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UNIFORMLY BOUNDED SET-VALUED COMPOSITION OPERATORS IN THE SPACES OF FUNCTIONS OF BOUNDED VARIATION IN THE SENSE OF SCHRAMM

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Abstract

We prove that, under some general assumptions, the one-sided regularizations of the generator of any uniformly bounded set-valued composition operator, acting in the spaces of functions of bounded variation in the sense of Schramm with nonempty bounded closed and convex values is an affine function. As a special case, we obtain an earlier result ([15]).

1. Introduction

Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be real normed spaces, C be a convex cone in X and $I = [a, b] \subset \mathbb{R}$ be an interval. Let $\mathrm{clb}(Y)$ be the family of all nonempty bounded, closed and convex subsets of Y. We consider the Nemytskij operator, i.e, for a function $F : I \longrightarrow C$ the composition operator is defined by

$$(HF)(t) = h(t, F(t)),$$

where $h : I \times C \longrightarrow \mathrm{clb}(Y)$ is a given set-valued function. It is shown that if operator H maps the space $\Phi BV(I; C)$ of functions of bounded Φ-variation in the sense of Schramm into the space $BS_\Psi(I; \mathrm{clb}(Y))$ of closed bounded convex valued functions of bounded Ψ-variation in the sense of Schramm and

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is uniformly bounded, then the one-sided regularizations h^- and h^+ of h, with respect to the first variable, are affine with respect to the second variable, i.e., Matkowski's representation holds. In particular,

$$h^-(t,x) = A(t)x \overset{*}{+} B(t), \qquad \text{for} \quad t \in I,\ x \in C,$$

for some function $A : I \longrightarrow \mathcal{L}(C, \mathrm{clb}(Y))$ and $B \in BS_\Psi(I; \mathrm{clb}(Y))$, where $\mathcal{L}(C, \mathrm{clb}(Y))$ stands for the space of all linear mappings acting from$C$ into $\mathrm{clb}(Y)$. This extends the main result of the paper [15].

2. Preliminaries

Let $\mathcal{F}$ be the set of all convex functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ vanishing at zero only (and, hence, continuous on $[0,\infty)$ and strictly increasing).

A sequence $\Phi = (\phi_i)_{i=1}^{\infty}$ of functions from $\mathcal{F}$ satisfying the following two conditions:

(i) $\phi_{n+1}(t) \le \phi_n(t)$ for all $t > 0$ and $n \in \mathbb{N}$,

(ii) $\sum\limits_{n=1}^{\infty} \phi_n(t)$ diverges for all $t > 0$,

is said to be a Φ-sequence.

Let $I \subset \mathbb{R}$ be an interval. For a set X we denote by X^I the set of all functions which map I into X.

If $I_n = [a_n, b_n]$ is a subinterval of the interval I $(n = 1, 2, ...)$, then we write $f(I_n) = f(b_n) - f(a_n)$.

Definition 1. [14] *Let $\Phi = (\phi_i)_{i=1}^{\infty}$ be a Φ-sequence and $(X, |\cdot|_X)$ be a real normed space. A function $f \in X^I$ is said to have bounded Φ-variation in the sense of Schramm in I, if*

$$v_\Phi(f) = v_\Phi(f, I) := \sup \sum_{n=1}^{m} \phi_n\left(|f(I_n)|\right) < \infty, \tag{1}$$

where the supremum is taken over all $m \in \mathbb{N}$ and all non-ordered collections of non-overlapping intervals $I_n = [a_n, b_n]$, $n = 1, ..., m$.

It is known that for all $a, b, c \in I$, such that $a \le c \le b$ we have

$$v_\Phi(f, [a,c]) \le v_\Phi(f, [a,b])$$

(that is, v_Φ is increasing with respect to the interval) and

$$v_\Phi(f, [a,c]) + v_\Phi(f, [c,b]) \le v_\Phi(f, [a,b]).$$

We will denote by $V_\Phi(I,X)$ the set of all bounded Φ-variation functions $f \in X^I$ in the Schramm sense. This is a symmetric and convex set; but it is not necessarily a linear space. In fact, Musielak-Orlicz proved the following statement: this class of functions is a vector space if and only if φ satisfies the δ_2 condition [11].

By $\Phi BV(I,X)$ we will denote the linear space of all functions $f \in X^I$ such that $v_\Phi(\lambda f) < \infty$ for some positive constant λ.

In the linear space $\Phi BV(I,X)$, the function $\|\cdot\|_\Phi$ defined by

$$\|f\|_\Phi := |f(a)| + p_\Phi(f), \quad f \in \Phi BV(I,X),$$

where

$$p_\Phi(f) := p_\Phi(f,I) = \inf\left\{\epsilon > 0 : v_\Phi\big(f/\epsilon\big) \le 1\right\}, \tag{2}$$

is a norm (see for instance [11]).

The linear normed space $(BV_\Phi(I,\mathbb{R}), \|\cdot\|_\Phi)$ was studied by Schramm (Theorem 2.3 [14]).

The functional $p_\varphi(\cdot)$ defined by (2) is called *the Luxemburg-Nakano-Orlicz seminorm* [5, 19, 20].

It is worth mentioning that the symbol $\Phi BV(I,C)$ stands for the set of all functions $f \in \Phi BV(I,X)$ such that $f : I \longrightarrow C$ and C is a subset of X.

Let $(Y, |\cdot|_Y)$ be a normed real vector space.

The family of all nonempty bounded closed convex subsets of Y equipped with the Hausdorff metric D generated by the norm in Y:

$$D(A,B) := \max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|_Y,\ \sup_{b\in B}\inf_{a\in A}|a-b|_Y\right\}, \quad A,B \in \mathrm{clb}(Y).$$

is denoted by $\mathrm{clb}(Y)$.

Given $A,B \in \mathrm{clb}(Y)$, we put $A+B := \{a+b : a \in A,\ b \in B\}$ and we introduce the operation $\stackrel{*}{+}$ in $\mathrm{clb}(Y)$ defined as follows:

$$A \stackrel{*}{+} B = \mathrm{cl}(A+B),$$

where cl stands for the closure in Y. The class $\mathrm{clb}(Y)$ with the operation $\stackrel{*}{+}$ is an Abelian semigroup, with $\{0\}$ as the zero element, which satisfies the cancelation law. Moreover, we can multiply elements of $\mathrm{clb}(Y)$ by nonnegative numbers and, for all $A,B \in \mathrm{clb}(Y)$ and $\lambda,\mu \ge 0$, the following conditions hold:

$$1\cdot A = A, \lambda(\mu A) = (\lambda\mu)A,\ \lambda(A \stackrel{*}{+} B) = \lambda A \stackrel{*}{+} \lambda B,\ (\lambda+\mu)A = \lambda A \stackrel{*}{+} \mu A.$$

In view of [4, lemma 3]

$$D(A \overset{*}{+} B, A \overset{*}{+} C) = D(A+B, A+C) = D(B,C);\ \ A,B,C \in \mathrm{clb}(Y), \tag{3}$$

$(\mathrm{clb}(Y), D, \overset{*}{+}, \cdot)$ is an abstract convex cone; this cone is complete provided Y is a Banach space.

Definition 2. *Let $\Phi = (\phi_n)_{n=1}^{\infty}$ be a Φ-sequence and $F : I \longrightarrow \mathrm{clb}(X)$. We say that F has bounded Φ-variation in the Schramm sense if*

$$w_\Phi(F) := \sup \sum_{n=1}^{m} \Phi_n \left(D(F(t_n), F(t_{n-1})) \right) < \infty, \tag{4}$$

where the least upper bound is taken over all $m \in \mathbb{N}$ and all collections of non-overlapping intervals $I_n = [a_n, b_n] \subset I, i = 1, \dots, m$.

From now on, let

$$BS_\Phi(I, \mathrm{clb}(X)) := \Big\{ F \in \mathrm{clb}(X)^I :\ w_\Phi(\lambda F) < \infty \text{ for some } \lambda > 0 \Big\}. \tag{5}$$

For $F_1, F_2 \in BS_\Phi(I, \mathrm{clb}(X))$ put

$$D_\Phi(F_1, F_2) := D\left(F_1(a), F_2(a)\right) + p_\Phi(F_1, F_2) \tag{6}$$

where

$$p_\Phi(F_1, F_2) := \inf \Big\{ \epsilon > 0 : S_\epsilon(F_1, F_2) \le 1 \Big\} \tag{7}$$

and

$$S_\epsilon(F_1, F_2) := \sup \sum_{n=1}^{m} \phi_n \left(\frac{1}{\epsilon} D\big(F_1(t_n) \overset{*}{+} F_2(t_{n-1}); F_2(t_n) \overset{*}{+} F_1(t_{n-1})\big) \right), \tag{8}$$

where the least upper bound is taken over the same collection $([a_n, b_n])_{n=1}^{m}$ as in Definition 2. Then $(BS_\Phi(I, \mathrm{clb}(X)), D_\Phi)$ is a metric space, and it is complete if X is a Banach space [18, Lemma 5.4].

Taking into account [17], Theorem 3.8 (d)] and [18, condition 5.6] we get the following:

Lemma 3. *Let $\Phi = (\phi_n)_{n=1}^{\infty}$ be a Φ-sequence and $F_1, F_2 \in BS_\Phi(I, \mathrm{clb}(X))$. Then, for $\lambda > 0$, $S_\lambda(F_1, F_2) \le 1$ if and only if $p_\Phi\big(F_1, F_2\big) \le \lambda$.*

Let $(X, |\cdot|_X)$, $(Y, |\cdot|_Y)$ be two real normed spaces. A subset $C \subset Y$ is said to be a convex cone if $\lambda C \subset C$ for all $\lambda \ge 0$ and $C + C \subset C$. It is obvious that $0 \in C$. Given a set-valued function $h : I \times C \longrightarrow \mathrm{clb}(Y)$ we consider the composition operator $H : C^I \longrightarrow \mathrm{clb}(Y)^I$ generated by h, i.e.,

$$(Hf)(t) := h(t, f(t)), \quad f \in C^I, \quad t \in I.$$

A set-valued function $F : C \to \mathrm{clb}(Y)$ is said to be *additive, if

$$F(x+y) = F(x) \overset{*}{+} F(y),$$

and *Jensen if

$$2F\left(\frac{x+y}{2}\right) = F(x) \overset{*}{+} F(y),$$

for all $x, y \in C$.

We will need the following

Lemma 4 (21, Cor. 4)**.** *Let C be a convex cone in a real linear space and let $(Y, |\cdot|_Y)$ be a Banach space. A set-valued function $F : C \to \mathrm{clb}(Y)$ is *Jensen if and only if there exists an *additive set-valued function $A : C \to \mathrm{clb}(Y)$ and a set $B \in \mathrm{clb}(Y)$ such that*

$$F(x) = A(x) \overset{*}{+} B,$$

for all $x \in C$.

For the normed spaces $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ by $\left(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}(X,Y)}\right)$, briefly $\mathcal{L}(X,Y)$, we denote the normed space of all additive and continuous mappings $A \in Y^X$.

Let C be a convex cone in a real normed space $(X, |\cdot|_X)$. From now on, let the set $\mathcal{L}(C, \mathrm{clb}(Y))$ consists of all set-valued functions $A : C \to \mathrm{clb}(Y)$ which are *additive and continuous (so positively homogeneous), i.e.,

$$\mathcal{L}(C, \mathrm{clb}(Y)) = \left\{A \in \mathrm{clb}(Y)^C : A \text{ is } {}^*\text{additive and continuous}\right\}.$$

The set $\mathcal{L}(C, \mathrm{clb}(Y))$ can be equipped with the metric defined by

$$d_{\mathcal{L}(C,\mathrm{clb}(Y))}(A, B) := \sup_{y \in C\setminus\{0\}} \frac{d(A(y), B(y))}{\|y\|_Y}.$$

Theorem 5. *Let $(X, |\cdot|_X)$ be a real normed space, $(Y, |\cdot|_Y)$ a real Banach space, C a convex cone in X and let Φ, $\Psi \in \mathcal{F}$. Suppose that the set-valued function is continuous with respect to the second variable. If the composition operator H generated by a set-valued function $h : I \times C \longrightarrow \mathrm{clb}(Y)$ maps $\Phi BV(I, C)$ into $BS_\Psi(I, \mathrm{clb}(Y))$ and satisfies the inequality,*

$$D_\Phi\left(H(f_1), H(f_2)\right) \le \omega\left(||f_1 - f_2||\right), \quad f_1, f_2 \in \Phi BV(I, C),$$

for some function $\omega : [0,\infty) \to [0,\infty)$*, then the left regularization of* h*, i.e. the function* $h^- : I \times X \longrightarrow Y$ *defined by*

$$h^-(t,x) := \lim_{s \uparrow t} h(s,x), \quad t \in I,\ x \in C,$$

exists and

$$h^-(t,x) = A(t)x \overset{*}{+} B(t), \quad t \in I,\ x \in C,$$

for some $A : I \longrightarrow \mathcal{A}(X, \mathrm{clb}(Y))$ *and* $B : I \longrightarrow \mathrm{clb}(Y)$*. Moreover, if* $0 \in C$*, then* $B \in BS_\Psi(I, \mathrm{clb}(Y))$ *and the linear set-valued function* $A(t)$ *is continuous.*

Proof. For every $x \in C$ the constant function $I \ni t \longrightarrow x$ belongs to $\Phi BV(I,C)$. Since H maps $\Phi BV(I,C)$ into $BS_\Psi(I, \mathrm{clb}(Y))$ for every $x \in C$ the function $I \ni t \longrightarrow h(t,x)$ belongs to $BS_\Psi(I, \mathrm{clb}(Y))$. The completeness of $\mathrm{clb}(Y)$ with respect to the Hausdorff metric [18, Lemma 6.12] implies the existence of the left regularization h^- of h.

Function H is uniformly continuous on $\Phi BV(I,C)$. Let $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the modulus of continuity of H, that is

$$\omega(\rho) := \sup\Big\{ D_\Psi\big(H(f_1), H(f_2)\big) : \|f_1 - f_2\|_\Phi \le \rho;\ f_1, f_2 \in \Phi BV(I,C)\Big\}$$

if $\rho > 0$. Hence we get

$$D_\Psi\big(H(f_1), H(f_2)\big) \le \omega\big(\|f_1 - f_2\|_\Phi\big) \quad \text{for} \quad f_1, f_2 \in \Phi BV(I,C). \tag{9}$$

From the definition of the metric D_Ψ and (9) we obtain

$$p_\Psi\big(H(f_1); H(f_2)\big) \le \omega\big(\|f_1 - f_2\|_\Phi\big) \quad \text{for} \quad f_1, f_2 \in \Phi BV(I,C). \tag{10}$$

From Lemma 3 if $\omega\big(\|f_1 - f_2\|_\Phi\big) > 0$ the inequality (10) is equivalent to

$$S_{\omega(\|f_1 - f_2\|_\Phi)}\Big(H(f_1), H(f_2)\Big) \le 1, \quad f_1, f_2 \in \Phi BV(I,C). \tag{11}$$

Therefore, for any

$$a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_m < \beta_m = b,$$

where α_i, $\beta_i \in I$, $i \in \{1, 2, \ldots, m\}$, $m \in \mathbb{N}$, the definitions of the operator H and the functional S_ϵ imply

$$\sum_{i=1}^{\infty} \psi_i \left(\frac{D\big(h(\beta_i, f_1(\beta_i)) \overset{*}{+} h(\alpha_i, f_2(\alpha_i)); h(\beta_i, f_2(\beta_i)) \overset{*}{+} h(\alpha_i, f_1(\alpha_i))\big)}{\omega\big(\|f_1 - f_2\|_\Phi\big)} \right) \le 1. \tag{12}$$

For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, we define functions $\eta_{\alpha,\beta} : \mathbb{R} \longrightarrow [0,1]$ by

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0 & \text{if} \quad t \le \alpha \\ \dfrac{t-\alpha}{\beta-\alpha} & \text{if} \quad \alpha \le t \le \beta \\ 1 & \text{if} \quad \beta \le t. \end{cases} \tag{13}$$

Let us fix $t \in I$. For arbitrary finite sequence

$$a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_m < \beta_m < t$$

and $x_1, x_2 \in C$, $x_1 \ne x_2$, the functions $f_1, f_2 : I \longrightarrow X$ defined by

$$f_\ell(\tau) := \frac{1}{2} \cdot \left[\eta_{\alpha_i,\beta_i}(\tau)(x_1 - x_2) + x_\ell + x_2\right], \quad \tau \in I,\ \ell = 1, 2 \tag{14}$$

belong to the space $\Phi BV(I, C)$. From (14) we have

$$f_1(\tau) - f_2(\tau) = \frac{x_1 - x_2}{2}, \qquad \tau \in I,$$

therefore

$$\|f_1 - f_2\|_\Phi = \left|\frac{x_1 - x_2}{2}\right|;$$

moreover

$$f_1(\beta_i) = x_1;\ f_2(\beta_i) = \frac{x_1 + x_2}{2};\ f_1(\alpha_i) = \frac{x_1 + x_2}{2};\ f_2(\alpha_i) = x_2.$$

Applying (12) we hence get

$$\sum_{i=1}^{\infty} \psi_i \left(\frac{D\left(h(\beta_i, x_1) \overset{*}{+} h(\alpha_i, x_2); h\left(\alpha_i, \frac{x_1+x_2}{2}\right) \overset{*}{+} h\left(\beta_i, \frac{x_1+x_2}{2}\right)\right)}{\omega\left(\left|\frac{x_1-x_2}{2}\right|\right)} \right) \le 1. \tag{15}$$

For a fixed positive integer m we have:

$$\sum_{i=1}^{m} \psi_i \left(\frac{D\left(h(\beta_i, x_1) + h(\alpha_i, x_2);\ h\left(\alpha_i, \frac{x_1+x_2}{2}\right) + h\left(\beta_i, \frac{x_1+x_2}{2}\right)\right)}{\omega\left(\left|\frac{x_1-x_2}{2}\right|\right)} \right) \le 1. \tag{16}$$

From the continuity of ψ_i passing to the limit in (16) when $\alpha_1 \uparrow t$ we infer that

$$\sum_{i=1}^{m} \psi_i \left(\frac{D\left(h^-(t, x_1) \overset{*}{+} h^-(t, x_2);\ 2h^-\left(t, \dfrac{x_1 + x_2}{2}\right)\right)}{\omega\left(\left|\dfrac{x_1 - x_2}{2}\right|\right)} \right) \le 1.$$

Hence,

$$\sum_{i=1}^{\infty} \psi_i \left(\frac{D\left(h^-(t,x_1) \overset{*}{+} h^-(t,x_2);\ 2h^-\left(t, \frac{x_1+x_2}{2}\right)\right)}{\omega\left(\left|\frac{x_1-x_2}{2}\right|\right)} \right) \le 1,$$

and by (ii)

$$D\Big(h^-(t,x_1) \overset{*}{+} h^-(t,x_2);\ 2h^-\Big(t, \frac{x_1+x_2}{2}\Big)\Big) = 0.$$

Therefore

$$h^-\Big(t, \frac{x_1+x_2}{2}\Big) = \frac{h^-(t,x_1) \overset{*}{+} h^-(t,x_2)}{2}$$

for all $t \in I$ and all $x_1,\ x_2 \in C$.

Thus for each $r \in I^-$ the function $h^-(r,\cdot)$ satisfies the *Jensen functional equation in C and by Lemma 4. for every $t \in I$ there exist an *additive set-valued function $A(t) : C \longrightarrow \mathrm{clb}(Y)$ and a set $B(t) \in \mathrm{clb}(Y)$ such that

$$h^-(t,x) = A(t)x \overset{*}{+} B(t) \quad \text{for} \quad x \in C,\ t \in I. \tag{17}$$

To show that $A(t)$ is continuous for any $t \in I$, let us fix $x, \overline{x} \in C$. By (3) and (17) we have

$$D(A(t)x, A(t)\overline{x}) = D(A(t)x \overset{*}{+} B(t), A(t)\overline{x} \overset{*}{+} B(t)) = D(h(t,x), h(t,\overline{x})).$$

Hence, the continuity of h with respect to the second variable implies the continuity of $A(t)$ and, consequently, being *additive, $A(t) \in \mathcal{L}(C, \mathrm{clb}(Y))$ for every $t \in I$. To prove that $B \in BW_\psi(I, \mathrm{clb}(Y))$ let us note that the *additivity of $A(t)$ implies $A(t)0 = \{0\}$. Therefore, putting $x = 0$ in (17), we get

$$h^-(t,0) = B(t), \quad t \in I,$$

which gives the required claim.

Remark 6. The counterpart of Theorem 5. for the right regularization h^+ of h defined by

$$h^+(t,x) := \lim_{s \downarrow t} h(s,x); \quad t \in I,$$

is also true.

Remark 7. Taking $\psi_n(t) = \psi(t)$, $(t \ge 0)$, we obtain the main result of [1].

3. Uniformly bounded set-valued composition operator

In this section we present the definition given recently by Matkowski [6] for uniformly bounded composition operators.

Definition 8. *Let X and Y be two metric (or normed) spaces. We say that a mapping $H : X \to Y$ is uniformly bounded if for any $t > 0$ there is a real number $\gamma(t)$ such that for any nonempty set $B \subset X$ we have*

$$\operatorname{diam} B \leq t \Longrightarrow \operatorname{diam} H(B) \leq \gamma(t).$$

Remark 9. Obviously, every uniformly continuous operator or Lipschitzian operator is uniformly bounded. Note that, under the assumptions of this definition, every bounded operator is uniformly bounded.

The following theorem represents our main result.

Theorem 10. *Let $(X, |\cdot|_X)$ be a real normed space, $(Y, |\cdot|_Y)$ a real Banach space, C a convex cone in X and let $\Phi, \Psi \in \mathcal{F}$. If the composition operator H generated by a set-valued function $h : I \times C \longrightarrow \operatorname{clb}(Y)$ maps $\Phi BV(I, C)$ into $BS_\Psi(I, \operatorname{clb}(Y))$ and is uniformly bounded, then the left regularization of h, i.e., the function $h^- : I \times X \longrightarrow Y$ defined by*

$$h^-(t, x) := \lim_{s \uparrow t} h(s, x), \quad t \in I,\ x \in C,$$

exists and

$$h^-(t, x) = A(t)x \overset{*}{+} B(t), \quad t \in I,\ x \in C,$$

for some $A : I \longrightarrow \mathcal{A}(X, \operatorname{clb}(Y))$ and $B : I \longrightarrow \operatorname{clb}(Y)$.

Proof. Let $t \geq 0$ and $f_1, f_2 \in \Phi BV(I; C)$ fulfill the condition

$$||f_1 - f_2||_\Phi \leq t.$$

Since $\operatorname{diam}\{f_1, f_2\} \leq t$, by the uniform boundedness of H, we have

$$\operatorname{diam} H\big(\{f_1, f_2\}\big) \leq \gamma(t),$$

that is

$$||H(f_1) - H(f_2)||_\Psi = \operatorname{diam} H\left(\{f_1, f_2\}\right) \leq \gamma(\|f_1 - f_2\|_\Phi)$$

and the result follows from Theorem 5. □

Remark 11. If the function $\gamma : [0, \infty) \to [0, \infty)$ in the Definition 8. is right continuous at 0 and $\gamma(0) = 0$ (or if only $\gamma(0+) = 0$), then, clearly the uniform boundedness of the involved operator reduces to its uniform continuity. It follows that Theorem 10. improves the result of [15, Th. 2.1] where H is assumed to be uniformly continuous.

Let us remark that the uniform boundedness of an operator (weaker than the usual boundedness) introduced and applied in [6] for the Nemytskij composition operators acting between spaces of Hölder functions in the single-valued case and then extended to the set-valued cases in [10] for the operators with convex and compact values.

Recall that the representation of Lipschitz continuous Nemytskij operators acting in the spaces of functions of bounded variation was first considered in [9] and then in [8] (in the single-valued case), and in [3] in the set-valued case. Let us add that A. Smajdor and W. Smajdor [2], extending the single-valued result of [7], initiated interesting and important study of the set-valued case.

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