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THE POISSON INTEGRALS OF FUNCTIONS OF TWO VARIABLES FOR HERMITE AND LAGUERRE EXPANSIONS

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Abstract. In this paper we consider the Poisson integrals of functions of two variables for Hermite and Laguerre expansions in the spaces $L^p(\mathbb{R}^2; \exp(-z_1^2 - z_2^2))$ and $L^p\left(\mathbb{R}_+^2; (z_1 z_2)^\alpha \exp(-z_1 - z_2)\right)$, respectively. We state some estimates of the rate of convergence of the Poisson integrals.

1. Introduction

In [1] Muckenhoupt considered the Poisson integral $A(f)$ of a function $f \in L^p\left(\mathbb{R}_+; z^\alpha \exp(-z)\right)$, $1 \le p \le \infty$, $\alpha > -1$, $\mathbb{R}_+ = [0, \infty)$, defined by

$$A(f)(r, y) = A(f; r, y) = \int_0^\infty K(r, y, z)\, f(z)\, z^\alpha \exp(-z)\, dz,$$

where

$$\begin{aligned} K(r, y, z) &= \sum_{n=0}^{\infty} \frac{r^n n!}{\Gamma(n + \alpha + 1)} L_n^\alpha(y) L_n^\alpha(z) \\ &= \frac{(ryz)^{-\frac{\alpha}{2}}}{1 - r} \exp\left(\frac{-r(y + z)}{1 - r}\right) I_\alpha\left(\frac{2(ryz)^{\frac{1}{2}}}{1 - r}\right), \qquad 0 < r < 1, \end{aligned}$$

L_n^α is the nth Laguerre polynomial and I_α is the modified Bessel function.

Reference [1] also considered the Poisson integral of a function $f \in L^p(\mathbb{R}; \exp(-z^2))$ for Hermite expansions defined by

$$B(f; r, x) = \int_{-\infty}^{\infty} P(r, x, z) f(z) \exp\left(-z^2\right) dz, \;\; 0 < r < 1,$$

with the Poisson kernel

$$P(r,x,z)=\sum_{n=0}^{\infty}\frac{r^n H_n(x)H_n(z)}{\sqrt{\pi}\,2^n n!}=\frac{1}{\sqrt{\pi(1-r^2)}}\exp\left(\frac{-r^2x^2+2rxz-r^2z^2}{1-r^2}\right),$$

where H_n is the nth Hermite polynomial. Some approximation properties of these operators are given in [2].

Let us consider the operator $U(f)$:

$$\begin{aligned} U(f)\quad (r,y_1,y_2) &= U(f;r,y_1,y_2)\\ &= \int_0^\infty\int_0^\infty K(r,y_1,z_1)\,K(r,y_2,z_2)\,f(z_1,z_2)\,(z_1z_2)^\alpha \exp(-z_1-z_2)\,dz_1\,dz_2, \end{aligned}$$

where $f\in L^p\left(\mathbb{R}^2_+;(z_1z_2)^\alpha\exp(-z_1-z_2)\right),\ 1\le p\le\infty,\ \alpha>-1.$

This paper contains some properties of the above operator and of operator $W(f)$ defined by

$$\begin{aligned} W(f)\quad (r,y_1,y_2) &= W(f;r,y_1,y_2)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(r,y_1,z_1)P(r,y_2,z_2)f(z_1,z_2)\exp\left(-z_1^2-z_2^2\right)\,dz_1\,dz_2, \end{aligned}$$

where $f\in L^p\left(\mathbb{R}^2;\exp(-z_1^2-z_2^2)\right)$, $0<r<1$. The norm of a function f in $L^p\left(X^2;w(z_1,z_2)\right)$ is given by

$$\|f\|_p=\begin{cases}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(t_1,t_2)|^p\,w(t_1,t_2)\,dt_1\,dt_2\right)^{\frac{1}{p}}, & 1\le p<\infty,\\ \sup\limits_{(t_1,t_2)\in\mathbb{R}^2}\text{ess}\,|f(t_1,t_2)|, & p=\infty,\end{cases}$$

where $X=\mathbb{R}_+$ and $w(z_1,z_2)=(z_1z_2)^\alpha\exp(-z_1-z_2)$, or $X=\mathbb{R}$ and $w(z_1,z_2)=\exp(-z_1^2-z_2^2)$, respectively.

We state some estimates of the rate of convergence of the integrals $U(f)$ and $W(f)$ using the classical moduli of continuity.

2. Auxiliary results

In this section we shall give some properties of the above operators which we shall apply to the proofs of the main theorems. We begin this section by recalling the following results of Toczek and Wachnicki [2].

Let $\varphi_{n,y}(z)=(z-y)^n,\ n\in\mathbb{N}=\{1,2,...\},\ y,z\in X$, where $X=\mathbb{R}_+$ or $X=\mathbb{R}$, respectively.

Lemma 1. *For each $y\in\mathbb{R}_+$ we have*

$$\begin{aligned}
A(1;r,y) &= 1,\\
A(\varphi_{1,y};r,y) &= (1-r)(1+\alpha-y),\\
A(\varphi_{2,y};r,y) &= (1-r)\left\{y^2(1-r)+2(\alpha+2)ry-2(\alpha+1)y\right.\\
&\quad \left.+(\alpha+2)(\alpha+1)(1-r)\right\},\\
A(\varphi_{4,y};r,y) &= (1-r)^2\left\{y^4(r-1)^2-4\alpha(r-1)^2y^3-4(4r^2-5r+1)\right.\\
&\quad +6(\alpha+4)(\alpha+3)r^2y^2-12(\alpha+3)(\alpha+2)ry^2\\
&\quad +6(\alpha+2)(\alpha+1)y^2-12(\alpha+3)(\alpha+2)(r-1)y\\
&\quad \left.+(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)(r-1)^2\right\}.
\end{aligned}$$

Lemma 2. *For each $y\in\mathbb{R}$*

$$\begin{aligned}
B(1;r,y) &= 1,\\
B(\varphi_{1,y};r,y) &= -y(1-r),\\
B(\varphi_{2,y};r,y) &= (1-r)\left\{y^2(1-r)+\frac{1}{2}(r+1)\right\},\\
B(\varphi_{4,y};r,y) &= (1-r)^2\left\{(r-1)^2y^4-3(r^2-1)y^2+\frac{3}{4}(r+1)^2\right\}
\end{aligned}$$

holds.

From the definitions of U and W we easily obtain:

Lemma 3. *If $f_1,f_2\in L^p\left(\mathbb{R}_+;z^\alpha\exp(-z)\right)$, $1\le p\le\infty$, $\alpha>-1$, then*

$$U(f;r,y_1,y_2)=A(f_1;r,y_1)\,A(f_2;r,y_2)$$

for $(y_1,y_2)\in\mathbb{R}^2_+$, $0<r<1$, where $f(z_1,z_2)=f_1(z_1)f_2(z_2)$, $z_1,z_2\in\mathbb{R}_+$.

Lemma 4. *If $f_1,f_2\in L^p\left(\mathbb{R};\exp(-z^2)\right)$, $1\le p\le\infty$, then*

$$W(f;r,y_1,y_2)=B(f_1;r,y_1)\,B(f_2;r,y_2)$$

for $(y_1,y_2)\in\mathbb{R}^2$, $0<r<1$, where $f(z_1,z_2)=f_1(z_1)f_2(z_2)$, $z_1,z_2\in\mathbb{R}$.

Applying Lemmas 1 and 2, it is easy to prove the following two lemmas.

Lemma 5. *For every* $(y_1, y_2) \in \mathbb{R}^2_+$ *it follows that*

$$\begin{aligned}
U(1; r, y_1, y_2) &= 1, \\
U(\varphi_{1,y_i}; r, y_1, y_2) &= (1-r)(1+\alpha-y_i), \\
U(\varphi_{2,y_i}; r, y_1, y_2) &= (1-r)\left\{y_i^2(1-r) + 2(\alpha+2)ry_i - 2(\alpha+1)y_i \right. \\
&\quad \left. + (\alpha+2)(\alpha+1)(1-r)\right\}, \\
U(\varphi_{4,y_i}; r, y_1, y_2) &= (1-r)^2\left\{y_i^4(r-1)^2 - 4\alpha(r-1)^2y_i^3 - 4(4r^2-5r+1) \right. \\
&\quad + 6(\alpha+4)(\alpha+3)r^2y_i^2 - 12(\alpha+3)(\alpha+2)ry_i^2 \\
&\quad + 6(\alpha+2)(\alpha+1)y_i^2 - 12(\alpha+3)(\alpha+2)(r-1)y_i \\
&\quad \left. + (\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)(r-1)^2\right\}
\end{aligned}$$

for $0 < r < 1$, $i = 1, 2$.

Lemma 6. *For every* $(y_1, y_2) \in \mathbb{R}^2$ *it follows that*

$$\begin{aligned}
W(1; r, y_1, y_2) &= 1, \\
W(\varphi_{1,y_i}; r, y_1, y_2) &= -y_i(1-r), \\
W(\varphi_{2,y_i}; r, y_1, y_2) &= (1-r)\left\{y_i^2(1-r) + \frac{1}{2}(r+1)\right\}, \\
W(\varphi_{4,y_1} + \varphi_{4,y_2}; r, y_1, y_2) &= (1-r)^2\left\{(r-1)^2(y_1^4 + y_2^4) \right. \\
&\quad \left. -3(r^2-1)(y_1^2+y_2^2) + \frac{3}{2}(r+1)^2\right\}
\end{aligned}$$

for $0 < r < 1$, $i = 1, 2$.

Using the Hölder inequality and Lemma 5, we obtain

Lemma 7. *For* $(y_1, y_2) \in \mathbb{R}^2_+$ *and* $0 < r < 1$ *we have*

$$\begin{aligned}
U(|\varphi_{1,y_i}|; r, y_1, y_2) &\le (1-r)^{\frac{1}{2}}\left\{y_i^2(1-r) + 2(\alpha+2)ry_i - 2(\alpha+1)y_i \right. \\
&\quad \left. + (\alpha+2)(\alpha+1)(1-r)\right\}^{\frac{1}{2}}, \quad i = 1, 2.
\end{aligned}$$

Similarly we get

Lemma 8. *For* $0 < r < 1$ *and* $(y_1, y_2) \in \mathbb{R}^2$ *we have*

$$W(|\varphi_{1,y_i}|; r, y_1, y_2) \le (1-r)^{\frac{1}{2}}\left\{y_i^2(1-r) + \frac{1}{2}(r+1)\right\}^{\frac{1}{2}}, \quad i = 1, 2.$$

Applying the Riesz-Thorin theorem, it is easy to prove

Lemma 9. *Let* $f \in L^p\left(\mathbb{R}_+^2; (z_1z_2)^\alpha \exp(-z_1-z_2)\right)$, *where* $\alpha > -1$ *and* $1 \le p \le \infty$. *Then* $U(f;r,\cdot,\cdot) \in L^p\left(\mathbb{R}_+^2; (z_1z_2)^\alpha \exp(-z_1-z_2)\right)$ *and*

$$\|U(f;r,\cdot,\cdot)\|_p \quad \le \quad \|f\|_p \tag{1}$$

for $0 < r < 1$.

In a similar fashion we can prove the following theorem for operator W.

Lemma 10. *Let* $f \in L^p\left(\mathbb{R}^2; \exp(-z_1^2-z_2^2)\right)$, *where* $1 \le p \le \infty$. *Then* $W(f;r,\cdot,\cdot) \in L^p\left(\mathbb{R}^2; \exp(-z_1^2-z_2^2)\right)$ *and* $\|W(f;r,\cdot,\cdot)\|_p \le \|f\|_p$ *for* $0 < r < 1$.

3. The rate of convergence

In this section we present some estimates of the rate of convergence of the integrals U and W. We state this estimates using the classical modulus of continuity defined by

$$\omega(f;\delta_1,\delta_2) = \sup_{\substack{0<h_1\le\delta_1\\0<h_2\le\delta_2}} \left\{ \sup_{(y_1,y_2)\in X^2} |f(y_1+h_1,y_2+h_2)-f(y_1,y_2)| \right\},$$

$\delta_1, \delta_2 > 0$, where $X = \mathbb{R}_+$ or $X = \mathbb{R}$, respectively.

Theorem 1. *Let* $f \in C(\mathbb{R}_+^2) \cap L^p\left(\mathbb{R}_+^2; (z_1z_2)^\alpha \exp(-z_1-z_2)\right)$, $1 \le p \le \infty$ *and* $\alpha > -1$. *Then*

$$|U(f;r,y_1,y_2) - f(y_1,y_2)| \le 6\omega\left(f;\delta_1,\delta_2\right)$$

for $0 < r < 1$ *and* $(y_1,y_2) \in \mathbb{R}_+^2$, *where*

$$\begin{aligned}\delta_i \;&=\; (1-r)^{\frac{1}{2}}\left\{y_i^2(1-r) + 2(\alpha+2)ry_i\right.\\ &\qquad \left.-2(\alpha+1)y_i + (\alpha+2)(\alpha+1)(1-r)\right\}^{\frac{1}{2}},\ i=1,2.\end{aligned}$$

Proof. First we suppose that $f \in C^1(\mathbb{R}_+^2) \cap L^p\left(\mathbb{R}_+^2; (z_1z_2)^\alpha \exp(-z_1-z_2)\right)$, $1 \le p \le \infty$, $\alpha > -1$. Let $f(z_1,z_2) - f(y_1,y_2) = \lambda_{y_1}(z_1,z_2) + \tau_{y_2}(z_1,z_2)$ for every $(z_1,z_2) \in \mathbb{R}_+^2$, where

$$\lambda_{y_1}(z_1,z_2) = \int_{y_1}^{z_1} \frac{\partial}{\partial u} f(u,z_2)\,du, \qquad \tau_{y_2}(z_1,z_2) = \int_{y_2}^{z_2} \frac{\partial}{\partial v} f(y_1,v)\,dv.$$

Observe that

$$|\lambda_{y_1}(z_1,z_2)| = \left|\int_{y_1}^{z_1} \frac{\partial}{\partial u} f(u,z_2)\,du\right| \leq \left|\int_{y_1}^{z_1} du\right| \cdot \sup_{(y_1,y_2)\in\mathbb{R}^2_+} \left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right|$$
$$= |z_1 - y_1| \cdot \sup_{(y_1,y_2)\in\mathbb{R}^2_+} \left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right|$$

and

$$|\tau_{y_2}(z_1,z_2)| \leq |z_2 - y_2| \cdot \sup_{(y_1,y_2)\in\mathbb{R}^2_+} \left|\frac{\partial f(y_1,y_2)}{\partial y_2}\right|.$$

Using Lemma 7, we get

$$U\left(|\lambda_{y_1}|; r, y_1, y_2\right) \leq U\left(|\varphi_{1,y_1}|; r, y_1, y_2\right) \sup_{(y_1,y_2)\in\mathbb{R}^2_+} \left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right|$$
$$\leq (1-r)^{\frac{1}{2}} \left\{y_1^2(1-r) + 2(\alpha+2)ry_1 - 2(\alpha+1)y_1\right.$$
$$\left.+(\alpha+2)(\alpha+1)(1-r)\right\}^{\frac{1}{2}} \sup_{(y_1,y_2)\in\mathbb{R}^2_+} \left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right|,$$

$$U\left(|\tau_{y_2}|; r, y_1, y_2\right) \leq (1-r)^{\frac{1}{2}} \left\{y_2^2(1-r) + 2(\alpha+2)ry_2 - 2(\alpha+1)y_2\right.$$
$$\left.+(\alpha+2)(\alpha+1)(1-r)\right\}^{\frac{1}{2}} \sup_{(y_1,y_2)\in\mathbb{R}^2_+} \left|\frac{\partial f(y_1,y_2)}{\partial y_2}\right|.$$

Hence we have

$$|U\left(f; r, y_1, y_2\right) - f(y_1,y_2)| \leq U\left(|\lambda_{y_1}|; r, y_1, y_2\right) + U\left(|\tau_{y_2}|; r, y_1, y_2\right)$$
$$\leq (1-r)^{\frac{1}{2}} \left\{y_1^2(1-r) + 2(\alpha+2)ry_1 - 2(\alpha+1)y_1\right.$$
$$\left.+ (\alpha+2)(\alpha+1)(1-r)\right\}^{\frac{1}{2}} \sup_{(y_1,y_2)\in\mathbb{R}^2_+} \left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right|$$
$$+ (1-r)^{\frac{1}{2}} \left\{y_2^2(1-r) + 2(\alpha+2)ry_2 - 2(\alpha+1)y_2\right.$$
$$\left.+ (\alpha+2)(\alpha+1)(1-r)\right\}^{\frac{1}{2}} \sup_{(y_1,y_2)\in\mathbb{R}^2_+} \left|\frac{\partial f(y_1,y_2)}{\partial y_2}\right|.$$

Let $f \in C(\mathbb{R}^2_+) \cap L^p\left(\mathbb{R}^2_+; (z_1 z_2)^\alpha \exp(-z_1 - z_2)\right)$, $1 \leq p \leq \infty$, $\alpha > -1$ and let f_{δ_1,δ_2} be the Steklov function of a function f of two variables given by the formula

$$f_{\delta_1,\delta_2}(y_1,y_2) = \frac{1}{\delta_1\delta_2}\int_0^{\delta_1}\int_0^{\delta_2} f(y_1+u,y_2+v)\,du\,dv \text{ for } (y_1,y_2)\in\mathbb{R}_+^2,\ \delta_1,\delta_2>0.$$

If $f\in C(\mathbb{R}_+^2)\cap L^p\left(\mathbb{R}_+^2;(z_1z_2)^\alpha\exp(-z_1-z_2)\right)$, then

$$f_{\delta_1,\delta_2}\in C^1(\mathbb{R}_+^2)\cap L^p\left(\mathbb{R}_+^2;(z_1z_2)^\alpha\exp(-z_1-z_2)\right)$$

for fixed $\delta_1,\delta_2>0$ and

$$\sup_{(y_1,y_2)\in\mathbb{R}_+^2}|f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)|\le\omega(f;\delta_1,\delta_2),$$

$$\sup_{(y_1,y_2)\in\mathbb{R}_+^2}\left|\frac{\partial}{\partial y_1}f_{\delta_1,\delta_2}(y_1,y_2)\right|\le 2\delta_1^{-1}\omega(f;\delta_1,\delta_2),$$

$$\sup_{(y_1,y_2)\in\mathbb{R}_+^2}\left|\frac{\partial}{\partial y_2}f_{\delta_1,\delta_2}(y_1,y_2)\right|\le 2\delta_2^{-1}\omega(f;\delta_1,\delta_2)$$

for all $\delta_1,\delta_2>0$.

Moreover, for $f\in L^p\left(\mathbb{R}_+^2;(z_1z_2)^\alpha\exp(-z_1-z_2)\right)$ we can write

$$\begin{aligned}
&|U(f;r,y_1,y_2)-f(y_1,y_2)|\\
&\le |U(f-f_{\delta_1,\delta_2};r,y_1,y_2)|+|U(f_{\delta_1,\delta_2};r,y_1,y_2)-f_{\delta_1,\delta_2}(y_1,y_2)| \qquad (2)\\
&\quad+|f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)|,
\end{aligned}$$

where $(y_1,y_2)\in\mathbb{R}_+^2$, $\delta_1,\delta_2>0$.

Now, from the first part of this proof we have

$$\begin{aligned}
&|U(f_{\delta_1,\delta_2};r,y_1,y_2)-f_{\delta_1,\delta_2}(y_1,y_2)|\\
&\le 2\omega(f;\delta_1,\delta_2)\Big\{\delta_1^{-1}(1-r)^{\frac12}\Big[y_1^2(1-r)+2(\alpha+2)ry_1-2(\alpha+1)y_1\\
&\quad+(\alpha+2)(\alpha+1)(1-r)\Big]^{\frac12}+\delta_2^{-1}(1-r)^{\frac12}\big[y_2^2(1-r)+2(\alpha+2)ry_2\\
&\quad-2(\alpha+1)y_2+(\alpha+2)(\alpha+1)(1-r)\Big]^{\frac12}\Big\}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&|U(f-f_{\delta_1,\delta_2};r,y_1,y_2)|\le\\
&\int_0^\infty\int_0^\infty K(r,y_1,z_1)\,K(r,y_2,z_2)\,(z_1z_2)^\alpha\exp(-z_1-z_2)\,dz_1\,dz_2\\
&\times\sup_{(y_1,y_2)\in\mathbb{R}_+^2}|f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)|\le U(1;r,y_1,y_2)\,\omega(f;\delta_1,\delta_2)\le\omega(f;\delta_1,\delta_2)
\end{aligned}$$

and

$$|f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)|\le\omega(f;\delta_1,\delta_2).$$

From (2) we have

$$\begin{aligned}&|U(f;r,y_1,y_2)-f(y_1,y_2)|\\ &\le \Big\{1+\delta_1^{-1}(1-r)^{\frac12}\left[y_1^2(1-r)+2(\alpha+2)ry_1-2(\alpha+1)y_1\right.\\ &\quad\left.+(\alpha+2)(\alpha+1)(1-r)\right]^{\frac12}+\delta_2^{-1}(1-r)^{\frac12}\left[y_2^2(1-r)\right.\\ &\quad\left.+2(\alpha+2)ry_2-2(\alpha+1)y_2+(\alpha+2)(\alpha+1)(1-r)\right]^{\frac12}\Big\}\,2\omega(f;\delta_1,\delta_2)\end{aligned}$$

for $0<r<1$, $\delta_1,\delta_2>0$ and all $(y_1,y_2)\in\mathbb{R}_+^2$. Setting

$$\delta_1=(1-r)^{\frac12}\left(y_1^2(1-r)+2(\alpha+2)ry_1-2(\alpha+1)y_1+(\alpha+2)(\alpha+1)(1-r)\right)^{\frac12},$$

$$\delta_2=(1-r)^{\frac12}\left(y_2^2(1-r)+2(\alpha+2)ry_2-2(\alpha+1)y_2+(\alpha+2)(\alpha+1)(1-r)\right)^{\frac12}$$

for fixed $(y_1,y_2)\in\mathbb{R}_+^2$, we get the required inequality.

In a similar fashion we obtain:

Theorem 2. *Let* $f\in C(\mathbb{R}^2)\cap L^p\left(\mathbb{R}^2;\exp(-z_1^2-z_2^2)\right)$, $1\le p\le\infty$. *Then*

$$|W(f;r,y_1,y_2)-f(y_1,y_2)|\le 6\omega\left(f;\delta_1,\delta_2\right)$$

for $0<r<1$ *and* $(y_1,y_2)\in\mathbb{R}^2$, *where*

$$\delta_i \;=\; (1-r)\left\{y_i^2(1-r)+\frac{1}{2}(r+1)\right\},\ i=1,2.$$

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