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AN INFLUENCE OF SERVICE DISCIPLINE ON CHARACTERISTICS OF A SINGLE-SERVER QUEUE WITH NON-HOMOGENEOUS CUSTOMERS

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Abstract. For single-server queueing systems with non-homogeneous customers having some random space requirements we compare processor-sharing and FIFO disciplines and investigate their influence on the total sum of space requirements characteristics (when this sum is not limited, i.e. $V = \infty$) and customers loss probability (when this sum is limeted, i.e. $V < \infty$), using analytical modeling and simulation.

1. Introduction

In the present work we investigate single-server queueing systems with non-homogeneous customers. This means that

1) each customer is characterized by some non-negative random capacity ζ;

2) customer's length ξ and his capacity ζ are generally dependent.

Note that we shall use the notion "customer length" instead of "service time". The difference between these notions is essential for processor sharing systems. The amount of work necessary for customer's service is called the customer length [5], i.e. the customer service time under condition that there are no other customers on service during this time period. Analogously, the residual length of the customer is referrred as his residual service time after some time instant under the same condition.

The total sum $\sigma(t)$ of capacities of all the customers present in the system at arbitrary time t may be limited by some constant value V $(0 < V \leq \infty)$ that is called the capacity of the system.

Such systems are used to model and solve various problems occurring in the design of computer and communicating nets and systems. It is clear that they differ from usual classical queueing systems in the case $V < \infty$. For example, we can analyze the non-classical system $M/G/1/(\infty, V)$ with limited capacity that differs from the classical system $M/G/1/\infty$.

Let

$$F(x,t) = \mathbf{P}\{\zeta < x, \xi < t\}$$

be the distribution function of the random vector (ζ, ξ). Then

$$L(x) = \mathbf{P}\{\zeta < x\} = F(x, \infty) \quad \text{and} \quad B(t) = \mathbf{P}\{\xi < t\} = F(\infty, t)$$

are the distribution functions of customer's capacity and length, respectively. The part of system capacity is occupied by a customer at the epoch he arrives and is released entirely at the epoch he completes service. The process $\sigma(t)$ is called the total customers capacity.

Total capacity limitation (in the case $V < \infty$) leads to losses of customers. A customer arriving at the epoch τ and having capacity x will be admitted to the system if $\sigma(\tau - 0) + x \leq V$. Otherwise $(\sigma(\tau - 0) + x > V)$, the customer will be lost.

Various single-server queueing systems with non-homogeneous (in the sense of assumptions 1, 2) customers were analyzed in [1–4].

The purpose of this paper is to compare processor sharing and FIFO or other conservative, not depending on customers capacity disciplines and investigate their influence on the stationary first moment of the total sum of customers capacities (when $V = \infty$) and customers loss probability (when $V < \infty$). To realize this purpose we use analytical modeling and simulation.

2. The case of unlimited system capacity

Suppose that customers intrance flow is Poisson. Let a be an arrival rate of entrance flow of customers. Assume that $V = \infty$. Then we have the classical $M/G/1/\infty$ and $M/G/1/\infty - EPS$ (processor sharing) systems without losses of customers. For such a system we can obtain the stationary characteristics of total customers capacity (see e.g. [2, 3]).

We shall use the following notation. Denote by

$$\alpha(s,q) = \int_0^\infty \int_0^\infty e^{-sx-qt} dF(x,t)$$

the double Laplace-Stieltjes transfom (LST) of the function $F(x,t)$. Let $\varphi(s) = \alpha(s,0)$ and $\beta(q) = \alpha(0,q)$ be the LST of the functions $L(x)$ and $B(t)$, respectively. Let $D(x) = \mathbf{P}\{\sigma < x\}$ be the distribution function of stationary total customers capacity σ. Let $\varphi_i = \mathbf{E}\zeta^i$, $\beta_i = \mathbf{E}\xi^i$ and $\alpha_{ij} = \mathbf{E}(\zeta^i\xi^j)$ be the ith moments of the random variables ζ, ξ and the mixed $(i+j)$th moment of the random variables ζ and ξ, respectively. $i,j = 1,2,\dots$, $\rho = a\beta_1 < 1$. Denote by $\delta(s) = \int_0^\infty e^{-sx}dD(x)$ the LST of the function $D(x)$ and by $\delta_i = \mathbf{E}\sigma^i$ the ith moment of total customers capacity σ, $i = 1,2,\dots$.

Then for the system $M/G/1/\infty$ (or for the discipline FIFO) we have [4]:

$$\delta_1^{\text{FIFO}} = \mathbf{E}\sigma^{\text{FIFO}} = a\alpha_{11} + \frac{a^2\beta_2\varphi_1}{2(1-\rho)}. \tag{1}$$

For the system $M/G/1/\infty - EPS$ (or for the discipline EPS) we get [2]:

$$\delta_1^{\text{EPS}} = \mathbf{E}\sigma^{\text{EPS}} = \frac{a\alpha_{11}}{1-\rho}. \tag{2}$$

From the simple relations (1) and (2), we obtain that $\delta_1^{\text{FIFO}} < \delta_1^{\text{EPS}}$ if the inequality $2\beta_1\alpha_{11} > \beta_2\varphi_1$ takes place. For example, if the random variables ζ and ξ are independent, i.e. $\alpha_{11} = \varphi_1\beta_1$, the last inequality takes the form $2\beta_1^2 > \beta_2$. Note that for exponential distributed customer length we have $2\beta_1^2 = \beta_2$. So, in this case for independent ζ and ξ we obtain that $\delta_1^{\text{FIFO}} = \delta_1^{\text{EPS}}$. If the customer length distribution is characterized by variation which is less than for exponential one, we always have $\delta_1^{\text{FIFO}} < \delta_1^{\text{EPS}}$. Evidently, this will be true for the case of positive correlated ζ and ξ (when $\alpha_{11} > \varphi_1\beta_1$).

For many real computer systems (for example, for communicating centers) the customer length can be defined by the relation $\xi = c\zeta + \xi_1$, where $c \geq 0$ and the random variables ζ and ξ_1 are independent.

Denote by κ_i the ith moment of the random variable ξ_1, $i = 1,2,\dots$. Then the first moments of the random variables σ^{FIFO} and σ^{EPS} can be calculated from relations (1) and (2), respectively, where [3]

$$\alpha_{11} = \varphi_1\kappa_1 + c\varphi_2,\ \beta_1 = c\varphi_1 + \kappa_1,\ \beta_2 = c^2\varphi_2 + 2c\varphi_1\kappa_1 + \kappa_2.$$

In this case we have that $\delta_1^{\text{FIFO}} < \delta_1^{\text{EPS}}$ if the following inequality takes place:

$$c^2\varphi_1\varphi_2 + 2\kappa_1(\varphi_1\kappa_1 + c\varphi_2) > \varphi_1\kappa_2. \tag{3}$$

In particular, if a customer length is proportional to his capacity, i.e. $\kappa_1 \equiv 0$, $\kappa_2 \equiv 0$, we have from (3) that $c^2\varphi_1\varphi_2 > 0$. Evidently, this inequality is always true. For example, if we assume additionally that the customer length ζ has an exponential distribution with parameter f, we obtain:

$$\delta_1^{\mathrm{FIFO}} = \frac{1}{f} \cdot \frac{\rho(2-\rho)}{1-\rho}, \qquad \delta_1^{\mathrm{EPS}} = \frac{1}{f} \cdot \frac{2\rho}{1-\rho}.$$

Intuitively this is clear, because in the case of EPS discipline short (or having small capacity) customers are for a small time in the system, while FIFO service organization does not depend on the customer capacity.

3. The case of limited system capacity

In this case, it is interesting to compare loss characteristics for EPS and FIFO disciplines.

If customer's length does not depend on his capacity and has an exponential distribution with parameter f, we obtain [6] for systems $M/M/1/(\infty, V)$ and $M/G/1/(\infty, V) - EPS$ with the same $\rho = a\beta_1$ that the loss probability P has the form:

$$P^{\mathrm{FIFO}} = P^{\mathrm{EPS}} = \begin{cases} \dfrac{1-\rho}{e^{(1-\rho)fV} - \rho} & \text{if } \rho \neq 1, \\ (1+fV)^{-1} & \text{if } \rho = 1. \end{cases}$$

Note that $\beta_1 = 1/\mu$ for the system $M/M/1/(\infty, V)$, where μ is the parameter of customer length.

Later on, we shall compare loss probabilities P and probabilities Q that unit of customer's capacity will be lost (see [7]) for cases of FIFO and EPS disciplines. It is clear (in this case) that probability Q is also the same for both systems under consideration. This fact can be confirmed by results of simulation (see Appendix, tables 1 and 2, where $f = 1$, $\mu = 1$). In our notation, we shall use the low indexes "an" or "sim" to demonstrate that an apropriate characteristic was obtained analytically or by simulation, respectively.

It can be conformed analytically and by simulation that we have the same results for loss characteristics P and Q in the systems $M/M/1/(\infty, V)$ and $M/G/1/(\infty, V) - EPS$ with the same ρ, when customer's length does not depend on his capacity and customer's capacity has the same distribution for both systems.

But if customer's length depends on his capacity, then service discipline has an influence on loss characteristics of the system. This influence depends on the character of this dependence and the value of ρ, but is inessential for small ρ. We demonstrate this fact in tables 3, 4, and 5 (in Appendix), when customer's length is proportional to his capacity and the capacity has an exponential distribution ($\xi = c\zeta$, $c = 1$, $\varphi_1 = \mathbf{E}\zeta = 1$).

It is interesting to compare the last results with those for the case of non-exponential customer volume and length distribution. We present them in tables 6–9 for independent random variables ζ and ξ having the uniform distribution on $[0; 2]$ (see tables 6, 7) and for the case when customer's length ξ is proportional to his capacity ζ (having the same distribution) with coefficient $c = 1$ (see tables 8 and 9).

4. Conclusion

In this paper we have analyzed the influence of service discipline on the first moment of total customers capacity in single-server queueing system with unlimited system capacity and on the loss characteristics for the system with limited total capacity. It was shown that

1) the discipline FIFO is better than EPS from the viewpoint of capacity occupied by customers in the system and loss characteristics;

2) the loss characteristics P and Q depend on service discipline and character of dependence between customer's capacity and his length.

However, the last dependence is inessential for rather small system capacities and small ρ; more precisely, in this case the influence of ζ and ξ dependence is inessential for loss characteristics calculation. Therefore, in practice we often need not to pay attention on this dependence and can use analytical methods to calculate the loss probability for queueing systems with customer length not depending on his capacity.

References

[1] A. M. Alexandrov, B. A. Kaz. Non-homogeneous demands flow service. *Izvestiya AN SSSR. Tekhnicheskaya Kibernetika*, No 2, 47–53, 1973. (In Russian).

[2] B. Sengupta. The spatial requirement of an $M/G/1$ queue or: how to design for buffer space. *Lect. Notes Contr. Inf. Sci.*, **60**, 547–562, 1984.

[3] O. M. Tikhonenko. *Queueing Models in Computer Systems.* Universitetskoe, Minsk, 1990. (In Russian).

[4] O. Tikhonenko. *Metody probabilistyczne analizy systemów informacyjnych.* Akademicka Oficyna Wydawnicza EXIT, Warszawa, 2006.

[5] S. F. Yashkov. *Analysis of Queues in Computers.* Radio i Svyaz, Moskow, 1989. (In Russian).

[6] O. Tikhonenko. Classical and non-classical processor sharing systems with non-homogeneous customers. *Scientific Issues of Jan Długosz University in Częstochowa. Ser. Mathematics*, **XIV**, 133–150, 2009.

[7] O. Tikhonenko, A. Gola, M. Ziółkowski. Estimations of loss characteristics of single-server queueing systems with non-homogeneous customers. *Scientific Issues of Jan Długosz University in Częstochowa. Ser. Mathematics*, **XIII**, 53–66, 2008.

Appendix

Table 1: Probabilities Q for $\rho = 0.2$

V	Q_{sim}^{FIFO}	Q_{sim}^{ESP}
0.0	1.0000	1.0000
1.0	0.7526	0.7526
2.0	0.4454	0.4451
2.5	0.3290	0.3289
3.0	0.2402	0.2401
4.0	0.1230	0.1231
5.0	0.0610	0.0607
6.0	0.0293	0.0295
8.0	0.0067	0.0069
10.0	0.0014	0.0014
12.0	0.0003	0.0003

Table 2: Probabilities Q for $\rho = 0.8$

V	Q_{sim}^{FIFO}	Q_{sim}^{EPS}
0.0	1.0000	1.0000
2.0	0.5774	0.5771
4.0	0.3083	0.3077
6.0	0.1776	0.1772
8.0	0.1085	0.1085
10.0	0.0688	0.0678
15.0	0.0233	0.0232
20.0	0.0085	0.0085
25.0	0.0031	0.0031
30.0	0.0012	0.0011
35.0	0.0004	0.0004

Table 3: Probabilities P and Q for $\rho = 0.2$

V	P_{sim}^{FIFO}	Q_{sim}^{FIFO}	P_{an}^{EPS}	Q_{sim}^{EPS}
0.0	1.0000	1.0000	1.0000	1.0000
1.0	0.3846	0.7454	0.3847	0.7454
2.0	0.1711	0.4445	0.1718	0.4455
3.0	0.0850	0.2531	0.0866	0.2549
4.0	0.0446	0.1419	0.0467	0.1448
5.0	0.0240	0.0790	0.0260	0.0824
6.0	0.0129	0.0435	0.0147	0.0469
8.0	0.0038	0.0130	0.0048	0.0153
10.0	0.0011	0.0038	0.0016	0.0051
12.0	0.0003	0.0011	0.0005	0.0017
15.0	0.0001	0.0002	0.0001	0.0003

Table 4: Probabilities P and Q for $\rho = 0.8$

V	P_{sim}^{FIFO}	Q_{sim}^{FIFO}	P_{an}^{EPS}	Q_{sim}^{EPS}
0.0	1.0000	1.0000	1.0000	1.0000
2.0	0.2570	0.5371	0.2641	0.5424
4.0	0.1272	0.2851	0.1475	0.3111
6.0	0.0715	0.1642	0.0964	0.2036
8.0	0.0429	0.0997	0.0676	0.1426
10.0	0.0267	0.0624	0.0493	0.1042
15.0	0.0090	0.0212	0.0248	0.0525
20.0	0.0032	0.0076	0.0135	0.0285
25.0	0.0012	0.0028	0.0076	0.0160
30.0	0.0004	0.0010	0.0044	0.0092
35.0	0.0002	0.0004	0.0025	0.0054
40.0	0.0001	0.0002	0.0015	0.0031
50.0	0.0000	0.0000	0.0005	0.0011

Table 5: Probabilities P and Q for $\rho = 1.0$

V	P_{sim}^{FIFO}	Q_{sim}^{FIFO}	P_{an}^{EPS}	Q_{sim}^{EPS}
0.0	1.0000	1.0000	1.0000	1.0000
2.0	0.2803	0.5616	0.2902	0.5687
4.0	0.1550	0.3303	0.1819	0.3624
5.0	0.1229	0.2654	0.1539	0.3070
10.0	0.0551	0.1220	0.0870	0.1736
15.0	0.0344	0.0764	0.0606	0.1212
20.0	0.0256	0.0572	0.0465	0.0929
30.0	0.0132	0.0290	0.0317	0.0635
35.0	0.0105	0.0246	0.0248	0.0548
40.0	0.0076	0.0217	0.0241	0.0481
50.0	0.0070	0.0164	0.0195	0.0389
60.0	0.0065	0.0144	0.0163	0.0325
70.0	0.0056	0.0120	0.0140	0.0279
80.0	0.0049	0.0104	0.0122	0.0250

Table 6: Probabilities P and Q for $\rho = 0,2$ when ζ and ξ are independent

V	P_{sim}^{FIFO}	Q_{sim}^{FIFO}	P_{an}^{EPS}	Q_{sim}^{EPS}
0.0	1.0000	1.0000	1.0000	1.0000
0.5	0.7565	0.9397	0.7565	0.9397
1.0	0.5240	0.7658	0.5241	0.7658
1.5	0.3031	0.4904	0.3039	0.4908
2.0	0.0923	0.1226	0.0938	0.1240
2.5	0.0577	0.0846	0.0603	0.0873
3.0	0.0314	0.0492	0.0344	0.0528
4.0	0.0051	0.0076	0.0079	0.0113
5.0	0.0012	0.0018	0.0023	0.0034
6.0	0.0002	0.0003	0.0006	0.0009
7.0	0.0000	0.0001	0.0002	0.0002

Table 7: Probabilities P and Q for $\rho = 0,8$ when ζ and ξ are independent

V	P_{sim}^{FIFO}	Q_{sim}^{FIFO}	P_{an}^{EPS}	Q_{sim}^{EPS}
0.0	1.0000	1.0000	1.0000	1.0000
1.0	0.5854	0.8065	0.5878	0.8075
2.0	0.3008	0.3944	0.3111	0.4016
3.0	0.1857	0.2622	0.2083	0.2854
4.0	0.1153	0.1605	0.1435	0.1936
5.0	0.0759	0.1068	0.1044	0.1418
6.0	0.0515	0.0723	0.0779	0.1057
8.0	0.0250	0.0352	0.0455	0.0618
10.0	0.0127	0.0178	0.0280	0.0380
15.0	0.0024	0.0034	0.0090	0.0122
20.0	0.0005	0.0007	0.0030	0.0040
25.0	0.0001	0.0001	0.0010	0.0014
30.0	0.0000	0.0000	0.0003	0.0004
35.0			0.0001	0.0001

Table 8: Probabilities P and Q for $\rho = 0,2$ when ξ is proportional to ζ

V	P_{sim}^{FIFO}	Q_{sim}^{FIFO}	P_{an}^{EPS}	Q_{sim}^{EPS}
0.0	1.0000	1.0000	1.0000	1.0000
1.0	0.5158	0.7597	0.5158	0.7596
2.0	0.1141	0.1423	0.1150	0.1431
3.0	0.0438	0.0686	0.0484	0.0740
4.0	0.0079	0.0112	0.0138	0.0192
5.0	0.0020	0.0031	0.0049	0.0072
6.0	0.0003	0.0005	0.0015	0.0022
7.0	0.0001	0.0001	0.0005	0.0007
8.0	0.0000	0.0000	0.0002	0.0002
9.0			0.0001	0.0001

Table 9: Probabilities P and Q for $\rho = 0,8$ when ξ is proportional to ζ

V	P_{sim}^{FIFO}	Q_{sim}^{FIFO}	P_{an}^{EPS}	Q_{sim}^{EPS}
0.0	1.0000	1.0000	1.0000	1.0000
0.5	0.7582	0.9400	0.7582	0.9400
1.0	0.5570	0.7855	0.5573	0.7857
2.0	0.3179	0.3963	0.3252	0.4023
3.0	0.1832	0.2621	0.2142	0.2956
4.0	0.1109	0.1536	0.1558	0.2084
5.0	0.0703	0.0997	0.1168	0.1578
6.0	0.0464	0.0658	0.0901	0.1221
8.0	0.0219	0.0310	0.0569	0.0770
10.0	0.0110	0.0155	0.0376	0.0508
15.0	0.0021	0.0030	0.0148	0.0200
20.0	0.0004	0.0006	0.0062	0.0084
25.0	0.0001	0.0001	0.0027	0.0036
30.0	0.0000	0.0000	0.0011	0.0015
35.0			0.0005	0.0007
40.0			0.0002	0.0003
45.0			0.0001	0.0001