E_α is nowhere dense in X.

A set $E \subset X = (X, \tau)$ is of the second κ-category if it is not of the first κ-category.

[1] Problems of this type in various settings and different terminology have been studied by many authors. Some results can be found in References which, however, are far from being complete.

Definition 2. *A topological space $X = (X, \tau)$ is called κ-Baire if the intersection of fewer than κ dense open subsets of X is dense in X.*

Recall that the definitions of a "usual" first (second) category set and of a Baire space correspond to $\kappa = \aleph_1$ and that each second κ-category set (κ-Baire space) is at the same time a "usual" second category set (resp. Baire space).

Definition 3. ([5]). *A space $X = (X, \tau)$ is called κ-SIB if it is a κ-Baire SI-space. We also say that X is a κ-SIB-space.*

In a similar way, we give

Definition 4. *A space $X = (X, \tau)$ is called κ-MIB (or κ-MIB-space) if X is a κ-Baire MI-space.*

Although initiated by the ω-problem, the propositions we are going to prove in this section were motivated by [5] and [6].

In [5] the authors obtained consistency and existence results concerning κ-SIB-spaces. Their methods used the theory of ideals on cardinals.

Our goal is far more simple. Namely, we are going only to show that if there exists a κ-SIB-space (or a second κ-category SI-space), then there exists a corresponding MI-space, i.e. a κ-MIB-space (or, respectively, a second κ-category MI-space). Some properties of functions and the ω-problem for such spaces will be discussed in Section 3.

Let $X = (X, \tau)$ be a topological space. Following [6], let $D(X, \tau)$ denote the family of all dense subsets of (X, τ).

By $\mathfrak{F}(X, \tau)$ we denote the family of filters $\mathscr{F}$ on (X, τ) consisting of dense subsets of (X, τ). It is clear that $\mathfrak{F}(X, \tau)$ is partially ordered by the usual inclusion relation.

Lemma 1. ([6], *Lemma* 3.3). Let $X = (X, \tau)$ be a topological space. Then there exists an ultrafilter $\mathscr{F}_m \in \mathfrak{F}(X, \tau)$.

Given a topological space (X, τ) and a filter $\mathscr{F} \in \mathfrak{F}(X, \tau)$, one may produce a finer topology $\hat{\tau}$ on X generated by the family $\tau \cup \mathscr{F}$. By definition, the basis for $\hat{\tau}$ consists of all intersections $U \cap E$, where $\emptyset \neq U \in \tau$ and $E \in \mathscr{F}$ (see [6]).

It is convenient to state the next two theorems of this section in the form of the following Proposition from [6]. Only category and baireness will be new items and this is exactly the object of our consideration.

Lemma 2. ([6], *Proposition* 3.4). Let $X = (X, \tau)$ be a dense-in-itself T_1 (or Hausdorff) space. Let $\mathscr{F}_m \in \mathfrak{F}(X, \tau)$ be an ultrafilter. Define $\hat{\tau}$ to be the topology generated by $\tau \cup \mathscr{F}_m$. Then

(i) $D(X, \hat{\tau}) = \mathscr{F}_m$;

(ii) $(X, \hat{\tau})$ is an MI–space which is T_1 (respectively, Hausdorff);

(iii) if (X, τ) is connected, then so is $(X, \hat{\tau})$.

Lemma 3. *(*[3]*, Theorem* 29*).* Every dense subset of an SI–space has dense interior.

Now we will prove the first main result of this section.

Theorem 1. *Assume that there exists a second κ-category T_1 (or Hausdorff) space (X, τ) which is SI. Let $\mathscr{F}_m \in \mathfrak{F}(X, \tau)$ be an ultrafilter and let $\hat{\tau}$ be a topology on X generated by $\tau \cup \mathscr{F}_m$. Then*

(i) $D(X, \hat{\tau}) = \mathscr{F}_m$;

(ii) $(X, \hat{\tau})$ is a T_1 (respectively, Hausdorff) MI-space;

(iii) $(X, \hat{\tau})$ is of second κ-category; thus $(X, \hat{\tau})$ is a second κ-category MI-space;

(iv) if (X, τ) is connected, then so is $(X, \hat{\tau})$.

Proof. Assertions (i), (ii), (iv) follow straightforward from Lemma 2. We only need to prove (iii). Assume that (iii) does not hold. Then there exists a set A, $\operatorname{card} A < \kappa$, such that

$$X = \bigcup_{\alpha \in A} E_\alpha,$$

where each E_α is $\hat{\tau}$-nowhere dense in X (i.e. nowhere dense in $(X, \hat{\tau})$). Therefore $X \setminus X_\alpha$ is $\hat{\tau}$-dense, hence τ-dense in X because $\tau \subset \hat{\tau}$. Since (X, τ) is SI, we have by Lemma 3 that $\operatorname{Int}_\tau(X \setminus E_\alpha)$ is τ-dense in X. It follows that $X \setminus \operatorname{Int}_\tau(X \setminus E_\alpha)$ is τ-closed and τ-nowhere dense in X. Since $E_\alpha \subset X \setminus \operatorname{Int}_\tau(X \setminus E_\alpha)$, we conclude that every E_α is τ-nowhere dense in X; a contradiction because (X, τ) is of the second κ-category. $\square$

Lemma 4. *(*[3]*, Theorem* 33*).* If X is an MI–space and $E \subset X$, then $\operatorname{Int} E = \emptyset$ if and only if E is closed and discrete (the empty set is considered as discrete).

Next we will prove our second main result replacing second κ-category spaces by κ-Baire spaces.

Theorem 2. *Assume that there exists a dense-in-itself T_1 (or Hausdorff) κ-SIB-space (X,τ). Let $\mathscr{F}_m \in \mathfrak{F}(X,\tau)$ be an ultrafilter and let $\hat{\tau}$ be a topology on X generated by $\tau \cup \mathscr{F}_m$. Then*

(i) $D(X,\hat{\tau}) = \mathscr{F}_m$;

(ii) $(X,\hat{\tau})$ is an MI-space which is T_1 (respectively, Hausdorff);

(iii) $(X,\hat{\tau})$ is a κ-Baire space;

Thus $(X,\hat{\tau})$ is a κ-MIB-space which is T_1 (respectively, Hausdorff);

(iv) moreover, if (X,τ) is connected, then so is $(X,\hat{\tau})$.

Proof. As in Theorem 1, claims (i), (ii), (iv) follow immediately from Lemma 2. It only remains to prove (iii). Assume that (iii) does not hold. Then there exists a nonempty set $G \in \hat{\tau}$ which is of the first κ-category in $(X,\hat{\tau})$.

Let us prove that in this case the set $X \setminus G$ should be dense in $(X,\hat{\tau})$.

Since the family $\{W \cap E : W \in \tau \setminus \{\emptyset\},\ E \in \mathscr{F}_m\}$ is a basis of the topology $\hat{\tau}$, it suffices to show that

$$\forall\, E \in \mathscr{F}_m\ \forall\, W \in \tau \setminus \{\emptyset\} :\ E \cap W \cap (X \setminus G) \neq \emptyset. \tag{3}$$

Assume that this does not hold. Then there exist $E_0 \in \mathscr{F}_m$ and $W_0 \in \tau \setminus \{\emptyset\}$ such that $E_0 \cap W_0 \cap (X \setminus G) = \emptyset$. It follows that $E_0 \subset (X \setminus W_0) \cup G$, and therefore $(X \setminus W_0) \cup G \in \mathscr{F}_m$, because $\mathscr{F}_m$ is a filter.

Then we have

$$\forall\, E \in \mathscr{F}_m\ : E \cap ((X \setminus W_0) \cup G) \in \mathscr{F}_m,$$

hence $E \cap ((X \setminus W_0) \cup G) = (E \setminus W_0) \cup (E \cap G)$ is dense in (X,τ) for each $E \in \mathscr{F}_m$. Since $\emptyset \neq W_0$ is τ-open, this yields that $E \cap G$ is τ-dense in W_0 for each $E \in \mathscr{F}_m$. In other words,

$$\forall\, E \in \mathscr{F}_m\ \forall\, V \in \tau \setminus \{\emptyset\},\ V \subset W_0,\ : V \cap (E \cap G) = (V \cap E) \cap G \neq \emptyset. \tag{4}$$

Since $V \cap E \in \hat{\tau} \setminus \{\emptyset\}$, Eq. (4) implies that a $\hat{\tau}$-open set $G \cap W_0$ is $\hat{\tau}$-dense in a τ-open, hence $\hat{\tau}$-open, set W_0. It follows that $W_0 \setminus G$ is $\hat{\tau}$-nowhere dense in a $\hat{\tau}$-open set W_0.

This implies, recalling that G is, by assumption, first κ-category in $(X,\hat{\tau})$, that W_0 is also first κ-category in $(X,\hat{\tau})$ what follows immediately in view of the equality

$$W_0 = (W_0 \setminus G) \cup (W_0 \cap G).$$

Therefore, there exists a set A, $\operatorname{card} A < \kappa$, such that

$$W_0 = \bigcup_{\alpha \in A} T_\alpha, \tag{5}$$

where each T_α is nowhere dense in $(X,\hat{\tau})$. Since $\mathrm{Int}_{\hat{\tau}} T_\alpha = \emptyset$ and $(X,\hat{\tau})$ is MI, we have that every T_α is $\hat{\tau}$-closed and $\hat{\tau}$-discrete (Lemma 4). As $(X,\hat{\tau})$ is dense-in-itself, each $X\setminus T_\alpha$ is dense in $(X,\hat{\tau})$. Since (X,τ) is κ-Baire, a τ-open set W_0 is of the second κ-category in (X,τ), therefore it follows by (5) that there exist $\beta\in A$ and $\Omega\subset W_0$, $\Omega\in\tau\setminus\{\emptyset\}$, such that T_β is τ-dense in Ω.

Since the set $X\setminus T_\beta$ is $\hat{\tau}$-dense in X, it is also τ-dense in X. In particular, $X\setminus T_\beta$ is τ-dense in Ω.

We have

$$\Omega = (\Omega\cap T_\beta)\cup(\Omega\cap(X\setminus T_\beta)),$$

where each of the two terms is τ-dense in Ω.

But this means that a τ-open set Ω is resolvable, which is impossible, because (X,τ) is an SI–space.

Consequently, we have shown that if (3) does not hold, then we get a contradiction. Thus $X\setminus G$ is $\hat{\tau}$-dense in X. But this is again a contradiction because G is nonempty and $\hat{\tau}$-open.

We finally conclude that $(X,\hat{\tau})$ has no nonempty first κ-category open subsets, i.e. $(X,\hat{\tau})$ is κ-Baire, as claimed. □

To complete this section, let us make the following

Remark 1. *In* [9] *it was shown that there is a model of the theory* **ZF** *in which all the subsets of the real line are Lebesgue measurable. Let* $\mathbb{R}_s$ *denote the real line in that model and* τ_d *denote the usual density topology on* $\mathbb{R}_s$.

QUESTION*: is* **ZF** *consistent with the conjunction of the following two statements:*
(a) each subset of $\mathbb{R}$ *is Lebesgue measurable,*
(b) almost each point of any set $E\subset\mathbb{R}$ *is its point of density?*

If the answer is in affirmative, then $(\mathbb{R}_s,\tau_d)$ *is a Baire space which is MI. Indeed, the complement of each* τ_d*-dense set* $E\subset\mathbb{R}_s$ *would be of measure zero, whence* E *is* τ_d*-open in* $\mathbb{R}_s$.

3. Some properties of real functions on Baire SI- and MI-spaces

Recall that if X is a topological space and $\varphi: X\to\mathbb{R}$ a USC (or LSC) function, then the set of points at which φ is discontinuous is of the first category (and F_σ) in X (see, e.g. [8], Theorem 1), and if X is a Baire space, then the complement of that set is dense in X. We also recall that by $\omega(F,x)$ we denote the oscillation of F at $x\in X$ (cf. ()). Since $\omega(F,\cdot)$ may take the value $\infty(:=+\infty)$, we consider $[0,\infty]$ with its standard topology of a one-point compactification of $[0,\infty)$.

Given a mapping $\varphi : X \to Y$ between topological spaces, we denote by $\mathscr{C}(\varphi)$ and $\mathscr{D}(\varphi)$ the sets of continuity and discontinuity points of φ, respectively.

Definition 5. ([1]). *A topological space X is said to be resolvable at a point $x_0 \in X$ if each open neighborhood of x_0 contains a nonempty open subset which is resolvable.*

We will use the following proposition which is the main result of [1].

Lemma 5. ([1], *Theorem* 3). *Let $X = (X, \tau)$ be a topological space. In order that X be resolvable at a point x_0, it is necessary and sufficient that the following condition be satisfied. There exist an open neighborhood G of x_0 and a function $F : G \to \mathbb{R}$ such that $0 < \omega(F, x_0) < \infty$ and $\omega(F, \cdot)$ is quasicontinuous at x_0.*

Theorem 3. *Let $X = (X, \tau)$ be a Baire SI-space. Then for each function $F : X \to \mathbb{R}$ we have*

(a) $\mathscr{C}(F) = \mathscr{C}(\omega(F, \cdot))$.

(b) The F_σ-set $\mathscr{D}(F)$ is nowhere dense.

Proof. The set $E_\infty = \{x \in X : \ \omega(F, x) = \infty\}$ is obviously closed. First we will show that E_∞ is nowhere dense. Indeed, assume that this is not the case. Then there exists an open set U such that $\omega(F, x) = \infty$ for each $x \in U$. It follows that $E_n = \{x \in U : \ F(x) > n\}$ is dense in U for each $n \in \mathbb{N}$. Since U is an SI-subspace of X, we have by Lemma 3 that $Int E_n$ is dense in U. The subspace U is a Baire subspace, this yields $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$. But then it follows that $F(x) = \infty$ at each $x \in \bigcap_{n=1}^{\infty} E_n$, which is clearly impossible. Thus, E_∞ is nowhere dense in X.

To prove (a), first observe that the inclusion $\mathscr{C}(F) \subset \mathscr{C}(\omega(F, \cdot))$ is obvious. The reverse inclusion may be proved as follows. Let $x_0 \in \mathscr{C}(\omega(F, \cdot))$. The case $\omega(F, x_0) = \infty$ is impossible what follows immediately from the fact that E_∞ is nowhere dense. So we have $\omega(F, x_0) < \infty$. We claim that $\omega(F, x_0) = 0$. Indeed, if not, we would get, by Lemma 5, that X is resolvable at x_0, a contradiction because X is SI. Thus $\omega(F, x_0) = 0$, i.e. $x_0 \in \mathscr{C}(F)$. This shows that $\mathscr{C}(\omega(F, \cdot)) \subset \mathscr{C}(F)$ which completes the proof of Claim (a).

Put $E_0 = X \setminus E_\infty$. Since $\omega(F, \cdot)$ is USC and finite on a dense open set E_0 (which is a Baire subspace of X), the set $E_0 \cap \mathscr{C}(\omega(F, \cdot)) = E_0 \cap \mathscr{C}(F)$ is dense in E_0, hence by Lemma 3, has a dense interior, because E_0 is SI. Therefore, $\mathscr{D}(F) = E_\infty \cup (E_0 \setminus \mathscr{C}(F)) = X \setminus \mathscr{C}(F)$ is a nowhere dense subset of X which proves Claim (b). □

Similar proposition holds for MI-spaces. Namely, we have

Theorem 4. *Let $X = (X, \tau)$ be a Baire MI-space. Then for each function $F : X \to \mathbb{R}$ we have*

(a)* $\mathscr{C}(F) = \mathscr{C}(\omega(F, \cdot))$.

(b)* $\mathscr{D}(F)$ *is a discrete closed set.*

Proof. Since each MI-space is an SI-space, Claim (a*) follows from Claim (a) of Theorem 3. By Claim (b) of Theorem 3, we have $\operatorname{Int}\mathscr{D}(F) = \emptyset$, whence by Lemma 4, Claim (b*) follows. □

As a consequence, we obtain the following simple criteria for the existence of ω-primitives on Baire SI- and MI-spaces.

Theorem 5. (A) *Let $X = (X, \tau)$ be a Baire SI-space. Then a USC function $f : X \to [0, \infty)$ has an ω-primitive $F : X \to \mathbb{R}$ if and only if f vanishes on a dense subset of X.*

(B) *Let $X = (X, \tau)$ be a Baire MI-space. Then a USC function $f : X \to [0, \infty)$ has an ω-primitive $F : X \to \mathbb{R}$ if and only if f vanishes outside of a closed and discrete subset of X.*

In either of the cases (A),(B) one may take $F = f$.

Proof of (A). Assume that F is an ω-primitive for f. Then applying Claim (a) of Theorem 3, we get $\mathscr{C}(F) = \mathscr{C}(\omega(F, \cdot)) = \mathscr{C}(f)$. This implies, in view of Claim (b) of Theorem 3, that $f(x) = \omega(F, x) = 0$ at each point x of the dense set $X \setminus \mathscr{D}(F)$.

Conversely, if a USC function $f : X \to [0, \infty)$ vanishes on a dense set E, then it is easy to see that $\forall x \in X : \omega(f, x) = f(x)$.

Proof of (B). Assume that a USC function $f : X \to [0, \infty)$ has an ω–primitive $F : X \to \mathbb{R}$. By Theorem 4, the set $\mathscr{D}(F)$ of points at which F is discontinuous is closed and discrete. Therefore, $f(x) = \omega(F, x) = 0$ at each $x \in X \setminus \mathscr{D}(F)$.

Conversely, assume that there is a closed and discrete set $E \subset X$ such that a USC function $f : X \to [0, \infty)$ vanishes outside E. Since X is dense in itself and $f \geq 0$ is USC, we easily deduce that the equality $\omega(f, x) = f(x)$ holds for each $x \in X$. In other words, f is an ω-primitive for itself. □

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