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UNIFORMLY CONTINUOUS COMPOSITION OPERATOR IN THE SPACE OF FUNCTIONS OF TWO VARIABLES OF BOUNDED Φ-VARIATION IN THE SENSE OF SCHRAMM

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Abstract

We prove in this paper that if the composition operator H, generated by a function $h : I_a^b \times C \to Y$, maps $\Phi_1 BV(I_a^b, C)$ into $\Phi_2 BV(I_a^b, Y)$ and is uniformly continuous, then the left-left regularization h^* of h is an affine function with respect to the third variable.

1. Introduction

Let I_a^b denote the rectangle $[a_1, b_1] \times [a_2, b_2]$. Let $(X, |\cdot|), (Y, |\cdot|)$ be real normed spaces and C be a convex cone in X. For a function $h : I_a^b \times C \to Y$, denote by $X^{I_a^b}$ the algebra of all functions $f : I_a^b \to X$ and by $H : X^{I_a^b} \to Y^{I_a^b}$ the Nemytskij operator generated by the function h defined by

$$(Hf)(t,s) = h(t,s,f(t,s)), \quad f \in X^{I_a^b}, (t,s) \in I_a^b.$$

Let $(\Phi BV(I_a^b, X), \|\cdot\|_\Phi)$ be a Banach space of functions $f \in X^{I_a^b}$ which have bounded Φ-variation in the sense of Schramm, where the norm $\|\cdot\|_\Phi$ is defined with the aid of Luxemburg-Nakano-Orlicz seminorm [14, 7, 15].

Assume that H maps the set of functions $f \in \Phi BV(I_a^b, X)$ such that $f(I_a^b) \subset C$ into $\Phi BV(I_a^b, Y)$. In the present paper, we prove that, if H is

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uniformly continuous, then the left-left, right-right, left-right and right-left regularizations of its generator h with respect to first two variables are affine functions with respect to the third variable. This extends the main results of [5] and [3]. In some spaces the representation theorems for the Lipschitzian Nemytskij operators have been established before, see [3-8, 11].

2. Preliminaries

In this section we recall some facts which will be in need in further considerations.

Denote by $\mathbb{R}$ the set of all real numbers and put $\mathbb{R}_+ = [0, \infty)$. We say that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a φ-function if φ is continuous on $\mathbb{R}_+$, $\varphi(0) = 0$, φ is increasing on $\mathbb{R}_+$ and $\varphi(t) \to \infty$ when $t \to \infty$. Let us recall first the concept of the bounded φ-variation in the sense of Wiener ([17]). Namely, we say that a function $u : [a, b] \to \mathbb{R}$ has a φ-bounded variation in the Wiener sense with respect to a φ-function φ provided the quality $V_\varphi^W(u)$ defined by the formula

$$V_\varphi^W(u) = V_\varphi^W(u; [a, b]) = \sup_\pi \sum_{j=1}^{n} \varphi\left(|u(t_j) - u\left(t_{j-1}\right)|\right)$$

is finite. Here the supremum is taken over all partitions π of the interval $[a, b]$.

Next, let $\Phi = \{\phi_n\}$ be a sequence of increasing convex functions, defined on the set of nonnegative real numbers and such that $\Phi_n(0) = 0$ and $\Phi_n(t) > 0$ for $t > 0$ and $n = 1, 2, \ldots$ We say that Φ is Φ^*-sequence if $\phi_{n+1}(t) \leq \phi_n(t)$ for all n, t and Φ-sequences and in addition

$$\sum_{n=1}^{\infty} \phi_n(t) \quad \text{diverges for all } t > 0. \tag{1}$$

If Φ is either a Φ^*-sequence or a Φ-sequence, we say that a function u is of Φ-bounded variation in the Schramm sense if the Φ-sum $\sum_n \phi_n\left(|u(I_n)|\right)$ is finite for any non-overlapping collection $\{I_n\}$ of I ([16]). If $I_n = [a_n, b_n]$ is a subinterval of the interval I $(n = 1, 2, \ldots)$ we write $u\left(I_n\right) := u\left(b_n\right) - u\left(a_n\right)$.

We introduce the $\Phi = \{\phi_{n,m}\}$ two dimensional sequence of increasing convex functions, such that $\phi_{n,m}(0) = 0$ and $\phi_{n,m}(t) > 0$ for $t > 0$ and

$n, m = 1, 2, \ldots$ We say that Φ is Φ-sequence [4, 3] if

$$\begin{aligned} &\phi_{n',m'}(t) \leq \phi_{n,m}(t) \text{ for each } n' \leq n,\ m' \leq m,\ t \in [0, \infty) \\ \text{and} \qquad &\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m}(t) \ \text{ diverges for } t > 0. \end{aligned} \tag{2}$$

3. Notation, definitions and auxiliary facts

At the beginning assume that $a = (a_1, a_2)$, $b = (b_1, b_2)$ are two fixed points in the plane $\mathbb{R}^2$. Denote by I_a^b the rectangle generated by the points a and b, i.e., $I_a^b = [a_1, b_1] \times [a_2, b_2]$.

Next, let us assume that $\{I_n\}$ and $\{J_m\}$ are two sequences of closed subintervals of the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively. It means $I_n = [a_1^n, b_1^n]$, $(n = 1, 2, \ldots)$, $J_m = [a_2^m, b_2^m]$, $(m = 1, 2, \ldots)$.

Finally assume that $f : I \to \mathbb{R}$ is a given function and let $\Phi = \{\phi_{n,m}\}$ be a fixed double Φ sequence.

Fix $x_2 \in J_1 = [a_2, b_2]$ and consider the function $f(\cdot, x_2) : [a_1, b_1] \to \mathbb{R}$. The quantity V_{Φ,I_1}^S defined by the formula

$$\begin{aligned} V_{\Phi,I_1}^S(u) &= \sup_{\pi_1} \sum_{n=1}^{\infty} \phi_{n,m}\left(|f(I_n, x_2)|\right) \\ &= \sup_{\pi_1} \sum_{n=1}^{\infty} \phi_{n,m}\left(|f(b_n, x_2) - f(a_n, x_2)|\right) \\ &= \sup_{\pi_1} \sum_{n=1}^{\infty} \phi_{n,m}\left(|f(b_n, x_2) - f(a_n, x_2)|\right), \end{aligned} \tag{3}$$

is said to be Φ-*variation in the sense of Schramm* of the function $f(\cdot, x_2)$. In the case when $V_{\Phi,I_1}^S(f) < \infty$ we will say that f has a *bounded* Φ*-variation in the sense of Schramm with respect to the first variable* (with fixed the second one). In the same way one can define the concept of the Φ-variation of the function $f(x_1, \cdot)$ in the Schramm sense. It is denoted by V_{Φ,J_1}^S. Obviously, if $V_{\Phi,J_1}^S(f) < \infty$ then one can say that f *has bounded* Φ*-variation in the sense of Schramm with respect to the second variable* (with fixed the first one).

Let us pay attention to the fact that the least upper bound in formula (3) is taken with respect to all sequences $\{I_n\}$ of subintervals of the interval I_1. Analogously we understand the least upper bound in the definition of the quantity V^S_{Φ,J_1} [4, 3]. Further, we provide the definition of the concept of two dimensional (or bi-dimensional) variation in the sense of Schramm.

Definition 1. *The quantity $V^S_{\Phi,I_a^b}(f)$ defined by the formula*

$$\begin{aligned} V^S_{\Phi,I_a^b}(f) &= \sup_{\pi_1,\pi_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m}\left(|f(I_n, J_m)|\right) = \\ &= \sup_{\pi_1,\pi_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m}\left(|f(b_n, J_m) - f(a_n, J_m)|\right) = \end{aligned}$$

$$= \sup_{\pi_1,\pi_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m}\left(|f(a_n, c_m) + f(b_n, d_m) - f(a_n, d_m) - f(b_n, c_m)|\right),$$

is said to be the bi-dimensional variation in the sense of Schramm of the function f where the least upper bound is considered on all collections of closed and bounded subintervals $\{I_n\}$, $\{J_m\}$ of intervals I_1 and J_1 respectively.

Finally, we introduce the definition of the main considered concept.

Definition 2. *We say that the quantity $TV^S_{\Phi}(f)$ defined by the formula*

$$TV^S_{\Phi}(f) = V^S_{\Phi,I_1}(f) + V^S_{\Phi,J_1}(f) + V^S_{\Phi,I_a^b}(f)$$

is the total Φ-variation of the function f in the sense of Schramm.

A function f is referred as a function with bounded total Φ-variation provided $TV^S_{\Phi}(f) < \infty$.

By $\Phi BV(I_a^b)$ we denote the set of all functions $f : I_a^b \to X$ which have bounded total Φ-variation in the sense of Schramm.

By P_Φ let us denote the functional defined on the set $\Phi BV(I_a^b)$ in the following way:

$$P_\Phi(f) = \inf\left\{\epsilon > 0 : TV^S_{\Phi}\left(\frac{f}{\epsilon}\right) \le 1\right\}. \tag{4}$$

The main result in [4] asserts that the set $\Phi BV\left(I_a^b\right)$ forms a Banach algebra with the norm defined by the formula

$$\|f\|_\Phi = |f(a)| + P_\Phi(f). \tag{5}$$

Observation 1. If we take the Φ-sequence defined as follows

$$\Phi = \{\phi_{n,m} : \phi_{n,m}(t) = t^p;\quad 1 < p < \infty,\, n, m = 1, 2, ...\}$$

then we can check that $P_\Phi(f) = \left(TV^S_{\Phi}(f)\right)^{1/p}$.

Our next result is contained in the following Lemma.

Lemma 1. *Let* $f \in \Phi BV\left(I_a^b; X\right)$ *and* $\Phi \in \Phi^*$. *Then* f *has the following properties:*

(1) *If* $(t,s),(t',s') \in I_b^a$ *then* $|f(t,s) - f(t',s')| \le 4\Phi_{n,m}^{-1}\left(\frac{1}{2}\right) P_\Phi(f)$.
(2) *If* $P_\Phi(f) > 0$ *then* $TV_\Phi^S(f/P_\Phi(f)) \le 1$.
(3) *Let* $r > 0$. *Then* $TV_\Phi^S(f/r) \le 1$ *if and only if* $P_\Phi(f) \le r$.

Observation 2. From part (1) of Lemma 1. we deduced that each function $f \in \Phi BV(I_a^b; X)$ is bounded. Moreover, the following estimation is satisfied

$$\|f\|_\infty = \sup\left\{|f(t,s)| : (t,s) \in I_a^b\right\} \le |f(a)| + 4\Phi_{n,m}^{-1}\left(\tfrac{1}{2}\right) P_\Phi(f) \qquad (6)$$

if $n, m = 1, 2, \ldots$ where the symbol $\|f\|_\infty$ denotes the supremum norm, i.e.

$$\|f\|_\infty = \sup\left\{|f(t,s)| : (t,s) \in I_a^b\right\}$$

Let us fix arbitrary $f \in \Phi BV(I_a^b)$. Then the function $f^* : I_a^b \to X$ defined by formula

$$f^*(x_1,x_2) = \begin{cases} \lim\limits_{(y_1,y_2)\to(x_1-0,x_2-0)} f(y_1,y_2), & (x_1,x_2) \in (a_1,b_1] \times (a_2,b_2], \\ \lim\limits_{(y_1,y_2)\to(x_1-0,a_2+0)} f(y_1,y_2), & x_1 \in (a_1,b_1] \text{ and } x_2 = a_2, \\ \lim\limits_{(y_1,y_2)\to(a_1+0,x_2-0)} f(y_1,y_2), & x_1 = a_1 \text{ and } x_2 \in (a_2,,b_2], \\ \lim\limits_{(y_1,y_2)\to(a_1+0,a_2+0)} f(y_1,y_2), & x_1 = a_1 \text{ and } x_2 = a_2 \end{cases}$$

is called the left-left regularization of the function f. The existence of all one-sided limits used above was proved in [2].

Definition 3. *A function* $f : I_a^b \to \mathbb{R}$ *is said to be left-left continuous if*

$$\lim_{y_1\to x_1-0,\, y_2\to x_2-0} f(y_1,y_2) = f(x_1,x_2) \text{ for all } (x_1,x_2) \in (a_1,b_1] \times (a_2,b_2].$$

By $\Phi BV^*\left(I_a^b\right)$ is denoted the subspace of $\Phi BV\left(I_a^b\right)$ consisting of those functions which are left-left continuous on $(a_1,b_1] \times (a_2,b_2]$ and by $\mathcal{L}(X,Y)$ the space defined by

$$\mathcal{L}(X,Y) := \{f : X \to Y : f \text{ is linear}\}$$

Lemma 2 ([3])**.** *If* $f \in \Phi BV\left(I_a^b\right)$, *then* $f^* \in \Phi BV^*\left(I_a^b\right)$.

In the sequel we are going to deal with the main result of this paper.

4. The Composition Operator

Our main result reads as follows:

Theorem 1. *Let $I_a^b \subset \mathbb{R}^2$ be a rectangle, $(X, |\cdot|_X)$ be a real normed space, $(Y, |\cdot|_Y)$ be a real Banach space, C be a convex cone in X. If the composition operator H generated by $h : I_a^b \times C \longrightarrow Y$ transforms $\Phi_1 BV\left(I_a^b, C\right)$ into $\Phi_2 BV\left(I_a^b, Y\right)$ and is uniformly continuous, then there exist functions $A \in \mathcal{L}(X, Y)$ and $B \in \Phi_2 BV\left(I_a^b, Y\right)$ such that*

$$h^*(t,s,y) = A(t,s)y + B(t,s), \quad (t,s) \in I_a^b, \quad y \in C,$$

where h^* is the left-left regularization of h.

Proof. For every $y \in C$ the constant function $f(t,s) = y$ with $(t,s) \in I_a^b$ belongs to $\Phi_1 BV\left(I_a^b, C\right)$. Since H maps $\Phi_1 BV\left(I_a^b, C\right)$ into $\Phi_2 BV\left(I_a^b, Y\right)$, it follows that the function $(t,s) \mapsto h(t,s,y)$, $(t,s) \in I_a^b$, belongs to $\Phi_2 BV\left(I_a^b, Y\right)$. Now the completeness of $\Phi_2 BV\left(I_a^b, Y\right)$ implies the existence of the left-left regularization h^* of h.

By assumption H is uniformly continuous on $\Phi_1 BV\left(I_a^b, C\right)$. Let ω be the modulus of continuity of H that is

$$\omega(\rho) := \sup\left\{\|H(f_1) - H(f_2)\|_{\Phi_2} : \|f_1 - f_2\|_{\Phi_1} \le \rho;\ f_1, f_2 \in \Phi_1 BV\left(I_a^b, C\right)\right\}$$

for $\rho > 0$. Hence we get

$$\|H(f_1) - H(f_2)\|_{\Phi_2} \le \omega\big(\|f_1 - f_2\|_{\Phi_1}\big), \quad \text{for} \quad f_1, f_2 \in \Phi_1 BV\left(I_a^b, C\right). \quad (7)$$

From the definition of the norm $\|\cdot\|_{\Phi_2}$ we obtain

$$P_{\Phi_2}\big(H(f_1) - H(f_2)\big) \le \|H(f_1) - H(f_2)\|_{\Phi_2}, \text{ for } f_1, f_2 \in \Phi_1 BV\left(I_a^b, C\right). \quad (8)$$

In view of (8), Definitions 1., 2. and Lemma 1.(3), if $\omega\big(\|f_1 - f_2\|_\Phi\big) > 0$, then

$$V^S_{\Phi, I_a^b}(f)\left(\frac{(H(f_1) - H(f_2))(\cdot, a_2)}{\omega\big(\|f_1 - f_2\|_{\Phi_1}\big)}\right) \le TV^S_{\Phi_2}\left(\frac{Hf_1 - Hf_2}{\omega\left(\|f_1 - f_2\|_{\Phi_1}\right)}\right) \le 1. \quad (9)$$

The definitions of the operator H and the functional $V^S_{\Phi, I_a^b}(f)$ imply that for any

$$a_1 \le \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n \le b_1,$$

and

$$a_2 \le \overline{\alpha}_1 < \overline{\beta}_1 < \overline{\alpha}_2 < \overline{\beta}_2 < \cdots < \overline{\alpha}_m < \overline{\beta}_m \le b_2,$$

if $n, m \in \mathbb{N}$, the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\phi_{i,j}\left(\frac{\left|h(\alpha_i,\overline{\alpha}_j,f_1(\alpha_i,\overline{\alpha}_j))-h(\alpha_i,\overline{\alpha}_j,f_2(\alpha_i,\overline{\alpha}_j))-h\left(\alpha_i,\overline{\beta}_j,f_1\left(\alpha_i,\overline{\beta}_j\right)\right)+h\left(\alpha_i,\overline{\beta}_j,f_2\left(\alpha_i,\overline{\beta}_j\right)\right)\right.}{\omega\left(\|f_1-f_2\|_{\Phi_1}\right)}\right. \tag{10}$$
$$\left.+\frac{\left.-h(\beta_i,\overline{\alpha}_j,f_1(\beta_i,\overline{\alpha}_j))+h(\beta_i,\overline{\alpha}_j,f_2(\beta_i,\overline{\alpha}_j))+h\left(\beta_i,\overline{\beta}_j,f_1\left(\beta_i,\overline{\beta}_j\right)\right)-h\left(\beta_i,\overline{\beta}_j,f_2\left(\beta_i,\overline{\beta}_j\right)\right)\right|}{\omega\left(\|f_1-f_2\|_{\Phi_1}\right)}\right)\le 1.$$

holds.

For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, we define functions $\eta_{\alpha,\beta} : \mathbb{R} \to [0,1]$ by the following formula:

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0 & \text{if} \quad t \le \alpha \\ \dfrac{t-\alpha}{\beta-\alpha} & \text{if} \quad \alpha \le t \le \beta \\ 1 & \text{if} \quad \beta \le t\,. \end{cases} \tag{11}$$

First let us fix $t \in (a_1, b_1]$, $s \in (a_2, b_2]$ and $n, m \in \mathbb{N}$. For arbitrary sequences

$$a_1 \le \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_n < \beta_n \le t$$
$$a_2 \le \overline{\alpha}_1 < \overline{\beta}_1 < \overline{\alpha}_2 < \overline{\beta}_2 < \ldots < \overline{\alpha}_m < \overline{\beta}_m \le s$$

and $y_1, y_2 \in C$, $y_1 \ne y_2$ the functions $f_1, f_2 : I \to X$ defined by

$$f_\ell(\tau,\gamma) := \frac{1}{2}\left[(\eta_{\alpha_i,\beta_i}(\tau) + \eta_{\overline{\alpha}_j,\overline{\beta}_j}(\gamma) - 1)(y_1 - y_2) + y_\ell + y_2\right], \tag{12}$$

for every $(\tau,\gamma) \in I_a^b$, $\ell = 1, 2$; belong to the space $\Phi_1 BV(I_a^b, C)$. From this we infer that

$$f_1(\cdot,\cdot) - f_2(\cdot,\cdot) = \frac{y_1 - y_2}{2},$$

therefore

$$\|f_1 - f_2\|_\Phi = \left|\frac{y_1 - y_2}{2}\right|;$$

moreover

$$f_1(\alpha_i,\overline{\alpha}_j) = y_2;\ f_2(\alpha_i,\overline{\alpha}_j) = \frac{-y_1 + 3y_2}{2};$$
$$f_1\left(\alpha_i,\overline{\beta}_j\right) = \frac{y_1 + y_2}{2};\ f_2\left(\alpha_i,\overline{\beta}_j\right) = y_2,$$
$$f_1(\beta_i,\overline{\alpha}_j) = y_2;\ f_2(\beta_i,\overline{\alpha}_j) = \frac{-y_1 + 3y_2}{2};$$
$$f_1\left(\beta_i,\overline{\beta}_j\right) = x_1;\ f_2\left(\beta_i,\overline{\beta}_j\right) = \frac{y_1 + y_2}{2}.$$

Applying (10), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\phi_{i,j}\left(\frac{\left|h(\alpha_i,\overline{\alpha}_j,y_2)-h\left(\alpha_i,\overline{\alpha}_j,\frac{-y_1+3y_2}{2}\right)-h\left(\alpha_i,\overline{\beta}_j,\frac{y_1+y_2}{2}\right)+h\left(\alpha_i,\overline{\beta}_j,y_2\right)\right.}{\omega\left(\|f_1-f_2\|_{\Phi_1}\right)}\right.$$
$$\left.+\ \frac{\left.-h(\beta_i,\overline{\alpha}_j,y_2)+h\left(\beta_i,\overline{\alpha}_j,\frac{-y_1+3y_2}{2}\right)+h\left(\beta_i,\overline{\beta}_j,x_1\right)-h\left(\beta_i,\overline{\beta}_j,\frac{y_1+y_2}{2}\right)\right|}{\omega\left(\|f_1-f_2\|_{\Phi_1}\right)}\right)\leq 1. \tag{13}$$

In view of continuity of $\phi_{i,j}$, and Lemma 2., the left-left continuity of h^* we infer that

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\phi_{i,j}(x)\leq 1 \text{ for } n,m=1,2,\dots, \tag{14}$$

where

$$x=\frac{\left|h^*(t,s,y_1)-2h^*\left(t,s,\frac{y_1+y_2}{2}\right)+h^*(t,s,y_2)\right|}{\omega\left(\left|\frac{y_1-y_2}{2}\right|\right)}.$$

Since $n,m\in\mathbb{N}$ are arbitrary, condition (14) implies inequality

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\phi_{i,j}(x)\leq 1.$$

In view of (2) we get $x=0$, i.e.

$$h^*\left(t,s,\frac{y_1+y_2}{2}\right)=\frac{h^*(t,s,y_1)+h^*(t,s,y_2)}{2} \tag{15}$$

for all $(t,s)\in(a_1,b_1]\times(a_2,b_2]$ and $y_1,y_2\in C$.

For $t\in(a_1,b_1]$ and $s=b_2$ let

$$a_1<\alpha_1<\beta_1<\alpha_2<\beta_2<\ldots<\alpha_n<\beta_n<t$$

and

$$a_2<\overline{\alpha}_1<\overline{\beta}_1<\overline{\alpha}_2<\overline{\beta}_2<\ldots<\overline{\alpha}_m<\overline{\beta}_m<b_2.$$

Proceeding as above we get (13).

If $\alpha_1\uparrow t$ and $\beta_m\downarrow s$ in (13), then we get (15).

The cases when $t=a_1$ and $s\in(a_2,b_2]$ or $t=a_1$ and $s=a_2$ can be treated similarly. Consequently

$$h^*\left(t,s,\frac{y_1+y_2}{2}\right)=\frac{h^*(t,s,y_1)+h^*(t,s,y_2)}{2}$$

is valid for all $(t,s)\in I_a^b$ and all $y_1,y_2\in C$.

Therefore, the function $h^*(t,s,\cdot)$ satisfies the Jensen functional equation in C for $(t,s) \in I_a^b$. Modifying the standard argument (Kuczma [6]), we conclude that for each $(t,s) \in I_a^b$ there exist additive functions $A(t,s) : C \to \mathcal{L}(X,Y)$ and $B(t,s) \in Y$ such that

$$h^*(\cdot, y) = A(\cdot)y + B(\cdot), \quad y \in C. \tag{16}$$

The uniform continuity of the operator $H : \Phi_1 BV\left(I_a^b, C\right) \to \Phi_2 BV\left(I_a^b, Y\right)$ implies the continuity of the additive function $A(t,s)$.

Consequently $A(t,s) \in \mathcal{L}(X,Y)$.

Finally, notice that $A(t,s)(0) = \{0\}$ for every $(t,s) \in I_a^b$. Therefore, putting $y = 0$ in (16), we get

$$h^*(t,s,0) = B(t,s), \quad (t,s) \in I_a^b,$$

which implies $B \in \Phi_2 BV\left(I_a^b, Y\right)$. □

Observation 3. A similar theorem to Theorem 1. is valid for the right-right, right-left and left-right regularizations of $h(\cdot, y)$, $y \in C$.

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