

SOME ALGEBRAIC PROPERTIES OF PREPONDERANTLY CONTINUOUS FUNCTIONS

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Abstract. In the presented paper we study some properties of preponderantly continuous functions and functions satisfying the property A_1 . For any family \mathcal{F} of real-valued functions we define $\mathcal{MAX}_{\mathcal{F}} = \{g: \max\{f, g\} \in \mathcal{F} \text{ for all } f \in \mathcal{F}\}$ and $\mathcal{MIN}_{\mathcal{F}} = \{g: \min\{f, g\} \in \mathcal{F} \text{ for all } f \in \mathcal{F}\}$. The aim of the paper is to find $\mathcal{MIN}_{\mathcal{F}}$ for two discussed classes of functions.

1. Preliminaries

Let \mathbb{R}, \mathbb{N} be the set of real numbers and natural numbers, respectively. Next, let I denote a closed interval, U any open subset of \mathbb{R} and $\text{Int}(A)$ is the interior of a set $A \subset \mathbb{R}$ in the natural metric. Let λ stand for Lebesgue measure in \mathbb{R} . For each measurable set $E \subset \mathbb{R}$ we define the lower and upper density of E at $x_0 \in \mathbb{R}$ by:

$$\underline{d}(E, x_0) = \liminf_{\lambda(I) \rightarrow 0, x_0 \in I} \frac{\lambda(I \cap E)}{\lambda(I)} \quad \text{and} \quad \bar{d}(E, x_0) = \limsup_{\lambda(I) \rightarrow 0, x_0 \in I} \frac{\lambda(I \cap E)}{\lambda(I)}.$$

If $\underline{d}(E, x_0) = \bar{d}(E, x_0)$, we denote this common value by $d(E, x_0)$ and call it the density of E at x_0 . In a similar way, we also define the one-sided lower and upper density of the set E at the point x_0 : $\underline{d}^+(E, x_0), \underline{d}^-(E, x_0), \bar{d}^+(E, x_0)$ and $\bar{d}^-(E, x_0)$. It is easy to check that $\underline{d}(E, x_0) = \min\{\underline{d}^+(E, x_0), \underline{d}^-(E, x_0)\}$ and $\bar{d}(E, x_0) = \max\{\bar{d}^+(E, x_0), \bar{d}^-(E, x_0)\}$. If $\underline{d}^+(E, x_0) = \bar{d}^+(E, x_0)$ ($\underline{d}^-(E, x_0) = \bar{d}^-(E, x_0)$), then we denote this common value by $d^+(E, x_0)$ ($d^-(E, x_0)$) and call it the right (the left) density of E at x_0 .

There are a few nonequivalent definitions of preponderant density and preponderant continuity [3]. We will use the following.

Definition 1. [1, 3] A point $x_0 \in \mathbb{R}$ is said to be the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ if $\underline{d}(E, x_0) > \frac{1}{2}$.

Similarly, we can define the preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ at the right and at the left. Moreover, a point $x_0 \in \mathbb{R}$ is the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ iff it is the point of preponderant density in Denjoy sense of the measurable set E at the right and at the left.

Definition 2. [1, 3] A function $f: U \rightarrow \mathbb{R}$ is said to be preponderantly continuous in Denjoy sense at $x_0 \in U$ if there exists a measurable set $E \subset U$ containing x_0 such that $\underline{d}(E, x_0) > \frac{1}{2}$ and $f|_E$ is continuous at x_0 . A function $f: U \rightarrow \mathbb{R}$ is said to be preponderantly continuous in Denjoy sense if it is preponderantly continuous in Denjoy sense at each point $x_0 \in U$. The class of all functions which are preponderantly continuous in Denjoy sense will be denoted by \mathcal{PD} .

Grande [2] defined a property of real functions called the property A_1 . Based on this, we may define a similar property, which extends the notion of preponderant continuity.

Definition 3. [2, 3] A function $f: U \rightarrow \mathbb{R}$ is said to have the property A_1 in Denjoy sense at $x_0 \in U$ if there exist measurable sets $E_1 \subset U$ and $E_2 \subset U$ containing x_0 such that x_0 is the point of preponderant density in Denjoy sense of both sets E_1 and E_2 , $f|_{E_1}$ is upper semi-continuous at x_0 and $f|_{E_2}$ is lower semi-continuous at x_0 . A function $f: U \rightarrow \mathbb{R}$ has the property A_1 in Denjoy sense if it has the property A_1 in Denjoy sense at each $x_0 \in U$. The class of all functions which have the property A_1 in Denjoy sense will be denoted by \mathcal{GPD} .

Corollary. $\mathcal{PD} \subset \mathcal{GPD}$.

2. Auxiliary lemmas

We will present some known facts and the useful lemma.

Theorem 1. [3, Corollary 9] $\mathcal{GPD} \subset \mathcal{B}_1$ and $\mathcal{PD} \subset \mathcal{B}_1$, where \mathcal{B}_1 is the set of Baire class 1 functions.

Theorem 2. [3, Theorem 2]

- (i) A measurable function $f: U \rightarrow \mathbb{R}$ is preponderantly continuous in Denjoy sense at $x_0 \in U$ iff $\lim_{n \rightarrow \infty} \underline{d}\left(\left\{x \in U: |f(x) - f(x_0)| < \frac{1}{n}\right\}, x_0\right) > \frac{1}{2}$,

- (ii) A measurable function $f: U \rightarrow \mathbb{R}$ has the property A_1 in Denjoy sense at $x_0 \in U$ iff $\lim_{n \rightarrow \infty} \underline{d}\left(\{x \in U: f(x) < f(x_0) + \frac{1}{n}\}, x_0\right) > \frac{1}{2}$ and $\lim_{n \rightarrow \infty} \underline{d}\left(\{x \in U: f(x) > f(x_0) - \frac{1}{n}\}, x_0\right) > \frac{1}{2}$.

Theorem 3. [3, Corollary 6] Let $E = \bigcup_{n=1}^{\infty} [a_n, b_n]$, where $b_{n+1} < a_n$ for every n and $x_0 = \lim_{n \rightarrow \infty} a_n$. Then

1. $\underline{d}^+(E, x_0) = \liminf_{n \rightarrow \infty} \frac{\lambda([x_0, a_n] \cap E)}{\lambda([x_0, a_n])}$
2. $\bar{d}^+(E, x_0) = \limsup_{n \rightarrow \infty} \frac{\lambda([x_0, b_n] \cap E)}{\lambda([x_0, b_n])}$.

Lemma 1. Let $\frac{1}{2} < \gamma < 1$, $x \in \mathbb{R}$ and let E be a measurable subset of \mathbb{R} such that $\bar{d}^+(E, x) = c > 0$. Then there exists a sequence of closed intervals $\{I_n = [a_n, b_n]: n \geq 1\}$ for which $x < \dots < b_{n+1} < a_n < \dots$, $d^+\left(\bigcup_{n=1}^{\infty} I_n, x\right) = \gamma$, and $\bar{d}^+\left(E \cap \bigcup_{n=1}^{\infty} I_n, x\right) \geq \frac{1}{2}c$.

Proof. Let $c_n = x + \frac{1}{n}$ for $n \in \mathbb{N}$. Hence $\lim_{n \rightarrow \infty} \frac{\lambda([c_{n+1}, c_n])}{\lambda([x, c_{n+1}])} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n(n+1)}}{\frac{1}{n+1}} = 0$.

Put $U_n^1 = [c_{n+1}, c_{n+1} + \gamma(c_n - c_{n+1})]$ and $U_n^2 = [c_n - \gamma(c_n - c_{n+1}), c_n]$ for $n \geq 1$. Then $\lambda(U_n^1) = \lambda(U_n^2) = \gamma\lambda([c_{n+1}, c_n])$, $[c_{n+1}, c_n] = U_n^1 \cup U_n^2$ and $\lambda(E \cap U_n^1) + \lambda(E \cap U_n^2) \geq \lambda(E \cap [c_{n+1}, c_n])$. It follows that for each $n \geq 1$ we can find a closed interval $J_n \subset [c_{n+1}, c_n]$ such that $\lambda(J_n) = \gamma\lambda([c_{n+1}, c_n])$ and $\lambda(E \cap J_n) \geq \frac{1}{2}\lambda(E \cap [c_{n+1}, c_n])$. Hence $\lambda\left(\bigcup_{n=1}^{\infty} J_n \cap [x, c_k]\right) = \gamma\lambda([x, c_k])$ for $k \geq 1$.

Let $z \in (x, c_1)$. There is $k \geq 1$ such that $z \in [c_{k+1}, c_k]$. Then

$$\lambda\left(\bigcup_{n=1}^{\infty} J_n \cap [x, z]\right) = \lambda\left(\bigcup_{n=k+1}^{\infty} J_n\right) + \lambda(J_k \cap [c_{k+1}, z]) \leq \gamma\lambda([x, z]) + \lambda([c_{k+1}, c_k]),$$

$$\lambda\left(\bigcup_{n=1}^{\infty} J_n \cap [x, z]\right) = \lambda\left(\bigcup_{n=k+1}^{\infty} J_n\right) + \lambda(J_k \cap [c_{k+1}, z]) \geq \gamma\lambda([x, z]) - \lambda([c_{k+1}, c_k])$$

and

$$\lambda\left(\bigcup_{n=1}^{\infty} J_n \cap E \cap [x, z]\right) \geq \lambda\left(\bigcup_{n=k+1}^{\infty} J_n \cap E\right) \geq \frac{1}{2}\lambda([x, z] \cap E) - \lambda([c_{n+1}, c_n]).$$

Therefore

$$\gamma - \frac{1}{n} = \gamma - \frac{\frac{1}{n} - \frac{1}{n+1}}{\frac{1}{n+1}} \leq \frac{\lambda\left(\bigcup_{n=1}^{\infty} J_n \cap [x, z]\right)}{\lambda([x, z])} \leq \gamma + \frac{\frac{1}{n} - \frac{1}{n+1}}{\frac{1}{n}} = \gamma + \frac{1}{n+1}$$

and

$$\frac{\lambda\left(\bigcup_{n=1}^{\infty} J_n \cap E \cap [x, z]\right)}{\lambda([x, z])} \geq \frac{1}{2} \frac{\lambda(E \cap [x, z])}{\lambda([x, z])} - \frac{\frac{1}{n} - \frac{1}{n+1}}{\frac{1}{n}} = \frac{1}{2} \frac{\lambda(E \cap [x, z])}{\lambda([x, z])} - \frac{1}{n}.$$

It follows that $d^+\left(\bigcup_{n=1}^{\infty} J_n, x\right) = \gamma$ and $\bar{d}^+\left(\bigcup_{n=1}^{\infty} J_n \cap E, x\right) \geq \frac{1}{2}\bar{d}^+(E, x)$.

We have proven that $d^+\left(\bigcup_{n=1}^{\infty} J_n, x\right) = \gamma$ and $\bar{d}^+\left(E \cap \bigcup_{n=1}^{\infty} J_n, x\right) \geq \frac{1}{2}\bar{d}^+(E, x)$, but the elements of the sequence need not be disjoint.

Let $\{I_n : n \geq 1\}$ be a sequence of closed intervals such that $I_n \subset \text{Int } J_n$ for all $n \in \mathbb{N}$ and $\bar{d}^+\left(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x\right) = 0$. Then the sequence $\{I_n : n \geq 1\}$ possesses all the required properties. \square

3. $\mathcal{MAX}_{\mathcal{F}}$ and $\mathcal{MIN}_{\mathcal{F}}$ for \mathcal{PD} and \mathcal{GPD}

Definition 4. For any family \mathcal{F} of functions from U to \mathbb{R} we define

$$\mathcal{MIN}_{\mathcal{F}} = \{g: U \rightarrow \mathbb{R} : \forall f \in \mathcal{F} \min\{f, g\} \in \mathcal{F}\}.$$

and

$$\mathcal{MAX}_{\mathcal{F}} = \{g: U \rightarrow \mathbb{R} : \forall f \in \mathcal{F} \max\{f, g\} \in \mathcal{F}\}.$$

Remark 1. Observe that $\max\{f, g\} = -\min\{-f, -g\}$ and if \mathcal{F} has the property $f \in \mathcal{F} \Rightarrow -f \in \mathcal{F}$, then

$$\mathcal{MAX}_{\mathcal{F}} = \{g: U \rightarrow \mathbb{R} : -g \in \mathcal{MIN}_{\mathcal{F}}\}.$$

We will find $\mathcal{MAX}_{\mathcal{F}}$ and $\mathcal{MIN}_{\mathcal{F}}$ for \mathcal{PD} and \mathcal{GPD} .

Lemma 2. $\mathcal{MIN}_{\mathcal{PD}} \subset \mathcal{PD}$ and $\mathcal{MIN}_{\mathcal{GPD}} \subset \mathcal{GPD}$.

Proof. To prove it, it suffices to take any $f \in \mathcal{MIN}_{\mathcal{PD}}$ ($f \in \mathcal{MIN}_{\mathcal{GPD}}$) and for each $x_0 \in U$ define a constant function $g(x) = f(x_0) + 1$. Then $g \in \mathcal{PD} \cap \mathcal{GPD}$ and, since $\min\{f, g\} \in \mathcal{PD}$ ($\min\{f, g\} \in \mathcal{GPD}$), it is easy to verify, applying Theorem 2, that f is preponderantly continuous in Denjoy sense at x_0 (g has the property A_1 in Denjoy sense at x_0). \square

Lemma 3. *If $f \in \mathcal{PD}$ ($f \in \mathcal{GPD}$) and g is approximately continuous, then $\max\{f, g\}, \min\{f, g\} \in \mathcal{PD}$ ($\max\{f, g\}, \min\{f, g\} \in \mathcal{GPD}$).*

Proof. Fix any $x_0 \in U$. Since $f \in \mathcal{PD}$ ($f \in \mathcal{GPD}$), there exists a measurable set E (there exist two measurable sets E_1 and E_2) such that $x_0 \in E$ ($x_0 \in E_1 \cap E_2$), x_0 is a point of Denjoy preponderant density of E (of both sets E_1 and E_2) and $f|_E$ is continuous at x_0 ($f|_{E_1}$ is upper semi-continuous at x_0 and $f|_{E_2}$ is lower semi-continuous at x_0). Similarly, since g is approximately continuous at x_0 , there exists a measurable set F such that $x_0 \in F$, $\underline{d}(F, x_0) = 1$ and $g|_F$ is continuous at x_0 . Then $\min\{f, g\}|_{E \cap F}$ and $\max\{f, g\}|_{E \cap F}$ are continuous at x_0 ($\min\{f, g\}|_{E_1 \cap F}$, $\max\{f, g\}|_{E_1 \cap F}$ are upper semi-continuous at x_0 and $\min\{f, g\}|_{E_2 \cap F}$, $\max\{f, g\}|_{E_2 \cap F}$ are lower semi-continuous at x_0). Moreover, $\underline{d}(E \cap F, x_0) \geq \underline{d}(E, x_0) - \overline{d}(\mathbb{R} \setminus F, x_0) > \frac{1}{2}$ ($\underline{d}(E_1 \cap F, x_0) > \frac{1}{2}$ and $\overline{d}(E_2 \cap F, x_0) > \frac{1}{2}$). It follows that $\min\{f, g\}$ and $\max\{f, g\}$ are preponderantly continuous in Denjoy sense at x_0 ($\min\{f, g\}$ and $\max\{f, g\}$ satisfy the property A_1 in Denjoy sense at x_0). Since x_0 was an arbitrary point, $\min\{f, g\}, \max\{f, g\} \in \mathcal{PD}$ ($\min\{f, g\}, \max\{f, g\} \in \mathcal{GPD}$). \square

Lemma 4. *If $g \in \mathcal{PD}$ is not approximately lower semi-continuous at $x_0 \in U$, then there exists $f \in \mathcal{PD}$ such that $\min\{f, g\} \notin \mathcal{GPD}$.*

Proof. We may assume that g is not approximately lower semi-continuous at x_0 at the right. Then there exists $\varepsilon > 0$ such that $\overline{d}^+(\{x > x_0 : f(x) < f(x_0) - \varepsilon\}, x_0) = c > 0$. Applying Lemma 1, we can find a sequence of closed intervals $\{I_n = [a_n, b_n] : n \geq 1\}$ such that $x_0 < \dots < b_{n+1} < a_n < \dots$, $d^+(\bigcup_{n=1}^{\infty} I_n, x_0) = \frac{1}{2} + \frac{1}{4}c$ and $\overline{d}(\bigcup_{n=1}^{\infty} I_n \cap \{x > x_0 : f(x) < f(x_0) - \varepsilon\}, x_0) > \frac{1}{2}c$. Pick a sequence of pairwise disjoint closed intervals $\{J_n = [c_n, d_n] : n \geq 1\}$ such that $I_n \subset \text{Int}(J_n)$ and $\overline{d}(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0) = 0$. Define a function $f : U \rightarrow \mathbb{R}$ letting

$$f(y) = \begin{cases} g(x_0) & \text{if } y \in (U \setminus (x_0, d_1)) \cup \bigcup_{n=1}^{\infty} I_n, \\ g(x_0) - 2\varepsilon & \text{if } y \in \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \text{linear on each interval } [c_n, a_n] \text{ and } [b_n, d_n], & n = 1, 2, \dots \end{cases}$$

Obviously, $\min\{f(x_0), g(x_0)\} = g(x_0)$ and $f \in \mathcal{PD}$, because f is continuous at each point except at x_0 and x_0 is a point of preponderant density in Denjoy sense of $(E \setminus (x_0, d_1)) \cup \bigcup_{n=1}^{\infty} I_n$. Let $E = \{y : \min\{f(y), g(y)\} > g(x_0) - \varepsilon\}$.

Then $E \cap \bigcup_{n=1}^{\infty} [d_{n+1}, c_n] = \emptyset$ and

$$\begin{aligned} \underline{d}(E, x_0) &\leq \underline{d}^+(E, x_0) \leq \underline{d}^+\left(E \cap \bigcup_{n=1}^{\infty} I_n, x_0\right) + \bar{d}^+\left(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0\right) = \\ &= d^+\left(\bigcup_{n=1}^{\infty} I_n, x_0\right) - \bar{d}^+\left(\bigcup_{n=1}^{\infty} I_n \cap \{x > x_0 : f(x) < f(x_0) - \varepsilon\}, x_0\right) \leq \\ &\leq \frac{1}{2} + \frac{1}{4}c - \frac{1}{2}c = \frac{1}{2} - \frac{1}{4}c < \frac{1}{2}. \end{aligned}$$

This implies that $\min\{f, g\}$ does not have the property A_1 in Denjoy sense at x_0 and $\min\{f, g\} \notin \mathcal{PGD}$, which completes the proof. \square

Theorem 4. $\mathcal{MLN}_{\mathcal{PD}} = \mathbf{A}$, where \mathbf{A} is the set of approximately continuous functions.

Proof. By Lemma 3, we have inclusion $\mathbf{A} \subset \mathcal{MLN}_{\mathcal{PD}}$.

Suppose that g is not approximately continuous at x_0 . If g is not approximately lower semi-continuous at x_0 , then applying Lemma 4, we obtain that $g \notin \mathcal{MLN}_{\mathcal{PD}}$. Assume that g is not approximately upper semi-continuous at $x_0 \in U$. Without loss of generality we may assume that g is not approximately upper semi-continuous at x_0 at the right. Then we can find $\varepsilon > 0$ such that $\bar{d}^+(\{x > x_0 : f(x) > f(x_0) + \varepsilon\}, x_0) = c > 0$.

As it was shown earlier, we can find $\varepsilon > 0$ and two sequences $\{I_n = [a_n, b_n] : n \geq 1\}$, $\{J_n = [c_n, d_n] : n \geq 1\}$ of closed intervals such that $x_0 < \dots < d_{n+1} < c_n < \dots$, $I_n \subset \text{Int}(J_n)$ for $n \in \mathbb{N}$, $d^+(\bigcup_{n=1}^{\infty} I_n, x_0) = \frac{1}{2} + \frac{1}{4}c$, $\bar{d}^+(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0) = 0$ and $\bar{d}^+(\bigcup_{n=1}^{\infty} I_n \cap \{x > x_0 : f(x) > f(x_0) + \varepsilon\}, x_0) > \frac{1}{2}c$. Define $f : U \rightarrow \mathbb{R}$ letting:

$$f(y) = \begin{cases} g(x_0) + 2 \cdot \varepsilon & \text{if } y \in (U \setminus (x_0, d_1)) \cup \bigcup_{n=1}^{\infty} I_n, \\ g(x_0) - 2 \cdot \varepsilon & \text{if } y \in \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \text{linear on the intervals } [c_n, a_n] \text{ and } [b_n, d_n], & n = 1, 2, \dots \end{cases}$$

It is clear that $f \in \mathcal{PD}$, since it is discontinuous only at x_0 and x_0 is a point of preponderant density in Denjoy sense of $(U \setminus (x_0, d_1)) \cup \bigcup_{n=1}^{\infty} I_n$. Moreover, $\min\{f(x_0), g(x_0)\} = g(x_0)$. Let $E = \{x \in U : |\min\{f(y), g(y)\} - g(x_0)| < \varepsilon\}$.

Then $E \cap \bigcup_{n=1}^{\infty} [d_{n+1}, c_n] = \emptyset$ and

$$\begin{aligned} d(E, x_0) &\leq \underline{d}^+(E, x_0) \leq \underline{d}^+\left(E \cap \bigcup_{n=1}^{\infty} I_n, x_0\right) + \bar{d}^+\left(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0\right) = \\ &= d^+\left(\bigcup_{n=1}^{\infty} I_n, x_0\right) - \bar{d}^+\left(\bigcup_{n=1}^{\infty} I_n \cap \{x > x_0 : f(x) < f(x_0) - \varepsilon\}, x_0\right) \leq \\ &\leq \frac{1}{2} + \frac{1}{4}c - \frac{1}{2}c = \frac{1}{2} - \frac{1}{4}c < \frac{1}{2}. \end{aligned}$$

Therefore $\min\{f, g\}$ is not Denjoy preponderantly continuous at x_0 . It follows that $\min\{f, g\} \notin \mathcal{PD}$. We have proven that if $g \notin \mathbf{A}$, then $g \notin \mathcal{MLN}_{\mathcal{PD}}$. Hence $\mathcal{MLN}_{\mathcal{PD}} \subset \mathbf{A}$, which completes the proof. \square

Applying Remark 1, we have:

Corollary.

$$\mathcal{MA}\mathcal{X}_{\mathcal{PD}} = \mathbf{A}.$$

Theorem 5. $\mathcal{MLN}_{\mathcal{GPD}} = \mathcal{GPD} \cap \{f : f \text{ is approximately lower semi-continuous}\}$.

Proof. Let $g \in \mathcal{MLN}_{\mathcal{GPD}}$. Remark 1 and Lemma 4 imply that $g \in \mathcal{GPD}$ and g is lower semi-continuous.

Let $f, g : U \rightarrow \mathbb{R}$, $f, g \in \mathcal{GPD}$, $x_0 \in I$ and g be approximately lower semi-continuous at x_0 . If $\min\{f(x_0), g(x_0)\} = g(x_0)$, then

$$\{y \in U : g(y) < g(x_0) + \varepsilon\} \subset \{y \in U : \min\{f(y), g(y)\} < g(x_0) + \varepsilon\}$$

and if $\min\{f(x_0), g(x_0)\} = f(x_0)$, then

$$\{y \in U : f(y) < f(x_0) + \varepsilon\} \subset \{y \in U : \min\{f(y), g(y)\} < f(x_0) + \varepsilon\}$$

for each $\varepsilon > 0$. In both cases x_0 is a point of preponderant density in Denjoy sense of

$$\{y \in U : \min\{f(y), g(y)\} < \min\{f(x_0), g(x_0)\} + \varepsilon\}$$

for each $x_0 \in I$ and each $\varepsilon > 0$, because $f, g \in \mathcal{GPD}$.

On the other hand, the set $\{y \in U : g(y) > g(x_0) - \varepsilon\} \cap \{y : f(y) > f(x_0) - \varepsilon\}$ is contained in $\{y \in U : \min\{f(y), g(y)\} > \min\{f(x_0), g(x_0)\} - \varepsilon\}$. Since $f \in \mathcal{GPD}$ and g is approximately lower semi-continuous at x_0 , we have

$$\underline{d}(\{y : f(y) > f(x_0) - \varepsilon\}, x_0) > \frac{1}{2} \quad \text{and} \quad \underline{d}(\{y : g(y) > g(x_0) - \varepsilon\}, x_0) = 1.$$

Therefore

$$\begin{aligned} & \underline{d}(\{y \in U : \min\{f(y), g(y)\} > \min\{f(x_0), g(x_0)\} - \varepsilon\}, x_0) \geq \\ & \geq \underline{d}(\{y \in U : f(y) > f(x_0) - \varepsilon\}, x_0) - \overline{d}(\mathbb{R} \setminus \{y \in U : g(y) > g(x_0) - \varepsilon\}, x_0) \geq \\ & \geq \underline{d}(\{y \in U : f(y) > f(x_0) - \varepsilon\}, x_0). \end{aligned}$$

Hence

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \underline{d}(\{y \in U : \min\{f(y), g(y)\} > \min\{f(x_0), g(x_0)\} - \varepsilon\}, x_0) \geq \\ & \geq \lim_{\varepsilon \rightarrow 0^+} \underline{d}(\{y \in U : f(y) > f(x_0) - \varepsilon\}, x_0) > \frac{1}{2}. \end{aligned}$$

It follows that $\min\{f, g\}$ has property A_1 in Denjoy sense at x_0 . Since x_0 was an arbitrary point of U , we have $\min\{f, g\} \in \mathcal{GPD}$. Therefore $g \in \mathcal{MIN}_{\mathcal{GPD}}$. This completes the proof. \square

Corollary.

$$\mathcal{MAX}_{\mathcal{GPD}} = \mathcal{GPD} \cap \{f : f \text{ is approximately upper semi-continuous}\}.$$

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