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ON APPROXIMATION OF SUBQUADRATIC FUNCTIONS BY QUADRATIC FUNCTIONS

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Abstract. In this paper we establish an approximation of subquadratic functions, which satisfy the condition

$$\exists \ \epsilon > 0 \quad \forall \ x \in X \quad |\varphi(2x) - 4\varphi(x)| \leq 3\epsilon,$$

by quadratic functions.

1. Introduction

Let X be a group and let $\mathbb{R}$ denotes the set of all reals. A function $\varphi\colon X \to \mathbb{R}$ is said to be subquadratic iff it satisfies the inequality

$$\varphi(x+y) + \varphi(x-y) \leq 2\varphi(x) + 2\varphi(y) \tag{1}$$

for all $x, y \in X$. If the sign " $\leq$ " in the inequality above is replaced by " $=$ ", then we say that φ is a quadratic one.

Section 2 of this paper contains some basic properties of subquadratic functions which play a crucial role in our proofs of the main theorems of this paper.

At the beginning of the third part of this paper, we will consider the problem of approximation of a subquadratic function $\varphi\colon X \to \mathbb{R}$, which satisfies the following condition

$$\exists \ \epsilon > 0 \quad \forall \ x \in X \quad |\varphi(2x) - 4\varphi(x)| \leq 3\epsilon, \tag{2}$$

by a quadratic function $\omega\colon X \to \mathbb{R}$.

At the end of this section, we will present some conditions for subquadratic functions defined on a topological group having additional properties or on $\mathbb{R}^N$, under which we will establish an approximation of functions of this type by continuous quadratic functions.

2. Some basic properties

At the beginning of this section we remind one of basic properties of subquadratic functions which is proved in [1], [2].

Lemma 1. [1], [2] *Let $X = (X, +)$ be a group and let $\varphi : X \to \mathbb{R}$ be a subquadratic function. Then*

$$\varphi(0) \geq 0$$

and

$$\varphi(kx) \leq k^2\varphi(x), \qquad x \in X,$$

for each positive integer k.

In [2] it was proved that if for some positive integer $k > 1$ a subquadratic function $\varphi : X \to \mathbb{R}$ satisfies equality

$$\varphi(kx) = k^2\varphi(x), \quad x \in X, \tag{3}$$

in the case when the domain of the function considered is a linear space, then it has to be a quadratic one. Essentially, the same argumentation yields the validity of the next lemma if the domain is a group and the condition (3) is replaced by another.

Lemma 2. *Let $X = (X, +)$ be a group and let $\varphi : X \to \mathbb{R}$ be a subquadratic function. If for every $x \in X$ there exists some positive integer $k > 1$ such that*

$$\varphi(k_x x) \geq {k_x}^2\varphi(x),$$

then $\varphi(2x) \geq 4\varphi(x)$ for every $x \in X$.

Lemma 3. *Let $X = (X, +)$ be an Abelian group. If a subquadratic function $\varphi\colon X \to \mathbb{R}$ satisfies the condition*

$$\varphi(2x) \geq 4\varphi(x)$$

for all $x \in X$, then it is a quadratic function.

Proof. Let $x, y \in X$. Then

$$\varphi(x+y)+\varphi(x-y) \leq 2\varphi(x)+2\varphi(y)=\frac{1}{2}\left[4\varphi(x)+4\varphi(y)\right] \leq$$

$$\leq \frac{1}{2}\left[\varphi(2x)+\varphi(2y)\right]=\frac{1}{2}\left[\varphi\left((x+y)+(x-y)\right)+\varphi\left((x+y)-(x-y)\right)\right] \leq$$

$$\leq \varphi(x+y)+\varphi(x-y).$$

Thus φ is a quadratic function. □

Now, using Lemmas 2 and 3, the Theorem 1 from [2] has the following form:

Theorem 1. *Let $X=(X,+)$ be an Abelian group and let $\varphi : X \to \mathbb{R}$ be a subquadratic function. If for every $x \in X$ there exists some positive integer $k > 1$ such that*

$$\varphi(k_x x) \geq {k_x}^2 \varphi(x),$$

then φ is a quadratic function.

By a topological group we mean a group endowed with a topology such that the group operation as well as taking inverses are continuous functions.

We adopt the following definition.

Definition 1. *We say that 2-divisible topological group X has the property $(\frac{1}{2})$ if and only if for every neighbourhood V of zero there exists a neighbourhood W of zero such that $\frac{1}{2}W \subset W \subset V$.*

At the end of this section we present Theorem 2 which was proved in [3].

Theorem 2. [3] *Let X be a uniquely 2-divisible topological Abelian group having the property $(\frac{1}{2})$, which is generated by any neighbourhood of zero in X. Assume that a subquadratic function $\varphi : X \to \mathbb{R}$ satisfies the following conditions:*

(i) $\varphi(0) \leq 0$;

(ii) *φ is locally bounded from below at a point of X;*

(iii) *φ is upper semicontinuous at zero.*

Then φ is continuous everywhere in X.

3. The main result

The next lemma plays a crucial role in our proofs. This lemma is valid for an arbitrary function defined on a semigroup with values in a normed space.

Lemma 4. *Let X be an arbitrary semigroup, Y a normed space and let $f\colon X \to Y$ be an arbitrary function. If there exists $\epsilon > 0$ such that the inequality*

$$\|f(2x) - 4f(x)\| \le 3\epsilon \tag{4}$$

holds for every $x \in X$, then the inequality

$$\|4^{-n} f(2^n x) - f(x)\| \le (1 - 4^{-n})\epsilon \tag{5}$$

holds for every $x \in X$ and $n \in \mathbb{N}$.

Proof. Let $x \in X$. By induction, we show that for every $n \in \mathbb{N}$ the inequality (5) holds. For $n = 1$, by (4), we have

$$4\left\|\frac{1}{4}f(2x) - f(x)\right\| \le 3\epsilon.$$

Thus

$$\left\|\frac{1}{4}f(2x) - f(x)\right\| \le \frac{3}{4}\epsilon = \left(1 - \frac{1}{4}\right)\epsilon.$$

Now assume that (5) holds for $n \in \mathbb{N}$. Then we have

$$\left\|4^{-n-1} f(2^{n+1}x) - f(x)\right\| \le$$

$$\le \left\|\frac{1}{4^{n+1}} f(2^{n+1}x) - \frac{1}{4^n} f(2^n x)\right\| + \left\|\frac{1}{4^n} f(2^n x) - f(x)\right\| =$$

$$= \frac{1}{4^n}\left\|\frac{1}{4} f(2(2^n x)) - f(2^n x)\right\| + \left\|\frac{1}{4^n} f(2^n x) - f(x)\right\| \le$$

$$\le \frac{1}{4^n}\frac{3}{4}\epsilon + \left(1 - \frac{1}{4^n}\right)\epsilon = \left(1 - \frac{1}{4^{n+1}}\right)\epsilon$$

for $n + 1$, which completes the induction. □

Applying Lemma 4, we will prove the following theorem.

Theorem 3. *Let X be an arbitrary Abelian group and let $\varphi\colon X \to \mathbb{R}$ be a subquadratic function. If there exists $\epsilon > 0$ such that the inequality*

$$|\varphi(2x) - 4\varphi(x)| \leq 3\epsilon \tag{6}$$

holds for every $x \in X$, then there exists a quadratic function $\omega\colon X \to \mathbb{R}$ such that

$$0 \leq \varphi(x) - \omega(x) \leq \epsilon$$

for every $x \in X$.

Proof. It follows by Lemma 1 that for arbitrary $x \in X$ and $n \in \mathbb{N}$ we have

$$\varphi(2^n x) \leq 4^n \varphi(x). \tag{7}$$

Whence

$$\frac{\varphi(2^n x)}{4^n} \leq \varphi(x), \qquad x \in X, \quad n \in \mathbb{N}. \tag{8}$$

Let us fix $x \in X$. We will consider the sequence $\left\{\frac{\varphi(2^n x)}{4^n}\right\}_{n\in\mathbb{N}}$. For arbitrary $m, n \in \mathbb{N}$, by Lemma 4, we have

$$\left|\frac{1}{4^{n+m}}\varphi(2^{n+m}x) - \frac{1}{4^n}\varphi(2^n x)\right| = \frac{1}{4^n}\left|\frac{1}{4^m}\varphi(2^{n+m}x) - \varphi(2^n x)\right| =$$

$$= \frac{1}{4^n}\left|\frac{1}{4^m}\varphi(2^m 2^n x) - \varphi(2^n x)\right| \leq \frac{1}{4^n}\left(1 - \frac{1}{4^m}\right)\epsilon < \frac{1}{4^n}\epsilon,$$

which means that for every $x \in X$ the sequence $\left\{\frac{\varphi(2^n x)}{4^n}\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence and consequently converges. Let

$$\omega(x) := \lim_{n\to\infty} \frac{\varphi(2^n x)}{4^n}, \qquad x \in X.$$

On letting $n \to \infty$ in (8), we obtain

$$\omega(x) \leq \varphi(x), \quad x \in X. \tag{9}$$

Since $\omega(2x) = 4\omega(x)$ for every $x \in X$, by Theorem 1, ω is a quadratic function.

Again, by Lemma 4, we have

$$\left|\frac{\varphi(2^n x)}{4^n} - \varphi(x)\right| \leq \left(1 - \frac{1}{4^n}\right)\epsilon, \quad x \in X. \tag{10}$$

Whence, on letting $n \to \infty$, we obtain

$$|\omega(x) - \varphi(x)| \leq \epsilon, \quad x \in X. \tag{11}$$

By (9) and (11), we have

$$0 \leq \varphi(x) - \omega(x) \leq \epsilon$$

for every $x \in X$. This ends the proof. □

As a consequence of Theorem 3, we have the following corollary.

Corollary 1. *Let X be a uniquely 2-divisible Abelian topological group having the property $\left(\frac{1}{2}\right)$, which is generated by any neighbourhood of zero in X. Assume that a subquadratic function $\varphi\colon X \to \mathbb{R}$ satisfies the following conditions:*

(i) $\exists\ \ \epsilon > 0\ \ \forall\ \ x \in X\ \ |\varphi(2x) - 4\varphi(x)| \leq 3\epsilon$;

(ii) φ is upper semicontinuous at zero in X;

(iii) φ is locally bounded from below at a point of X.

Then the function ω, which appears in thesis of Theorem 3, is continuous.

Proof. Due to the upper semicontinuoity of φ at zero, the function ω is also upper semicontinuous at zero [5, p. 131].

If φ is locally bounded from below at some point $x_0 \in X$, then by the inequality

$$\varphi(x) - \epsilon \leq \omega(x) \leq \varphi(x) + \epsilon, \qquad x \in X,$$

the function ω is also locally bounded from below at the point $x_0 \in X$. Since the quadratic function ω satisfies the condition $\omega(0) = 0$, then according to Theorem 2, it is continuous. This completes the proof. □

Now, let $X = \mathbb{R}^N$. In this case we have the following theorem.

Theorem 4. *Let $A \subset \mathbb{R}^N$ be a set of positive inner Lebesgue measure or of the second category with the Baire property. If a subquadratic function $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:*

(i) $\exists\ \ \epsilon > 0\ \ \forall\ \ x \in X\ \ |\varphi(2x) - 4\varphi(x)| \leq 3\epsilon$;

(ii) φ is bounded on A,

then there exists a continuous quadratic function $\omega\colon \mathbb{R}^N \to \mathbb{R}$ such that

$$0 \leq \varphi(x) - \omega(x) \leq \epsilon$$

for every $x \in \mathbb{R}^N$.

Proof. Due to Theorem 3, there exists a quadratic function $\omega\colon \mathbb{R}^N \to \mathbb{R}$ such that

$$0 \le \varphi(x) - \omega(x) \le \epsilon, \quad x \in \mathbb{R}^N. \tag{12}$$

Therefore, we have

$$\varphi(x) - \epsilon \le \omega(x) \le \varphi(x), \quad x \in \mathbb{R}^N. \tag{13}$$

Since φ is bounded on the set A, there exist real constants m, M such that

$$m \le \varphi(x) \le M, \quad x \in A. \tag{14}$$

Inequalities (13) and (14) imply

$$m - \epsilon \le \omega(x) \le M, \quad x \in A. \tag{15}$$

Finally, ω is a bounded function on the set A. A quadratic function is a polynomial function of degree 2. Due to Theorem 3 from [4, p. 386], ω is continuous in $\mathbb{R}^N$. □

We end our paper with the following corollary.

Corollary 2. *Let $A \subset \mathbb{R}$ be a set of positive inner Lebesgue measure or of the second category with the Baire property. If a subquadratic function $\varphi\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:*

(i) $\exists \;\; \epsilon > 0 \;\; \forall \;\; x \in X \;\; |\varphi(2x) - 4\varphi(x)| \le 3\epsilon$;

(ii) φ *is bounded on* A,

then

$$\varphi(x) \ge cx^2$$

for every $x \in \mathbb{R}$, where c is a real constant .

Proof. Due to Theorem 4, there exists a continuous quadratic function $\omega\colon \mathbb{R} \to \mathbb{R}$ such that

$$\omega(x) \le \varphi(x), \quad x \in \mathbb{R}. \tag{16}$$

Since $\omega\colon \mathbb{R} \to \mathbb{R}$ is a continuous quadratic function, then it takes the form

$$\omega(x) = cx^2, \quad x \in \mathbb{R}, \tag{17}$$

where $c \in \mathbb{R}$ is a constant. By (16) and (17), we obtain the thesis of the Corollary 2. □

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