SOLUTIONS OF THE DHOMBRES-TYPE TRIGONOMETRIC FUNCTIONAL EQUATION

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Abstract. Let $(G, +)$ be a uniquely 2-divisible Abelian group. In the present paper we will consider the solutions of functional equation
$[f(x + y)]^2 - [f(x - y)]^2 + f(2x + 2y) + f(2x - 2y) = f(2x)[f(2y) + 2g(2y)]$, $x, y \in G$,
where f and g are complex-valued functions defined on G.

1. Introduction

We know many trigonometric identities. To us, important will be the following:

$$\left[\sin\left(\frac{x + y}{2}\right)\right]^2 - \left[\sin\left(\frac{x - y}{2}\right)\right]^2 = \sin(x)\sin(y), \quad x, y \in \mathbb{R}, \qquad (1)$$

$$\sin(x + y) + \sin(x - y) = 2\sin(x)\cos(y), \quad x, y \in \mathbb{R}, \qquad (2)$$

$$\sinh(x - y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y), \quad x, y \in \mathbb{R}. \qquad (3)$$

Let $(G, +)$ be a uniquely 2-divisible Abelian group and $f, g : G \to \mathbb{C}$. Equation (1) translates into the well known *sine functional equation* [1, 8]

$$\left[f\left(\frac{x + y}{2}\right)\right]^2 - \left[f\left(\frac{x - y}{2}\right)\right]^2 = f(x)f(y) \quad \text{for all} \quad x, y \in G, \qquad (4)$$

and (2) gives rise to the familiar *cosine functional equation* [1, 6, 7]

$$f(x+y) + f(x-y) = 2f(x)g(y) \quad \text{for all} \quad x, y \in G, \tag{5}$$

and (3) leads to the *Aczel-Dhombres functional equation* [1]

$$f(x-y) = f(x)g(y) - g(x)f(y) \quad \text{for all} \quad x, y \in G. \tag{6}$$

From now on, f_o and f_e stand for the odd and the even part of a function f.

Theorem 1 (Aczél and Dhombres [1]). *Let $(G, +)$ be a uniquely 2-divisible Abelian group. Then $f, g : G \to \mathbb{C}$ satisfy equation (6) if and only if*

(i) *$f = 0$ and g is arbitrary; or*

(ii) *there exists an additive function $A : G \to \mathbb{C}$ and a constant $\alpha \in \mathbb{C}$ such that $f(x) = A(x)$, $g(x) = \alpha A(x) + 1, x \in G$; or*

(iii) *there exists an exponential function $m : G \to \mathbb{C}$ and constants $\beta, \gamma \in \mathbb{C}$ such that $f(x) = \beta m_o(x)$, $g(x) = \gamma m_o(x) + m_e(x)$, $x \in G$.*

From the system of equations

$$\begin{cases} f(x+y) = f(x) + f(y), \\ f(xy) = f(x)f(y), \end{cases}$$

we get *the Dhombres functional equation* (see [2])

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y)$$

for functions f mapping a given ring into another one. A different system of the functional equations has been studied by Ger [3, 4, 5]. Here we consider the sum of equations (4) and (5).

2. Main results

We replace x by $2x$ and y by $2y$ in (4) and (5). Summing up these functional equations side by side, for all $x, y \in G$, we get

$$[f(x+y)]^2 - [f(x-y)]^2 + f(2x+2y) + f(2x-2y) = f(2x)[f(2y) + 2g(2y)]. \tag{7}$$

Remark 1. *Put $x = y = 0$ in (7), so that we have $f(0) = 0 \vee f(0) = 2 - 2g(0)$.*

Lemma 1. *Let $(G, +)$ be a uniquely 2-divisible Abelian group and let functions $f, g : G \to \mathbb{C}$ satisfy equation (7). In this case*

(i) *if $f = 0$, then g is arbitrary;*

(ii) *if $g = 0$, then $f = 0$ or $f = 2$.*

Proof. Ad (ii). For $g = 0$, putting $y = x$ in (7), we get

$$f(x) = [f(0)]^2 - f(0) = f(0)[f(0) - 1] = \gamma, \quad x \in G. \tag{8}$$

From equation (7) we obtain

$$\gamma^2 - \gamma^2 + 2\gamma = \gamma^2,$$

whence $\gamma = 0 \vee \gamma = 2$. By (8), we conclude that $f = 0 \vee f = 2$. $\square$

Lemma 2. *Let $(G, +)$ be a uniquely 2-divisible Abelian group and let nonzero functions $f, g : G \to \mathbb{C}$ be functions defined by*

$$f(x) = aA(x) + f(0), \ g(x) = bA(x) + g(0), \quad x \in G, \tag{9}$$

with some additive function $A : G \to \mathbb{C}$ and $a, b \in \mathbb{C}$ satisfy equation (7). In this case we have the following possibilities:
(i) *If $f(0) = 0$, then $f(x) = aA(x)$, $g(x) = 1$, $x \in G$.*
(ii) *If $f(0) \neq 0$, then $f(x) = f(0)$, $g(x) = 1 - \frac{1}{2}f(0)$, $x \in G$.*

Proof. Applying (9) to (7), we have

$$[aA(x+y) + f(0)]^2 - [aA(x-y) + f(0)]^2 + aA(2x+2y) + 2f(0) + aA(2x-2y)$$

$$= [aA(2x) + f(0)][(a+2b)A(2y) + f(0) + 2g(0)], \ x, y \in G.$$

From the properties of additive function A for all $x, y \in G$, we infer that

$$[2 - f(0) - 2g(0)][2aA(x) + f(0)] + 2f(0)[a - 2b]A(y) = 8abA(x)A(y). \tag{10}$$

Case 1. Assume that $f(0) = 0$. Then equation (10) has a form

$$aA(x)[1 - g(0) - 2bA(y)] = 0, \quad x, y \in G. \tag{11}$$

If $a = 0 \vee A = 0 \vee (a \neq 0 \wedge A \neq 0 \wedge b \neq 0)$, then we get $f = 0$, a contradiction. Hence we have only one possibility $(a \neq 0 \wedge A \neq 0 \wedge b = 0)$. Consequently, equation (11) gives $g(0) = 1$. From (9) we obtain (i).

Case 2. Let $f(0) \neq 0$. By Remark 1 and equation (10), we get the relation

$$A(y)[f(0)(a - 2b) - 4abA(x)] = 0, \quad x, y \in G.$$

If $A = 0$, then (ii). Assume that $A \neq 0$. From above we have

$$f(0)(a - 2b) = 4abA(x), \quad x \in G.$$

Therefore $(a = 0 \Rightarrow b = 0) \vee (b = 0 \Rightarrow a = 0)$, the case (ii). If $a \neq 0 \wedge b \neq 0$, then $A = 0$, a contradiction. $\square$

Now, we formulate some properties of the exponential function without a proof.

Lemma 3. *Let $(G, +)$ be a uniquely 2-divisible Abelian group. Then a nonzero exponential function $m : G \to \mathbb{C}$ has the following properties*

(i) $m_e(x + y) + m_e(x - y) = 2m_e(x)m_e(y), \quad x, y \in G$;

(ii) $[m_o(x + y)]^2 - [m_o(x - y)]^2 = m_o(2x)m_o(2y), \quad x, y \in G$;

(iii) $m_o(x + y) + m_o(x - y) = 2m_o(x)m_e(y), \quad x, y \in G$;

(iv) $m_o(2x) = 2m_o(x)m_e(x), \quad x \in G$;

(v) $[m_e(x + y)]^2 - [m_e(x - y)]^2 = m_o(2x)m_o(2y), \quad x, y \in G$;

(vi) $m_o(x + y) - m_o(x - y) = 2m_e(x)m_o(y), \quad x, y \in G$.

Lemma 4. *Let $(G, +)$ be a uniquely 2-divisible Abelian group and let nonzero functions $f, g : G \to \mathbb{C}$ be functions defined by*

$$f(x) = am_o(x) + bm_e(x), \; g(x) = cm_o(x) + dm_e(x), \; x \in G, \qquad (12)$$

with some exponential function $m : G \to \mathbb{C}$ and $a, b, c, d \in \mathbb{C}$ satisfying equation (7). Then we have the following possibilities:

(i) $f(x) = am_o(x), \; g(x) = m_e(x), \; x \in G$; or

(ii) $f(x) = b \neq 0, \; g(x) = 1 - \frac{1}{2}b, \; x \in G$; or

(iii) $f(x) = bm_o(x) + bm_e(x), g(x) = \frac{1}{2}bm_o(x) + (1 - \frac{1}{2}b)m_e(x), x \in G$; or

(iv) $f(x) = -bm_o(x) + bm_e(x), g(x) = -\frac{1}{2}bm_o(x) + (1 - \frac{1}{2}b)m_e(x), x \in G$.

Proof. Inserting functions (12) into equation (7), for all $x, y \in G$, we obtain

$$[am_o(x + y) + bm_e(x + y)]^2 - [am_o(x - y) + bm_e(x - y)]^2 + a[m_o(2x + 2y)$$

$$+m_o(2x - 2y)] + b[m_e(2x + 2y) + m_e(2x - 2y)]$$

$$= [am_o(2x) + bm_e(2x)][(a + 2c)m_o(2y) + (b + 2d)m_e(2y)].$$

From above and Lemma 3, we get

$$[b^2 - 2ac]m_o(x)m_o(y) + b[a - 2c]m_e(x)m_o(y)$$

$$+a[2 - b - 2d]m_o(x)m_e(y) + b[2 - b - 2d]m_e(x)m_e(y) = 0, \; x, y \in G. \quad (13)$$

Directly from the definition (12), we see that $f(0) = b$ and $g(0) = d$. Moreover, from Remark 1 we infer that $b = 0$ or $b = 2(1 - d)$.

Now we shall distinguish two cases regarding the value of function f at zero.

Case 1. Let $f(0) = b = 0$. Then, by (13), we conclude that

$$am_o(x)[-cm_o(y) + (1-d)m_e(y)] = 0, \ x, y \in G.$$

If $a = 0$ or $m_o = 0$, then also $f = 0$. Hence

$$-cm_o(y) + (1-d)m_e(y) = 0, \ y \in G. \tag{14}$$

Putting $y = 0$ in (14) and using $m_o(0) = 0$, we have $d = 1$. Jointly with (14), for all $y \in G$, this implies that $-cm_o(y) = 0$, whence $c = 0$, which ends the proof of (i).

Case 2. Assume that $f(0) = b \neq 0$. Set $b = 2(1-d)$ in (13). Then, we get

$$m_o(y)[(b^2 - 2ac)m_o(x) + b(a - 2c)m_e(x)] = 0, \ x, y \in G. \tag{15}$$

Subcase 2.1. Let $m_o = 0$. By equation (12), we conclude that $f = bm_e$, $g = dm_e$. Replacing y by $-y$ in (7), we arrive at

$$[f(x-y)]^2 - [f(x+y)]^2 + f(2x-2y) + f(2x+2y) = f(2x)[f(2y) + 2g(2y)]. \tag{16}$$

Subtracting (7) and (16), we get

$$[f(x+y)]^2 = [f(x-y)]^2, \quad x, y \in G.$$

Putting here $y = x$ and replacing x by $\frac{x}{2}$, we obtain $f^2 = b^2$. The case $f = -b$ is impossible. In other words, we have (ii): $f = b$, $m_e = 1$, $g = d = \frac{2-b}{2} = 1 - \frac{1}{2}b$.

Subcase 2.2. Suppose $m_o \neq 0$. Then (15) yields

$$(b^2 - 2ac)m_o(x) + b(a - 2c)m_e(x) = 0, \quad x \in G. \tag{17}$$

Putting $x = 0$, we have $a = 2c$. From (17), for all $x \in G$, we get $(b^2 - 4c^2)m_o(x) = 0$, i.e. $b^2 = 4c^2$. If $a = 2c \wedge b = 2c$, then we have the case (iii). However, $a = 2c \wedge b = -2c$ yields (iv). $\qquad\square$

Theorem 2. *Let $(G, +)$ be a uniquely 2-divisible Abelian group. Then functions $f, g : G \to \mathbb{C}$ satisfy equation (7) if and only if*

(i) *$f = 0$ and g is arbitrary; or*

(ii) *$f(x) = \alpha \neq 0$, $g(x) = 1 - \frac{1}{2}\alpha$, $x \in G$; or*

(iii) *there exists an additive function $A : G \to \mathbb{C}$ such that $f = A, g = 1$; or*

(iv) *there exists an exponential function $m : G \to \mathbb{C}$ and some constant $\beta \in \mathbb{C}$ such that $f = \beta m_o$, $g = m_e$; or*

(v) *there exists an exponential function* $m \colon G \to \mathbb{C}$ *such that* $f(x) = f(0)m_o(x)$
$+ f(0)m_e(x)$, $g(x) = \frac{f(0)}{2}m_o(x) + \left(1 - \frac{f(0)}{2}\right)m_e(x)$, $x \in G$; *or*

(vi) *there exists an exponential function* $m \colon G \to \mathbb{C}$ *such that* $f(x) =$
$-f(0)m_o(x) + f(0)m_e(x)$, $g(x) = -\frac{f(0)}{2}m_o(x) + \left(1 - \frac{f(0)}{2}\right)m_e(x)$, $x \in G$.

Proof. From Lemma 1 we obtain (i) and (ii) for $\alpha = 2$. Assume that $f \neq 0$ and $g \neq 0$. Putting $x = 0$ in (7), we get

$$[f(y)]^2 - [f(-y)]^2 + f(2y) + f(-2y) = f(0)[f(2y) + 2g(2y)], \quad y \in G.$$

Let $2C := f(0)$. Thus, from above

$$f(2y) + f(-2y) - 2C[f(2y) + 2g(2y)] = [f(-y)]^2 - [f(y)]^2, \quad y \in G. \quad (18)$$

Interchanging the roles of x and y in (7), we obtain

$$[f(y+x)]^2 - [f(y-x)]^2 + f(2y+2x) + f(2y-2x)$$
$$= f(2y)f(2x) + 2f(2y)g(2x), x, y \in G. \quad (19)$$

Subtracting (7) and (19), we get

$$[f(y-x)]^2 - [f(x-y)]^2 + f(2x-2y) - f(2y-2x)$$
$$= 2f(2x)g(2y) - 2f(2y)g(2x), x, y \in G. \quad (20)$$

Applying (18) for y equal $x - y$, we recive

$$f(2x-2y) + f(-2x+2y) - 2C[f(2x-2y) + 2g(2x-2y)]$$
$$= [f(-x+y)]^2 - [f(x-y)]^2, x, y \in G. \quad (21)$$

By (20) and (21), we get the relation

$$(1 - C)f(2x - 2y) - 2Cg(2x - 2y) = f(2x)g(2y) - f(2y)g(2x), \quad x, y \in G.$$

Replacing x by $\frac{x}{2}$ and y by $\frac{y}{2}$, we obtain

$$(1 - C)f(x - y) - 2Cg(x - y) = f(x)g(y) - f(y)g(x), \quad x, y \in G. \quad (22)$$

Case 1. Let $f(0) = 0 \Rightarrow C = 0$. Thus, from (22) we get

$$f(x - y) = f(x)g(y) - f(y)g(x), \quad x, y \in G,$$

By Theorem 1 (ii), we infer that $f(x) = A(x), g(x) = \alpha A(x) + 1$ for some additive function A and some constant α. In view of Lemma 2 for $a = 1$, $b = \alpha$, $f(0) = 0$, $g(0) = 1$, we deduce that

$$f(x) = A(x), \quad g(x) = 1, \quad x \in G.$$

This is the case (iii) of our theorem. By Theorem 1 (iii), we get

$$f(x) = \beta m_o(x), \quad g(x) = \gamma m_o(x) + m_e(x), \quad x \in G.$$

For $a = \beta$, $b = 0$, $c = \gamma$, $d = 1$ in Lemma 4 (i) we have the case (iv), i.e.

$$f(x) = \beta m_o(x), \quad g(x) = m_e(x), \quad x \in G.$$

Case 2. Assume that $f(0) \neq 0$. Then $g(0) = 1 - C$, and (22) gives

$$g(0)f(x - y) - f(0)g(x - y) = f(x)g(y) - f(y)g(x), \quad x, y \in G. \qquad (23)$$

Subcase 2.1. If $g(0) = 0$, then $f(0) = 2$. By (23), we infer that

$$g(x - y) = g(x)\frac{f(y)}{2} - g(y)\frac{f(x)}{2}, \quad x, y \in G. \qquad (24)$$

Theorem 1 (ii) yields $g(x) = A(x), \frac{f(x)}{2} = \alpha A(x) + 1$ for some additive function A and some constant α. Thus

$$f(x) = 2\alpha A(x) + 2, \quad g(x) = A(x), \quad x \in G.$$

By Lemma 2 for $a = 2\alpha$, $b = 1$, $f(0) = 2$, $g(0) = 0$, we get $f = 2$, $g = 0$. This is the case (ii). Theorem 1 (iii) leads us to

$$f(x) = 2\gamma m_o(x) + 2m_e(x), \quad g(x) = \beta m_o(x), \quad x \in G.$$

From Lemma 4 (ii) for $a = 2\gamma$, $b = 2, c = \beta, d = 0$, we get (ii) of the theorem. The case (iii) for $f(0) = 2$ gives (v), and (iv) gives (vi).
Subcase 2.2. Let $f(0) \neq 0$ and $g(0) \neq 0$. Thus, from (23) for

$$F(x) := g(0)f(x) - f(0)g(x), \quad G(x) := \frac{g(x)}{g(0)}, \quad x \in G,$$

we conclude that

$$F(x - y) = F(x)G(y) - F(y)G(x), \quad x, y \in G. \qquad (25)$$

Again, by Theorem 1 (ii), we obtain

$$g(x) = g(0)\alpha A(x) + g(0), \quad f(x) = \frac{1 + f(0)g(0)\alpha}{g(0)}A(x) + f(0), \quad x \in G.$$

By Lemma 2 (ii) for $a = \frac{1+f(0)g(0)\alpha}{g(0)}$, $b = g(0)\alpha$, we get (ii) of the theorem. Further, Theorem 1 (iii) yields

$$F(x) := g(0)f(x) - f(0)g(x) = \beta m_o(x), \quad x \in G,$$

$$G(x) := \frac{g(x)}{g(0)} = \gamma m_o(x) + m_e(x), \quad x \in G,$$

or, equivalently,

$$g(x) = g(0)\gamma m_o(x) + g(0)m_e(x), \quad x \in G,$$

$$f(x) = \frac{\beta + \gamma f(0)g(0)}{g(0)} m_o(x) + f(0)m_e(x), \quad x \in G.$$

Now, using Lemma 4 for $a = \frac{\beta + \gamma f(0)g(0)}{g(0)}$, $b = f(0)$, $c = \gamma g(0)$, $d = g(0)$, the case (ii) gives (ii) of our theorem, however (iii) yields (v), and (iv) gives (vi). $\qquad\square$

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