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ON UPPER AND LOWER STRONG QUASI-UNIFORM CONVERGENCE OF MULTIVALUED MAPS

IRENA DOMNIK, AGATA SOCHACZEWSKA

Abstract

The concept of strong convergence of functions and multifunctions was introduced by I. Kupka, V. Toma and A. Sochaczewska. In this paper we consider new definitions of convergence for the nets of multifunctions – upper and lower strong quasi-uniform convergence.

1. Introduction

In literature we can find different kinds of convergence of functions and multifunctions in topological spaces. We will introduce these types which are formulated in the terms of open covers and stars.

Definition 1. *Let (Y, τ) be a topological space. The set $\bigcup\{A \in \mathcal{A} : A \cap E \neq \emptyset\}$ is called a star of a set E with respect to the cover $\mathcal{A}$ of the space Y. This set is denoted by* $\mathrm{St}(E, \mathcal{A})$.

The concept of strong convergence of functions was considered by I. Kupka and V. Toma in ([4]).

Definition 2. ([4]) *Let X be an arbitrary set and (Y, τ) be a topological space. If $\{f_j : j \in J\}$ is a net of functions from X to Y, then we say that $\{f_j : j \in J\}$*

Irena Domnik — Pomeranian University in Słupsk,
Agata Sochaczewska — Pomeranian University in Słupsk,

strongly converges to a function $f : X \longrightarrow Y$ *if for every open cover* $\mathcal{A}$ *of* Y *there exists an index* $j_0 \in J$ *such that* $f_j(x) \in \mathrm{St}(f(x), \mathcal{A})$ *for every* $x \in X$ *and for every* j *such that* $j \geq j_0$.

The next definition of the strong convergence has been introduced by A. Sochaczewska in [6].

Definition 3. ([6]) *Let* (X, τ_X), (Y, τ_Y) *be topological spaces. Let moreover* $f_n, f : X \to Y$ *if* $n \in \mathbb{N}$. *We say that a sequence* $(f_n)_{n\in\mathbb{N}}$ *is strongly quasi-uniformly convergent to the function* f *if it is pointwise convergent and for each open cover* $\mathcal{A}$ *of* Y *and for each* $x_0 \in X$ *there exists* $n_0 \in \mathbb{N}$ *such that for every* n, $n \geq n_0$ *there exists a neighbourhood* U *of* x_0 *such that*

$$f_n(x) \in \mathrm{St}(f(x), \mathcal{A})$$

for all $x \in U$.

The convergence in the sense of I. Kupka and V. Toma implies the strong quasi-uniform convergence, but not conversely ([6]).

Let X and Y be two sets and $S(Y)$ denote the set of all nonempty subsets of Y. A multivalued function F from X into Y can be considered as a function from X to $S(Y)$. Nevertheless, we will write $F : X \to Y$.

The strong convergence was considered for nets of multivalued maps by I. Kupka in ([3]). Namely:

Definition 4. ([3]) *Let* $(X, \tau_X), (Y, \tau_Y)$ *be topological spaces. Let* $\{F_j : j \in J\}$ *be a net of multifunctions from* X *into* Y. *Let* F *be a multifunction from* X *to* Y. *We say that* $\{F_j : j \in J\}$ *strongly converges to* F *if for every open cover* $\mathcal{A}$ *of* Y *there exists an index* $j_0 \in J$ *such that the inclusions*

$$F(x) \subset \mathrm{St}(F_j(x), \mathcal{A}), \qquad F_j(x) \subset \mathrm{St}(F(x), \mathcal{A})$$

hold for all j, $j \geq j_0$ *and* $x \in X$.

I. Domnik in ([1]) introduced the following kind of convergence of nets of multivalued maps:

Definition 5. ([1]) *Let* X *be a nonempty set and let* (Y, τ) *be a topological space. A net* $\{F_j : j \in J\}$ *of multivalued maps* $F_j : X \to Y$ *is said to be upper (respectively: lower) strongly convergent to a multivalued map* $F : X \to Y$ *if*

for each open cover $\mathcal{A}$ of Y there exists $j_0 \in J$ such that $F_j(x) \subset \mathrm{St}(F(x), \mathcal{A})$ (respectively: $F(x) \subset \mathrm{St}(F_j(x), \mathcal{A})$) for every $j \in J$, $j \geq j_0$, and $x \in X$.

I. Domnik considered properties of this kind of convergence, for example preservation of continuity ([1]).

2. Upper and lower strong quasi-uniform convergence

In this paper we will consider new types of convergence for the net of multivalued maps — upper and lower strong quasi-uniform convergence.

Definition 6. *Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces. A net $\{F_j : j \in J\}$ of multivalued maps $F_j : X \to Y$ is said to be upper (respectively: lower) strongly quasi-uniformly convergent to a map $F : X \to Y$ if for every open cover $\mathcal{A}$ of the space Y and for each point $x_0 \in X$ there exists $j_0 \in J$ such that for every j, $j \geq j_0$, there exists a neighbourhood U of the point x_0 with the inclusion $F_j(x) \subset \mathrm{St}(F(x), \mathcal{A})$ (respectively: $F(x) \subset \mathrm{St}(F_j(x), \mathcal{A})$) for every point x of the set U.*

Let observe that this kind of convergence is closely connected with the convergence of nets of multivalued maps in the sense of I. Domnik. Moreover, the upper strong quasi-uniform convergence follows from the convergence in the sense of A. Sochaczewska for sequences of functions.

In further part of the article we will apply the following definition and lemmas.

Definition 7. ([2]) *Let (Y, τ_Y) be a topological space. A subset A of Y is called α-paracompact if for every τ_Y-open cover $\mathcal{A}$ of A there is a τ_Y-open locally finite cover $\mathcal{B}$ such that $\mathcal{B}$ is a refinement of $\mathcal{A}$.*

Remark 1. ([2]) *Every compact set is α-paracompact.*

Lemma 1. ([2]) *Every α-paracompact subset of a Hausdorff space is closed.*

Lemma 2. ([1]) *Let (Y, τ_Y) be a regular space. If A is α-paracompact subset of Y, U is open in Y and A is the subset of U, then there exists an open set V satisfying inclusions $A \subset V \subset \mathrm{cl}(V) \subset U$.*

For an open subset U of Y we define

$$U^+ := \{B \in S(Y) : B \subset U\}, \qquad U^- := \{B \in S(Y) : B \cap U \neq \emptyset\}.$$

The families

$$\mathcal{B} := \{U^+ : U \in \tau_Y\}, \qquad \mathcal{P} := \{U^- : U \in \tau_Y\}$$

form a base and subbase of the upper and lower Vietoris topology, respectively. These topologies will be denoted by τ_Y^+ and τ_Y^-.

Let $F, F_j : X \to Y, j \in J$ be multivalued maps. We will write

$$F \in (\tau_Y^+) \lim F_j, \quad \text{respectively} \quad F \in (\tau_Y^-) \lim F_j$$

if the net $\{F_j : j \in J\}$ is pointwise convergent to F with respect to topology τ_Y^+, respectively to τ_Y^-.

Theorem 1. *Let (X, τ_X) be a topological space, (Y, τ_Y) — a regular space. If a multivalued map $F\colon X \to Y$ has α-paracompact values, then the upper strong quasi-uniform convergence of a net $\{F_j : j \in J\}$ to F implies τ^+-pointwise convergence.*

Proof. Let $x_0 \in X$ and $V^+ \in \mathcal{B}$ be such that $F(x_0) \in V^+$. Then V is an open subset of Y and $F(x_0) \subset V$. By the assumption the values of the multivalued map F are α-paracompact and Y is a regular space. It implies the existence of an open subset W of Y with properties $F(x_0) \subset W \subset \mathrm{cl}(W) \subset V$. The family $\mathcal{A} = \{V, Y \setminus \mathrm{cl}(W)\}$ forms an open cover of the space Y.

The net $\{F_j : j \in J\}$ is upper strongly quasi-uniformly convergent to F. It means that for a cover $\mathcal{A}$ there exists $j_0 \in J$ such that for $j, j \geq j_0$ there is a neighbourhood U of x_0 such that the inclusion $F_j(x) \subset \mathrm{St}(F(x), \mathcal{A})$ holds for every point x from the set U. Assume $j \geq j_0$. In particular, $x_0 \in U$ and we have the condition $F_j(x_0) \subset \mathrm{St}(F(x_0), \mathcal{A}) = V$. It follows that $F_j(x_0) \in V^+$ for all j, $j \geq j_0$. Hence we obtain

$$F \in (\tau_Y^+) \lim F_j.$$

□

Theorem 2. *Let (X, τ_X), (Y, τ_Y) be topological spaces. If Y is a regular space and a net $\{F_j : j \in J\}$ is lower strongly quasi-uniformly convergent to F, then $F \in (\tau_Y^-) \lim F_j$.*

Proof. Let $x_0 \in X$ and V^- be a set from subbase $\mathcal{P}$ of lower Vietoris topology. It means that V is an open subset of Y such that $V \cap F(x_0) \neq \emptyset$. Let us take an element y from this intersection. Since $y \in V$ and Y is a regular space,

then there exists an open set W satisfying condition $y \in W \subset \mathrm{cl}(W) \subset V$. The family $\mathcal{A} := \{V, Y \setminus \mathrm{cl}(W)\}$ forms an open cover of the space Y. From lower strong quasi-uniform convergence of the net $\{F_j : j \in J\}$ to F there exists $j_0 \in J$ such that for every j, $j \geq j_0$, there is a neighbourhood U of the point x_0 with the inclusion $F(x) \subset \mathrm{St}(F_j(x), \mathcal{A})$ for every $x \in U$. Let j, $j \geq j_0$, be fixed. There exists a neighbourhood U of the point x_0 such that $F(x) \subset \mathrm{St}(F_j(x), \mathcal{A})$ for every $x \in U$. This condition is satisfied for x_0, too. Hence $F(x_0) \subset \mathrm{St}(F_j(x_0), \mathcal{A})$. Since $y \in \mathrm{St}(F_j(x_0), \mathcal{A})$, then there exists a set from the cover $\mathcal{A}$ containing y.

We assumed that $y \in W$, so $y \notin Y \setminus \mathrm{cl}(W)$. Hence $y \in V$ and $V \cap F_j(x_0) \neq \emptyset$ for j, $j \geq j_0$. In consequence, $F \in (\tau_Y^-) \lim F_j$. □

Definition 8. ([5]) *Let (X, τ_X), (Y, τ_Y) be topological spaces. A multivalued map $F : X \to Y$ is said to be upper (respectively: lower) semi-continuous at the point $x_0 \in X$ if for every open set V of Y such that $F(x_0) \subset V$ $[F(x_0) \cap V \neq \emptyset]$ there exists a neighbourhood U of the point x_0 with the property $F(x) \subset V$ $[F(x) \cap V \neq \emptyset]$ for every point x of the set U.*

The symbol $C^+(F)$ (respectively: $C^-(F)$) denotes the set of all points, in which the multivalued map F is upper (lower) semi-continuous.

Theorem 3. *Let (X, τ_X) be a topological space. If (Y, τ_Y) is a regular space, a net $\{F_j : j \in J\}$ of multivalued maps is τ_Y^--pointwise and upper strongly quasi-uniformly convergent to a multivalued map F, then the inclusion*

$$\bigcap_{i \in J} \bigcup_{j \geq i} C^-(F_j) \subset C^-(F)$$

holds.

Proof. Let $x_0 \in \bigcap_{i \in J} \bigcup_{j \geq i} C^-(F_j)$. Therefore for every $i \in J$ there exists j, $j \geq i$, such that

$$x_0 \in C^-(F_j). \tag{1}$$

Let V be an open subset of Y such that $F(x_0) \cap V \neq \emptyset$. Let us assume that $y \in F(x_0) \cap V$. From the regularity of Y we can find an open subset W of Y with properties $y \in W \subset \mathrm{cl(W)} \subset V$. Moreover $F \in (\tau_Y^-) \lim F_j$, so there exists $j_0 \in J$ such that $F_j(x_0) \cap W \neq \emptyset$ for every j, $j \geq j_0$. The

family $\mathcal{A} := \{V, Y \setminus \mathrm{cl(W)}\}$ is an open cover of Y. The upper strong quasi-uniform convergence to F implies the existence $j_1 \in J$ such that for every j, $j \geq j_1$ there exists the neighbourhood U_1 of x_0 fulfilling the inclusion $F_j(x) \subset \mathrm{St}(F(x), \mathcal{A})$ if $x \in U_1$. We can choose j_2 such that $j_2 \geq j_0$ and $j_2 \geq j_1$. In view of condition (1) we will find an index j, $j \geq j_2$, such that $x_0 \in C^-(F_j)$. Then there exists a neighbourhood U_2 of x_0 such that $F_j(x) \cap W \neq \emptyset$ for every $x \in U_2$. Let us consider $U = U_1 \cap U_2$. The set U is a neighbourhood of x_0 satisfying the conditions

$$F_j(x) \subset \mathrm{St}(F(x), \mathcal{A}) \tag{2}$$

$$F_j(x) \cap W \neq \emptyset \tag{3}$$

for every $x \in U$.

We will prove that $F(x) \cap W \neq \emptyset$ for each point x of the set U. On the contrary, suppose that $F(x_1) \cap V = \emptyset$ for some $x_1 \in U$. It means that

$$F(x_1) \subset Y \setminus V \subset Y \setminus \mathrm{cl}(W)$$

$$\mathrm{St}(F(x_1), \mathcal{A}) = Y \setminus \mathrm{cl}(W).$$

From (2) we obtain

$$F_j(x_1) \subset \mathrm{St}(F(x_1), \mathcal{A}) = Y \setminus \mathrm{cl}(W) \subset Y \setminus W.$$

Therefore $F_j(x_1) \cap W = \emptyset$, which is contrary to (3). In this way, we have proved that for every open set V such that $F(x_0) \cap V \neq \emptyset$ there is a neighbourhood U of x_0 with condition $F(x) \cap V \neq \emptyset$ for every $x \in U$. It means that the map F is lower semi-continuous at the point x_0 and $x_0 \in C^-(F)$. In consequence

$$\bigcap_{i \in J} \bigcup_{j \geq i} C^-(F_j) \subset C^-(F),$$

which finishes the proof. □

Corollary 1. *Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces, and Y be a regular space, a net $\{F_j : j \in J\}$ of multivalued maps $F_j : X \to Y$ be τ_Y^--pointwise and upper strongly quasi-uniformly convergent to a map F. If F_j are lower semi-continuous for every $j \in J$, then F is lower semi-continuous.*

Proof. The assumptions of Theorem 3 are satisfied, so we have the inclusion

$$\bigcap_{i\in J}\bigcup_{j\geq i} C^-(F_j) \subset C^-(F).$$

Since F_j are lower semi-continuous, so $C^-(F_j) = X$ for every $j \in J$. Therefore $X \subset C^-(F)$. In result $C^-(F) = X$ and F is the lower semi-continuous multivalued map. □

Theorem 4. *Let (X, τ_X) be a topological space, and , (Y, τ_Y) be a regular space and $F : X \to Y$ be a multivalued map with α-paracompact values. If a net $\{F_j : j \in J\}$ of multivalued maps is τ_Y^+-pointwise and lower strongly quasi-uniformly convergent to F, then*

$$\bigcap_{i\in J}\bigcup_{j\geq i} C^+(F_j) \subset C^+(F).$$

Proof. Assume that

$$x_0 \in \bigcap_{i\in J}\bigcup_{j\geq i} C^+(F_j). \tag{4}$$

Let V be an open set with the property $F(x_0) \subset V$.

There exists an open set W such that

$$F(x_0) \subset W \subset \mathrm{cl(W)} \subset V.$$

because of $F(x_0)$ is α-paracompact set in the regular space Y and Lemma 2.

Then the family $\mathcal{A} := \{V, Y \setminus \mathrm{cl(W)}\}$ is an open cover of the space Y. The net $\{F_j : j \in J\}$ is τ_Y^+-pointwise convergent to the map F. Therefore there exists an index $j_1 \in J$ such that $F_j(x_0) \subset W$ for every j, $j \geq j_1$. From lower strong quasi-uniform convergence of the net $\{F_j : j \in J\}$ to F follows that there is $j_2 \in J$ such that for every j, $j \geq j_2$ there exists a neighbourhood U_1 of the point x_0 such that $F(x) \subset \mathrm{St}(F_j(x), \mathcal{A})$ for every $x \in U_1$. Let j_0 be such that $j_0 \geq j_1$ and $j_0 \geq j_2$. By (4) there exists j, $j \geq j_0$, such that $x_0 \in C^+(F_j)$. Since the map F is upper semi-continuous at the point x_0 and $F_j(x_0) \subset W$ we can choose a neighbourhood U_2 of the point x_0 such that $F_j(x) \subset W$ for every $x \in U_2$. Then the set $U := U_1 \cap U_2$ is open, $x_0 \in U$ and

$$F(x) \subset \mathrm{St}(F_j(x), \mathcal{A}) \text{ and } F_j(x) \subset W$$

Therefore

$$\mathrm{St}(F_j(x), \mathcal{A}) = V.$$

and in consequence $F(x) \subset V$ for every $x \in U$. We obtained that for every open set V such that $F(x_0) \subset V$ there exists a neighbourhood U of x_0 such that $F(x) \subset V$ for every $x \in U$. It means that $x_0 \in C^+(F)$. Then

$$\bigcap_{i \in J} \bigcup_{j \geq i} C^+(F_j) \subset C^+(F).$$

□

Corollary 2. *Let (X, τ_X) be a topological space, (Y, τ_Y) be a regular space and a map $F : X \to Y$ has α-paracompact values. If multifuctions $F_j : X \to Y$ are upper semi-continuous for $j \in J$ and the net $\{F_j : j \in J\}$ is τ_Y^+-pointwise and lower strongly quasi-uniformly convergent to the map F, then F is upper semi-continuous.*

Proof. All assumptions of Theorem 4 are satisfied. Hence we have

$$X = \bigcap_{i \in J} \bigcup_{j \geq i} C^+(F_j) \subset C^+(F) \subset X.$$

Thus $C^+(F) = X$ and it is equivalent to upper semi-continuity of the map F on the space X. □

Definition 9. ([3]) *Let X and Y be topological spaces. A net $\{F_j : j \in J\}$ of multivalued maps from X to Y is called frequently upper (lower) semi-continuous at a point $x_0 \in X$ if for every $j_0 \in J$ there exists $j \in J$, $j \geq j_0$, such that the multivalued map F_j is upper (lower) semi-continuous at the point x_0.*

Theorem 5. *Let (X, τ_X) be a topological space, (Y, τ_Y) be a regular space. Let F_j, F be multivalued maps from X to Y for every $j \in J$. Assume that there exists $x_0 \in X$ such that $F(x_0)$ is an $\alpha-$paracompact set. If the net $\{F_j : j \in J\}$ is upper and lower strongly quasi-uniformly convergent to F and it is frequently upper semi-continuous at the point x_0, then the map F is upper semi-continuous at x_0.*

Proof. Let V be an open subset of the regular space Y such that $F(x_0) \subset V$. The set $F(x_0)$ is $\alpha-$paracompact, so by virtue of Lemma 2 there exist open sets G, H satisfying inclusions

$$F(x_0) \subset G \subset \mathrm{cl}(G) \subset H \subset \mathrm{cl}(H) \subset V.$$

Let us consider open covers $\mathcal{A} := \{V, Y \setminus \mathrm{cl}(H)\}$ and $\mathcal{B} := \{H, Y \setminus \mathrm{cl}(G)\}$ of the space Y. Assumptions about upper and lower strong quasi-uniform convergence of the net $\{F_j : j \in J\}$ to F imply conditions:

$$\exists_{j_0 \in J} \forall_{j \geq j_0} \exists_{U_1 - \text{neighbourhood of } x_0} \forall_{x \in U_1} F_j(x) \subset \mathrm{St}(F(x), \mathcal{B}) \tag{5}$$

$$\exists_{j_1 \in J} \forall_{j \geq j_1} \exists_{U_2 - \text{neighbourhood of } x_0} \forall_{x \in U_2} F(x) \subset \mathrm{St}(F_j(x), \mathcal{A}). \tag{6}$$

The set J is directed, so there exists j_2 such that $j_2 \geq j_0$ and $j_2 \geq j_1$. Moreover, the net $\{F_j : j \in J\}$ is frequently upper semi-continuous at x_0. Therefore, there exists an index j, $j \geq j_2$ such that a map F_j is upper semi-continuous at x_0. Then conditions (5) and (6) are satisfying for j. It means that there exists a neighbourhood G_1 of x_0 such that $F_j(x) \subset \mathrm{St}(F(x), \mathcal{B})$ for each $x \in G_1$ and there exists a neighbourhood G_2 of the point x_0 such that $F(x) \subset \mathrm{St}(F_j(x), \mathcal{A})$ for each $x \in G_2$. If $G = G_1 \cap G_2$, then G is a neighbourhood of x_0. In particular $F_j(x_0) \subset \mathrm{St}(F(x_0), \mathcal{B}) = H$. The map F_j is upper semi-continuous at x_0. Therefore there exists a neighbourhood $U_3, U_3 \subset U$, of the point x_0, such that $F_j(x) \subset H$ for every $x \in U_3$. For $x \in U_3$ we have $F(x) \subset \mathrm{St}(F_j(x), \mathcal{A}) = V$.

Thus we proved that for every open set V such that $F(x_0) \subset V$ there is a neighbourhood U_3 of the point x_0 with the property $F(x) \subset V$ for every $x \in U_3$. It means that F is upper semi-continuous at the point x_0. □

Theorem 6. *Let (X, τ_X) be a topological space, (Y, τ_Y) be a regular space. Assume that F_j, F are multivalued maps from X to Y for every $j \in J$. Let $x_0 \in X$. If the net $\{F_j : j \in J\}$ is frequently lower semi-continuous at the point x_0 and it converges upper and lower strongly quasi-uniformly to F, then the map F is lower semi-continuous at x_0.*

Proof. Let V be an open subset of the space Y such that $F(x_0) \cap V \neq \emptyset$. Let $z \in F(x_0) \cap V$. Since the space Y is regular, $z \in V$ and V is the open set, so there exist open sets G, W such that

$$z \in G \subset \mathrm{cl(G)} \subset W \subset \mathrm{cl}(W) \subset V.$$

We will consider the following open covers of the space Y:

$$\mathcal{A} := \{W, Y \setminus \mathrm{cl}(G)\}, \qquad \mathcal{B} := \{V, Y \setminus \mathrm{cl}(W)\}.$$

Conditions of upper and lower strong quasi-uniform convergence imply the facts:

$$\exists_{j_1\in J}\forall_{j\geq j_1}\exists_{U_1-\text{neighbourhood of } x_0}\forall_{x\in U_1}F_j(x)\subset \mathrm{St}(F(x),\mathcal{B}) \tag{7}$$

$$\exists_{j_2\in J}\forall_{j\geq j_2}\exists_{U_2-\text{neighbourhood of } x_0}\forall_{x\in U_2}F(x)\subset \mathrm{St}(F_j(x),\mathcal{A}). \tag{8}$$

Since J is a directed set, there exists $j_3\in J$ such that $j_3\geq j_1$ and $j_3\geq j_2$. The net $\{F_j : j\in J\}$ is frequently lower semi-continuous at the point x_0. Therefore, there is an index j, $j\geq j_3$ such that a map F_j is lower semi-continuous at the point x_0. From conditions (7) and (8) we obtain

$$\exists_{G_1-\text{neighbourhood of } x_0}\forall_{x\in G_1}F_j(x)\subset \mathrm{St}(F(x),\mathcal{B}) \tag{9}$$

$$\exists_{G_2-\text{neighbourhood of } x_0}\forall_{x\in G_2}F(x)\subset \mathrm{St}(F_j(x),\mathcal{A}). \tag{10}$$

Let us consider the set $U_3 := U_1\cap U_2$, which is a neighbourhood of x_0. Now, we will prove that $F_j(x_0)\cap W\neq\emptyset$. Suppose that $F_j(x_0)\cap W=\emptyset$. It implies inclusions $F_j(x_0)\subset Y\setminus W\subset Y\setminus \mathrm{cl}(G)$. From (10) we infer that

$$F(x_0)\subset \mathrm{St}(F_j(x_0),\mathcal{A}) = Y\setminus \mathrm{cl}(V)\subset Y\setminus V,$$

which contradicts to $z\in F(x_0)\cap V$. Thus $F(x_0)\cap V\neq\emptyset$.

In consequence we showed that $F_j(x_0)\cap W\neq\emptyset$. The map F_j is lower semi-continuous at x_0, so there exists a neighbourhood $U\subset U_3$ such that

$$F_j(x)\cap W\neq\emptyset \tag{11}$$

for every $x\in U$.

By (9) we obtain $F_j(x)\subset \mathrm{St}(F(x),\mathcal{B})$. Then $F(x)\cap V\neq\emptyset$ for each $x\in U$. Indeed, if $F(x')\cap V=\emptyset$ for any $x'\in U$, then

$$F_j(x')\subset \mathrm{St}(F(x'),\mathcal{B}) = Y\setminus \mathrm{cl}(W)\subset Y\setminus W,$$

which contradicts to (11). Thus we proved the following condition:
for every open set V such that $F(x_0)\cap V\neq\emptyset$, there exists a neighbourhood U of the point x_0 such that $F(x)\cap V\neq\emptyset$ for every $x\in U$.

It means that F is lower semi-continuous at the point x_0. □

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Irena Domnik
Pomeranian University in Słupsk,
Institute of Mathematics,
76-200 Słupsk, ul. Arciszewskiego 22d, Poland
E-mail address: irkadomnik@wp.pl

Agata Sochaczewska
Pomeranian University in Słupsk,
Institute of Mathematics,
76-200 Słupsk, ul. Arciszewskiego 22d, Poland
E-mail address: agata_sochaczewska@wp.pl