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UNIFORMLY CONTINUOUS SET-VALUED COMPOSITION OPERATORS IN THE SPACES OF FUNCTIONS OF BOUNDED VARIATION IN THE SENSE OF SCHRAMM

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Abstract. We show that the one-sided regularizations of the generator of any uniformly continuous set–valued Nemytskij operator, acting between the spaces of functions of bounded variation in the sense of Schramm, is an affine function. Results along these lines extend the study [1].

1. Introduction

Let $(X, |\cdot|)$ and $(Y, |\cdot|)$ be two real normed spaces, C be a convex cone in X and $I = [a, b] \subset \mathbb{R}$ $(a, b \in \mathbb{R}, a < b)$ be an interval. Let $cc(Y)$ be the family of all non-empty convex and compact subsets of Y. We consider the Nemytskij operator, i.e. the composition operator defined by $(HF)(t) = h(t, F(t))$, where $F : I \longrightarrow C,\ h : I \times C \longrightarrow cc(Y)$ is a given set-valued function. It is shown that if the operator H maps the space $\Phi BV(I; C)$ of functions of bounded Φ-variation in the sense of Schramm into the space $BS_\Psi(I; cc(Y))$ of set–valued functions of bounded Ψ-variation in the sense of Schramm, and is uniformly continuous, then the one-sided regularizations h^- and h^+ of h with respect to the first variable, are affine with respect to the second variable. In particular,

$$h^-(t,x) = A(t)x + B(t) \quad \text{for } t \in I,\ x \in C,$$

for some function $A : I \longrightarrow \mathcal{L}(C, cc(Y))$ and $B \in BS_\Psi(I; cc(Y))$, where $\mathcal{L}(C, cc(Y))$ stands for the space of all linear mappings acting from C into $cc(Y)$.

2. Preliminaries

We start by recalling some very basic facts as definitions and known results concerning the space of functions of bounded variation in the sense of Schramm.

Let $\mathcal{F}$ be the set of all convex functions $\phi : [0,\infty) \longrightarrow [0,\infty)$ such that $\phi(0) = \phi(0^+) = 0$ and $\lim_{t\longrightarrow\infty} \phi(t) = \infty$. Then we have

Remark 1. If $\phi \in \mathcal{F}$, then ϕ is continuous and strictly increasing (see [1, 7]).

A sequence $\Phi = (\phi_i)_{i=1}^\infty$ of functions from $\mathcal{F}$ satisfying the following two conditions:

(i) $\phi_{n+1}(t) \leq \phi_n(t)$ for all $t > 0$ and $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty} \phi_n(t)$ diverges for all $x > 0$,

is said to be a Φ–*sequence.*

Let $I = [a,b]$ $(a, b \in \mathbb{R}, a < b)$ be an interval. For a set X we denote by X^I the set of all functions $f : I \to \mathbb{R}$.

If $I_n = [a_n, b_n]$ is a subinterval of the interval I $(n = 1, 2, \dots)$, then we write $f(I_n) := f(b_n) - f(a_n)$.

Definition 1. Let $\Phi = (\phi_n)_{n=1}^\infty$ be a Φ–sequence and $(X, |\cdot|)$ be a real normed space. A function $f \in X^I$ is of *bounded Φ-variation in the sense of Schramm in I* if

$$v_\Phi(f) = v_\Phi(f, I) := \sup \sum_{n=1}^{m} \phi_n\big(|f(I_n)|\big) < \infty, \tag{1}$$

where the supremum is taken over all $m \in \mathbb{N}$ and all non–ordered collections of non–overlapping intervals $I_n = [a_n, b_n] \subset I, n = 1, \dots, m$ ([18]).

It is known that for all $a, b, c \in I$, $a \le c \le b$ we have $v_\Phi(f, [a, c]) \le v_\Phi(f, [a, b])$ (that is v_Φ is increasing with respect to the interval) and $v_\Phi(f, [a, c]) + v_\Phi(f, [c, b]) \le v_\Phi(f, [a, b])$.

In what follows we denote by $V_\Phi(I, X)$ the set of all functions $f \in X^I$ of bounded Φ-variation in the Schramm sense and by $\Phi BV(I, X)$ the linear space of all functions $f \in X^I$ such that $v_\Phi(\lambda f) < \infty$ for some constant $\lambda > 0$.

In the space $\Phi BV(I, X)$ the function $\|\cdot\|_\Phi$ defined by

$$\|f\|_\Phi := |f(a)| + p_\Phi(f), \quad f \in \Phi BV(I, X),$$

where

$$p_\Phi(f) := p_\Phi(f, I) = \inf\Big\{\epsilon > 0 : v_\Phi\big(f/\epsilon\big) \le 1\Big\}, \quad f \in \Phi BV(I, X), \tag{2}$$

is a norm (see for instance [14]).

For $X = \mathbb{R}$, the linear normed space $(\Phi BV(I, \mathbb{R}), \|\cdot\|_\Phi)$ was studied by Schramm [18, Theorem 2.3]. The functional $p_\Phi(\cdot)$ defined by (2) is called *the Luxemburg-Nakano-Orlicz seminorm* [5, 25, 26].

It is worth mentioning that the symbol $\Phi BV(I, C)$ stands for the set of all functions $f \in \Phi BV(I, X)$ such that $f : I \longrightarrow C$ and C is a subset of X.

Let $cc(X)$ be the family of all non-empty convex compact subsets of X, and let D be the *Pompeiu-Hausdorff metric* in $cc(X)$, i.e.

$$D(A, B) := \max\Big\{e(A, B),\ e(B, A)\Big\}, \quad A, B \in cc(X), \tag{3}$$

where

$$e(A, B) = \sup\Big\{d(x, B) : x \in A\Big\}, \qquad d(x, B) = \inf\Big\{d(x, y) : y \in B\Big\}. \tag{4}$$

It is easy to check that the Pompeiu-Hausdorff metric D is invariant with respect to translation, i.e.

$$D(A, B) = D(A + Q, B + Q) \tag{5}$$

(see [4, Lemma 3]) for all $A, B \in cc(X)$ and bounded nonempty subset Q of X.

Definition 2. Let $\Phi = (\phi_n)_{n=1}^\infty$ be a Φ–sequence and $F : I \longrightarrow cc(X)$. We say that F has bounded Φ-variation in the Schramm sense if

$$w_\Phi(F) := \sup \sum_{n=1}^{m} \Phi_n\left(D(F(t_n), F(t_{n-1}))\right) < \infty, \tag{6}$$

where the supremum is taken over all $m \in \mathbb{N}$ and all non–ordered collections of non–overlaping intervals $I_n = [a_n, b_n] \subset I, i = 1, \dots, m$.

From now on, let

$$BS_\Phi(I, cc(X)) := \Big\{ F \in cc(X)^I : \ w_\Phi(\lambda F) < \infty \text{ for some } \lambda > 0 \Big\}. \qquad (7)$$

For $F_1, F_2 \in BS_\Phi(I, cc(X))$ put

$$D_\Phi(F_1, F_2) := D\left(F_1(a), F_2(a)\right) + p_\Phi(F_1, F_2), \qquad (8)$$

where

$$p_\Phi(F_1, F_2) := \inf \Big\{ \epsilon > 0 : S_\epsilon(F_1, F_2) \le 1 \Big\} \qquad (9)$$

and

$$S_\epsilon(F_1, F_2) := \sup \sum_{n=1}^{m} \phi_n \left(\frac{1}{\epsilon} D\big(F_1(t_n) + F_2(t_{n-1}); F_2(t_n) + F_1(t_{n-1})\big) \right), \qquad (10)$$

where the supremum is taken over the same collection $([a_n, b_n])_{n=1}^m$ as in Definition 2. Then $(BS_\Phi(I, cc(X)), D_\Phi)$ is a metric space, and it is complete if X is a Banach space [24, Lemma 5.4].

Taking into account [23, Theorem 3.8 (d)] and [24, condition 5.6], we get the following

Lemma 1. Let $\Phi = (\phi_n)_{n=1}^\infty$ be a Φ–sequence and $F_1, F_2 \in BS_\Phi(I, cc(X))$. Then, for $\lambda > 0$,

$$S_\lambda(F_1, F_2) \le 1 \ \text{ if and only if } \ p_\Phi\big(F_1, F_2\big) \le \lambda.$$ ■

In what follows, let$(X, |\cdot|)$, $(Y, |\cdot|)$ be two real normed spaces and C be a convex cone in X. Given a set-valued function $h : I \times C \longrightarrow cc(Y)$ we set the composition operator $H : C^I \to cc(Y)^I$ generated by h as:

$$(Hf)(t) := h(t, f(t)), \quad f \in C^I, \quad t \in I. \qquad (11)$$

Moreover, let us denote by $\mathcal{A}(C, cc(Y))$ the space of all additive functions and by $\mathcal{L}(C, cc(Y))$ the space of all set-valued linear functions, i.e. the space of all set-valued functions $A \in \mathcal{A}(C, cc(Y))$ which are positively homogeneous [1].

Now we quote the following lemma given by Nikodem.

Lemma 2. ([15, Theorem 5.6]). Let $(X, |\cdot|)$, $(Y, |\cdot|)$ be normed spaces and C a convex cone in X. A set-valued function $F : C \longrightarrow cc(Y)$ satisfies the Jensen equation

$$F\Big(\frac{x+y}{2}\Big) = \frac{1}{2}\Big(F(x) + F(y)\Big), \quad x, y \in C, \qquad (12)$$

if and only if there exist an additive set-valued function $A : C \longrightarrow cc(Y)$ and a set $B \in cc(Y)$ such that $F(x) = A(x) + B$ for all $x \in C$. ■

3. The composition operator

Now we will present our main result.

Theorem 1. Let $(X, |\cdot|)$ be a real normed space, $(Y, |\cdot|)$ a real Banach space, C a convex cone in X and suppose that $\Phi = (\phi_n)_{n=1}^{\infty}$ and $\Psi = (\psi_n)_{n=1}^{\infty}$ are Φ–sequences. If the composition operator H generated by a set-valued function $h : I \times C \longrightarrow cc(Y)$ maps $\Phi BV(I, C)$ into $BS_\Psi(I, cc(Y))$ and is uniformly continuous, then the left regularization of h, i.e. the function $h^- : I \times C \longrightarrow cc(Y)$ defined by

$$h^-(t, x) := \lim_{s \uparrow t} h(s, x), \quad t \in I, \; x \in C,$$

exists and

$$h^-(t, x) = A(t)x + B(t), \quad t \in I, \quad x \in C,$$

for some $A : I \longrightarrow \mathcal{A}(X, cc(Y))$ and $B : I \longrightarrow cc(Y)$. Moreover, if $0 \in C$, then $B \in BS_\Psi(I, cc(Y))$ and the linear set-valued function $A(t)$ is continuous.

Proof. For every $x \in C$, the constant function $I \ni t \longrightarrow x$ belongs to $\Phi BV(I, C)$. Since H maps $\Phi BV(I, C)$ into $BS_\Psi(I, cc(Y))$ for every $x \in C$, the function $I \ni t \longrightarrow h(t, x)$ belongs to $BS_\Psi(I, cc(Y))$. Now the completeness of $cc(Y)$ with respect to the Pompeiu-Hausdorff metric [24, Lemma 6.12] implies the existence of the left regularization h^- of h.

By the assumption, H is uniformly continuous on $\Phi BV(I, C)$. Let $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the modulus of continuity of H, that is

$$\omega(\rho) := \sup\Big\{D_\Psi\big(H(f_1), H(f_2)\big) : \|f_1 - f_2\|_\Phi \le \rho; \; f_1, f_2 \in \Phi BV(I, C)\Big\}, \; \rho > 0.$$

Hence we get

$$D_\Psi\big(H(f_1), H(f_2)\big) \le \omega\big(\|f_1 - f_2\|_\Phi\big) \quad \text{for} \quad f_1, f_2 \in \Phi BV(I, C). \tag{13}$$

From the definition of the metric D_Ψ and (13), we obtain

$$p_\Psi\big(H(f_1); H(f_2)\big) \le \omega\big(\|f_1 - f_2\|_\Phi\big) \quad \text{for} \quad f_1, f_2 \in \Phi BV(I, C). \tag{14}$$

From Lemma 1, if $\omega\big(\|f_1 - f_2\|_\Phi\big) > 0$, the inequality (14) is equivalent to

$$S_{\omega(\|f_1 - f_2\|_\Phi)}\Big(H(f_1), H(f_2)\Big) \le 1, \quad f_1, f_2 \in \Phi BV(I, C). \tag{15}$$

Therefore, for any $a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m = b,\ \alpha_i,\ \beta_i \in I$, $i \in \{1, 2, \cdots, m\},\ m \in \mathbb{N},$ the definitions of the operator H and the functional S_ϵ, imply

$$\sum_{i=1}^{\infty} \psi_i\Big(\frac{D\big(h(\beta_i, f_1(\beta_i)) + h(\alpha_i, f_2(\alpha_i)); h(\beta_i, f_2(\beta_i)) + h(\alpha_i, f_1(\alpha_i))\big)}{\omega\big(\|f_1 - f_2\|_\Phi\big)}\Big) \le 1. \tag{16}$$

For $\alpha, \beta \in \mathbb{R},\ \alpha < \beta$, we define functions $\eta_{\alpha,\beta} : \mathbb{R} \longrightarrow [0, 1]$ by

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0 & \text{if} \quad t \le \alpha \\ \dfrac{t-\alpha}{\beta-\alpha} & \text{if} \quad \alpha \le t \le \beta \\ 1 & \text{if} \quad \beta \le t\,. \end{cases} \tag{17}$$

Let us fix $t \in I$. For arbitrary finite sequence $a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < t$ and $x_1, x_2 \in C,\ x_1 \ne x_2$, the functions $f_1, f_2 : I \longrightarrow X$ defined by

$$f_\ell(\tau) := \frac{1}{2}\big[\eta_{\alpha_i,\beta_i}(\tau)(x_1 - x_2) + x_\ell + x_2\big], \quad \tau \in I,\ \ell = 1, 2, \tag{18}$$

belong to the space $\Phi BV(I, C)$. From (18) we have

$$f_1(\tau) - f_2(\tau) = \frac{x_1 - x_2}{2}\,, \qquad \tau \in I,$$

therefore

$$\|f_1 - f_2\|_\Phi = \Big|\frac{x_1 - x_2}{2}\Big|;$$

moreover

$$f_1(\beta_i) = x_1;\ f_2(\beta_i) = \frac{x_1 + x_2}{2};\ f_1(\alpha_i) = \frac{x_1 + x_2}{2};\ f_2(\alpha_i) = x_2.$$

Using (16), we get

$$\sum_{i=1}^{\infty} \psi_i\left(\frac{D\left(h(\beta_i, x_1) + h(\alpha_i, x_2); h\left(\alpha_i, \frac{x_1+x_2}{2}\right) + h\left(\beta_i, \frac{x_1+x_2}{2}\right)\right)}{\omega\left(\left|\frac{x_1-x_2}{2}\right|\right)}\right) \le 1. \tag{19}$$

Fix a positive integer m. We have

$$\sum_{i=1}^{m} \psi_i\left(\frac{D\left(h(\beta_i, x_1) + h(\alpha_i, x_2); h\left(\alpha_i, \frac{x_1+x_2}{2}\right) + h\left(\beta_i, \frac{x_1+x_2}{2}\right)\right)}{\omega\left(\left|\frac{x_1-x_2}{2}\right|\right)}\right) \le 1. \tag{20}$$

From the continuity of ψ_i, passing to the limit in (20) when $\alpha_1 \uparrow t$, we obtain that

$$\sum_{i=1}^{m} \psi_i \left(\frac{D\Big(h^-(t,x_1)+h^-(t,x_2); 2h^-\Big(t, \dfrac{x_1+x_2}{2}\Big)\Big)}{\omega\Big(\Big|\dfrac{x_1-x_2}{2}\Big|\Big)} \right) \leq 1.$$

Hence,

$$\sum_{i=1}^{\infty} \psi_i \left(\frac{D\left(h^-(t,x_1)+h^-(t,x_2); 2h^-\left(t, \frac{x_1+x_2}{2}\right)\right)}{\omega\left(\left|\frac{x_1-x_2}{2}\right|\right)} \right) \leq 1,$$

and, by (ii),

$$D\Big(h^-(t,x_1)+h^-(t,x_2); 2h^-\Big(t, \frac{x_1+x_2}{2}\Big)\Big) = 0.$$

Therefore,

$$h^-\Big(t, \frac{x_1+x_2}{2}\Big) = \frac{h^-(t,x_1)+h^-(t,x_2)}{2}$$

for all $t \in I$ and all $x_1,\ x_2 \in C$.

Thus, for each $t \in I$, the function $h^-(t,\cdot)$ satisfies the Jensen functional equation in C. Consequently, by Lemma 2, for every $t \in I$ there exist an additive set–valued function $A(t) : C \longrightarrow cc(Y)$ and a set $B(t) \in cc(Y)$ such that

$$h^-(t,x) = A(t)x + B(t) \quad \text{for} \quad x \in C,\ t \in I, \tag{21}$$

which proves the first part of our result.

The uniform continuity of the operator $H : \Phi BV(I,C) \longrightarrow BS_\Psi(I, cc(Y))$ implies the continuity of the function $A(t)$ so that $A(t) \in \mathcal{L}(C, cc(Y))$ [15, Theorem 5.3]. Putting $x = 0$ in (21) and taking into account that $A(t)0 = \{0\}$ for $t \in I$, we get

$$h^-(t,0) = B(t), \quad t \in I,$$

which implies that $B \in BS_\Psi(I, cc(Y))$. ∎

Remark 2. The counterpart of Theorem 1 for the right regularization h^+ of h defined by

$$h^+(t,x) := \lim_{s \downarrow t} h(s,x), \quad t \in I,$$

is also true.

Remark 3. Taking $\psi_n(t) = \psi(t)$ $(t \geq 0)$, we obtain the main result of [1].

Remark 4. Denote by S the set of all functions $f \in \Phi BV(I, C)$ such that

$$f(t) = \frac{1}{2}\left[\eta_{\alpha,\beta}(t)(x_1 - x_2) + x + x_2\right],$$

where $\eta_{\alpha,\beta}$ is defined by (17) and $x = x_1$ or $x = x_2$. It follows from the argument used in the proof that Theorem 1 remains valid if the uniform continuity of the operator H is postulated only on the set S.

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