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TWISTED GROUP ALGEBRAS OF SUR-TYPE OF FINITE GROUPS OVER AN INTEGRAL DOMAIN OF CHARACTERISTIC p

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Abstract

Let S be an integral domain of positive characteristic p, which is not a field, S^* the unit group of S, G a finite group, and $S^\lambda G$ the twisted group algebra of the group G over S with a 2-cocycle $\lambda \in Z^2(G, S^*)$. Denote by $\mathrm{Ind}_m(S^\lambda G)$ the set of isomorphism classes of indecomposable $S^\lambda G$-modules of S-rank m. We exhibit algebras $S^\lambda G$ of SUR-type, in the sense that there exists a function $f_\lambda\colon \mathbb{N} \to \mathbb{N}$ such that $f_\lambda(n) \geq n$ and $\mathrm{Ind}_{f_\lambda(n)}(S^\lambda G)$ is an infinite set for every integer $n > 1$.

1. Introduction

Let $p \geq 2$ be a prime. Gudyvok [4] and Janusz [8], [9] showed that if K is an infinite field of characteristic p and G is a non-cyclic p-group for which $|G/G'| \neq 4$, then $\mathrm{Ind}_n(KG)$ is an infinite set for every integer $n > 1$. Let G be a finite p-group of order $|G| > 2$, K a commutative local ring of characteristic p^n, and $\mathrm{rad}\, K \neq 0$. Gudyvok and Chukhray [5], [6] proved that if $\overline{K} := K/\mathrm{rad}\, K$ is an infinite field or K is an integral domain, then $\mathrm{Ind}_n(KG)$ is infinite for every integer $n > 1$. In paper [7], jointly with Sygetij, they obtained a similar result in the case where G is a non-cyclic p-group, $p \neq 2$ and K is an infinite ring of characteristic p or $\overline{K}$ is an infinite field. The similar problem was studied in [2], [3] for twisted group algebras $K^\lambda G$, where K is a field of characteristic p or a commutative local ring of characteristic p.

In this paper we exhibit twisted group algebras $S^\lambda G$ of SUR-type, where S is an integral domain of characteristic p and G is a finite group.

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2. Preliminaries

Let K be a commutative ring of characteristic p, K^* the unit group of K, G a finite group, e the identity element of G, G_p a Sylow p-subgroup of G and G'_p the commutator subgroup of G_p. We suppose that p divides $|G|$ and G_p is a normal subgroup of G. *The twisted group algebra* of G over K with a 2-cocycle $\lambda \in H^2(G, K^*)$ is the free K-algebra $K^\lambda G$ with a K-basis $\{u_g \colon g \in G\}$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in G$. The K-basis $\{u_g \colon g \in G\}$ is called canonical (corresponding to λ). By a $K^\lambda G$-module we mean a finitely generated left $K^\lambda G$-module that is K-free. Denote by $\mathrm{Ind}_m(K^\lambda G)$ the set of isomorphism classes of indecomposable $K^\lambda G$-modules of K-rank m. An algebra $K^\lambda G$ is defined to be of SUR-type (Strongly Unbounded Representation type) if there is a function $f_\lambda \colon \mathbb{N} \to \mathbb{N}$ such that $f_\lambda(n) \geq n$ and $\mathrm{Ind}_{f_\lambda(n)}(K^\lambda G)$ is an infinite set for every $n > 1$. A function f_λ is called an SUR-dimension-valued function. Given a $K^\lambda H$-module V, we write $\mathrm{End}_{K^\lambda H}(V)$ for the ring of all $K^\lambda H$-endomorphisms of V, $\mathrm{rad}_{K^\lambda H}(V)$ for the Jacobson radical of $\mathrm{End}_{K^\lambda H}(V)$ and $\overline{\mathrm{End}_{K^\lambda H}(V)}$ for the quotient ring

$$\mathrm{End}_{K^\lambda H}(V)/\operatorname{rad}\mathrm{End}_{K^\lambda H}(V).$$

Given a subgroup Ω of K^*, we denote by $Z^2(H, \Omega)$ the group of all Ω-valued normalized 2-cocycles of the group H, where we assume that H acts trivially on Ω. If D is a subgroup of a group H, the restriction of $\lambda \in Z^2(H, K^*)$ to $D \times D$ is also denoted by λ. In this case, $K^\lambda D$ is the K-subalgebra of $K^\lambda H$ consisting of all K-linear combinations of the elements $\{u_d \colon d \in D\}$, where $\{u_h \colon h \in H\}$ is a canonical K-basis of $K^\lambda H$ corresponding to λ.

Throughout the paper, S denotes an arbitrary integral domain of characteristic p, which is not a field, $\mathfrak{m}$ is a maximal ideal of S and $R = S_\mathfrak{m}$ is the localization of S at $\mathfrak{m}$. The ring R is a local ring and $\mathfrak{m}R$ is a unique maximal ideal of R. Moreover, $S/\mathfrak{m} \cong R/\mathfrak{m}R$ as fields, and as R-modules. Given $\mu \in Z^2(G_p, S^*)$, the kernel $\mathrm{Ker}(\mu)$ of μ is the union of all cyclic subgroups $\langle g \rangle$ of G_p such that the restriction of μ to $\langle g \rangle \times \langle g \rangle$ is a coboundary. We recall from [[3], p. 196] that $G'_p \subset \mathrm{Ker}(\mu)$, $\mathrm{Ker}(\mu)$ is a normal subgroup of G_p and the restriction of μ to $\mathrm{Ker}(\mu) \times \mathrm{Ker}(\mu)$ is a coboundary.

Let $H = \langle a \rangle$ be a cyclic p-group of order $|H| > 2$, and K a commutative local ring of characteristic p. Suppose that there exists a non-zero element

$t \in \operatorname{rad} K$ which is not a zero-divisor. Let E_m be the identity matrix of order m, $J_m(0)$ the upper Jordan block of order m with zeros on the main diagonal, and $\langle 1 \rangle$ the $m \times 1$-matrix of the form

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Denote by Γ_i the matrix K-representation of degree n of the group H defined in the following way:

1) if $n = 2$ then

$$\Gamma_i(a) = \begin{pmatrix} 1 & t^i \\ 0 & 1 \end{pmatrix} \quad (i \in \mathbb{N});$$

2) if $n = 3m$ $(m \geq 1)$ then

$$\Gamma_i(a) = \begin{pmatrix} E_m & t^i E_m & J_m(0) \\ 0 & E_m & t^i E_m \\ 0 & 0 & E_m \end{pmatrix} \quad (i \in \mathbb{N});$$

3) if $n = 3m + 1$ $(m \geq 1)$ then

$$\Gamma_i(a) = \begin{pmatrix} E_m & t^{2i} E_m & J_m(0) & t\langle 1 \rangle \\ 0 & E_m & t^i E_m & 0 \\ 0 & 0 & E_m & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (i \in \mathbb{N});$$

4) if $n = 3m + 2$ $(m \geq 1)$ then

$$\Gamma_i(a) = \begin{pmatrix} E_m & t^{i+2} E_m & J_m(0) & t^{2i+4}\langle 1 \rangle & t\langle 1 \rangle \\ 0 & E_m & t^{2i+4} E_m & 0 & t^2\langle 1 \rangle \\ 0 & 0 & E_m & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (i \in \mathbb{N});$$

Lemma 1 ([3], p. 272)**.** *Let V_i be the underlying KH-module of the representation Γ_i. If $i \neq j$, then the KH-modules V_i and V_j are non-isomorphic. The algebra $\operatorname{End}_{KH}(V_i)$ is finitely generated as a K-module and there is an algebra isomorphism $\overline{\operatorname{End}_{KH}(V_i)} \cong K/\operatorname{rad} K$ for every $i \in \mathbb{N}$.*

Lemma 2 ([3], p. 275). *Let $H = \langle a\rangle \times \langle b\rangle$ be an abelian group of type $(2,2)$, $t \in \operatorname{rad} K$, $t \neq 0$ and assume that t is not a zero-divisor. Denote by V_i the underlying KH-module of the matrix representation Δ_i of degree n of the group H defined as follows:*

1) *if $n = 2m$ $(m \geq 1)$, then*

$$\Delta_i(a) = \begin{pmatrix} E_m & t^iE_m \\ 0 & E_m \end{pmatrix} \qquad \Delta_i(b) = \begin{pmatrix} E_m & J_m(0) \\ 0 & E_m \end{pmatrix} \quad (i \in \mathbb{N});$$

2) *if $n = 2m+1$ $(m \geq 1)$, then*

$$\Delta_i(a) = \begin{pmatrix} E_m & t^iE_m & 0 \\ 0 & E_m & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Delta_i(b) = \begin{pmatrix} E_m & J_m(0) & t^i\langle 1\rangle \\ 0 & E_m & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (i \in \mathbb{N});$$

If $i \neq j$, then the modules V_i and V_j are non-isomorphic.

Moreover, $\operatorname{End}_{KH}(V_i)$ is finitely generated as a K-module and there is an algebra isomorphism $\overline{\operatorname{End}_{KH}(V_i)} \cong K/\operatorname{rad} K$ for all $i \in \mathbb{N}$.

Let H be a finite p-group and $|H| > 2$. Denote by $[M]$ the isomorphism class of RH-modules which contains M and by $\sum_n(RH)$ the set of all $[M]$ satisfying the following conditions:

(i) $M \cong R \otimes_S W$ for some SH-module W;
(ii) the R-rank of M equals n;
(iii) $\operatorname{End}_{RH}(M)$ is finitely generated as an R-module;
(iv) $\overline{\operatorname{End}_{RH}(M)} \cong R/\operatorname{rad} R$.

Lemma 3. *The set $\sum_n(RH)$ is infinite for every integer $n > 1$.*

Proof. Let t be a non-zero element of $\mathfrak{m}$. Then $t \in \operatorname{rad} R$ and t is not a zero-divisor in R. Next apply Lemmas 1 and 2. $\square$

Lemma 4. *Let K be a commutative local ring of characteristic p, B a finite abelian p-group, D a subgroup of B, $\lambda \in Z^2(B, K^*)$ and M an indecomposable $K^\lambda D$-module. Assume that $\operatorname{End}_{K^\lambda D}(M)$ is a finitely generated K-algebra and $\overline{\operatorname{End}_{K^\lambda D}(M)}$ is isomorphic to a field L containing $\overline{K} = K/\operatorname{rad} K$. Then*

$$M^B = K^\lambda B \otimes_{K^\lambda D} M$$

is an indecomposable $K^\lambda B$-module, $\operatorname{End}_{K^\lambda B}(M^B)$ *is a finitely generated K-algebra and the quotient algebra* $\overline{\operatorname{End}_{K^\lambda B}(M^B)}$ *is isomorphic to a field that is a finite purely inseparable field extension of L.*

The proof is similar to those of Lemma 2.2 in [1].

Let $\lambda \in Z^2(G, S^*)$. Denote by H_p the kernel of the restriction of λ to $G_p \times G_p$. If $h \in H_p$ and $x \in G$, then $x^{-1}hx \in G_p$ and $|x^{-1}hx| = |h|$. From the equality $u_x^{-1}u_h u_x = \gamma u_{x^{-1}hx}$ $(\gamma \in S^*)$ follows

$$u_x^{-1}u_h^{|h|}u_x = \gamma^{|h|} \cdot u_{x^{-1}hx}^{|h|},$$

hence

$$u_{x^{-1}hx}^{|h|} = \gamma^{-|h|}u_e.$$

We obtain $x^{-1}hx \in H_p$, therefore H_p is a normal subgroup of G. Since the restriction of λ to $H_p \times H_p$ is a coboundary, we may assume that $\lambda_{h_1,h_2} = 1$ for all $h_1, h_2 \in H_p$. Then $\gamma^{|h|} = 1$, hence $\gamma = 1$. Consequently, we may suppose that $\lambda_{a,g} = \lambda_{g,a} = 1$ for arbitrary $a \in H_p$ and $g \in G$.

3. On twisted group algebras of SUR-type

We recall that S is an integral domain of characteristic p, which is not a field, and R is the localization of S at a maximal ideal $\mathfrak{m}$. Denote by F a subfield of S. We assume that G is a finite group, and G_p is a normal subgroup of G. Given $\lambda \in Z^2(G, S^*)$, we denote by H_p the kernel of the restriction of λ to $G_p \times G_p$.

Theorem 1. *Let G be a finite group and $\lambda \in Z^2(G, S^*)$. If $|H_p| > 2$ then $S^\lambda G$ is of SUR-type with an SUR-dimension-valued function $f_\lambda(n) = nt_n$, where $1 \le t_n \le |G\colon H_p|$.*

Proof. By Lemma 3, $\sum_n(RH_p)$ is infinite for each $n > 1$.

Let $[V] \in \sum_n(RH_p)$ and $V^G = R^\lambda G \otimes_{RH_p} V$. Denote by $\{g_1 = e, g_2, \ldots, g_t\}$ a cross section of H_p in G. Then

$$V^G = \otimes_{i=1}^t V_i, \quad V_i = u_{g_i} \otimes V.$$

The RH_p-module V_i is called a conjugate of V. Denote $V^{(g_i)} = V_i$. Since $\operatorname{End}_{RH_p}(V_i) \cong \operatorname{End}_{RH_p}(V)$, the ring $\operatorname{End}_{RH_p}(V_i)$ is local for every $i \in \{1, \ldots, t\}$. Hence V_i is an indecomposable RH_p-module. By Krull-Schmidt Theorem [[11], Sect. 7.3], the RH_p-module V^G has a unique decomposition into a finite sum of indecomposable RH_p-modules, up to isomorphism and the order of summands.

Let $[L] \in \sum_n(RH_p)$. If V is isomorphic to an RH_p-module $L^{(g)}$, then L is isomorphic to the RH_p-module $V^{(g^{-1})}$. Hence there are infinitely many classes $[L_1], \ldots, [L_i], \ldots$ in $\sum_n(RH_p)$ such that every indecomposable RH_p-component of $\left(L_i^G\right)_{H_p}$ is isomorphic to none of the indecomposable RH_p-component of $\left(L_j^G\right)_{H_p}$ if $i \neq j$. Therefore there are infinitely many non-isomorphic indecomposable $R^\lambda G$-modules M such that M is an $R^\lambda G$-component of a module of the form V^G. The R-rank of any $R^\lambda G$-component of V^G is divisible by n and does not exceed $n \cdot |G\colon H_p|$. Since

$$V \cong R \otimes_S W, \quad V^G \cong R \otimes_S W^G$$

for some SH_p-module W, there exists an integer t_n such that $1 \leq t_n \leq |G\colon H_p|$ and $\mathrm{Ind}_{nt_n}(S^\lambda G)$ is an infinite set. $\square$

Theorem 2. *Let G be a finite group and $\lambda \in Z^2(G, S^*)$ and assume that $|H_p\colon G_p'| > 2$. Then $f_\lambda(n) := ndt_n$, where $d = |G_p\colon H_p|$ and $1 \leq t_n \leq |G\colon G_p|$, is an SUR-dimension-valued function for $S^\lambda G$.*

Proof. Let $A = G/G_p'$ and

$$U = \bigoplus_{g \in G_p' \setminus \{e\}} S(u_g - u_e).$$

The set $V := S^\lambda G \cdot U$ is a two-sided ideal of $S^\lambda G$. The quotient algebra $S^\lambda G/V$ is isomorphic to $S^\mu A$, where $\mu_{xG_p', yG_p'} = \lambda_{x,y}$ for all $x, y \in G$.

It contains the group algebra SB_p, where $B_p = H_p/G_p'$. Since $|B_p| > 2$, by Lemma 3, $\sum_n(RB_p)$ is infinite for each positive integer n. The abelian group $A_p = G_p/G_p'$ is a Sylow p-subgroup of A.

Let $[M] \in \sum_n(RB_p)$. By Lemma 4, the $R^\mu A_p$-module

$$M^{A_p} = R^\mu A_p \otimes_{RB_p} M$$

is indecomposable and $\mathrm{End}_{R^\mu A_p}\left(M^{A_p}\right)$ is a local ring. The R-rank of M^{A_p} equals $n \cdot |A_p\colon B_p| = n \cdot |G_p\colon H_p|$. Arguing similarly as in the proof of Theorem 1, we conclude that if $[M]$ and $[N]$ belong to $\sum_n(RB_p)$ and $M \ncong N$, then $M^{A_p} \ncong N^{A_p}$. Let

$$\left(M^{A_p}\right)^A := R^\mu A \otimes_{R^\mu A_p} M^{A_p}.$$

By the same arguments as in the proof of Theorem 1, we can prove that there exist infinitely many pairwise non-isomorphic indecomposable $R^{\mu}A$-modules Ω such that Ω is an $R^{\mu}A$-component of a module of the form $\left(M^{A_p}\right)^A$. Note that the R-rank of Ω is divisible by $n \cdot |G_p \colon H_p|$ and does not exceed

$$n \cdot |G_p \colon H_p| \cdot |G \colon G_p| = nd \cdot |G \colon G_p|.$$

Hence for every $n > 1$ there is an integer t_n such that $1 \leq t_n \leq |G \colon G_p|$ and the set $\mathrm{Ind}_{ndt_n}(S^{\mu}A)$ is infinite.

If M is an $S^{\mu}A$-module, then M is as well an $S^{\lambda}G$-module. $S^{\mu}A$-modules M and N are isomorphic if and only if M and N are isomorphic as $S^{\lambda}G$-modules. Consequently, the set $\mathrm{Ind}_{ndt_n}(S^{\lambda}G)$ is also infinite for any $n > 1$. □

Theorem 3. *Let $p \neq 2$, G be a finite group and $\lambda \in Z^2(G, F^*)$. If the algebra $F^{\lambda}G$ is not semisimple, then the algebra $S^{\lambda}G$ is of SUR-type. Moreover, if $d = \dim_F(F^{\lambda}G_p/\operatorname{rad} F^{\lambda}G_p)$ and $d < |G_p \colon G'_p|$, then a function $f_{\lambda}(n) = ndt_n$, where $1 \leq t_n \leq |G \colon G_p|$, is an SUR-dimension-valued function for $S^{\lambda}G$.*

Proof. Applying Lemma 3 and arguing as in the proof of Theorem 2 in [3], we prove that, for every $n > 1$, there are infinitely many pairwise non-isomorphic indecomposable $R^{\lambda}G_p$-modules $V_1, V_2, \dots$ satisfying the following conditions:

1) the R-rank of V_i is equal to nd;

2) $\mathrm{End}_{R^{\lambda}G_p}(V_i)$ is finitely generated as an R-module;

3) $\overline{\mathrm{End}_{R^{\lambda}G_p}(V_i)}$ is isomorphic to a field which is a finite purely inseparable field extension of $R/\operatorname{rad} R$;

4) $V_i \cong R \otimes_S W_i$, where W_i is an $S^{\lambda}G_p$-module.

Let $V_i^G := R^{\lambda}G \otimes_{R^{\lambda}G_p} V_i$ and $\left(V_i^G\right)_{G_p}$ be the module V_i^G viewed as an $R^{\lambda}G_p$-module. The $R^{\lambda}G_p$-module $(V_i^G)_{G_p}$ is a direct sum of conjugates of V_i. By the Krull-Schmidt Theorem [[11], Sect. 7.3], $\left(V_i^G\right)_{G_p}$ has a unique decomposition into a finite sum of indecomposable $R^{\lambda}G_p$-modules, up to isomorphism and the order of summands. Hence the R-rank of each indecomposable component of $R^{\lambda}G$-module V_i^G is divisible by nd. It follows that the S-rank of each indecomposable component of $S^{\lambda}G$-module W_i^G is divisible by nd. Therefore, there exists an integer t_n such that $1 \leq t_n \leq |G \colon G_p|$ and $\mathrm{Ind}_{ndt_n}(S^{\lambda}G)$ is an infinite set. □

Theorem 4. *Let $p = 2$, G be a finite group, $\lambda \in Z^2(G, F^*)$ and moreover $d = \dim_F(F^{\lambda}G_2/\operatorname{rad} F^{\lambda}G_2)$.*

(i) If the algebra $F^\lambda G$ is not semisimple, then the set $\mathrm{Ind}_l(S^\lambda G)$ *is infinite for some* $l \leq |G|$.

(ii) If $d < \frac{1}{2}|G_2\colon G_2'|$, *then* $S^\lambda G$ *is of SUR-type with an SUR-dimension-valued function* $f_\lambda(n) = ndt_n$, *where* $1 \leq t_n \leq |G\colon G_2|$.

Proof. Apply Lemma 3 and proceed as in the proof of Theorem 3 in [3]. □

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