Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XVI, Częstochowa 2011

A STOCHASTIC GRAPH AS A SPECIFIC TOOL OF MATHEMATIZATION AND ARGUMENTATION

Ireneusz Krech

Institute of Mathematics, Pedagogical University of Cracow
ul. Podchorążych 2, PL-30-084 Cracow, Poland
e-mail: irekre@tlen.pl

Abstract. The article presents a stochastic graph as a tool enabling us to show the equality of the event probability without calculating the probability as such. A very important factor here is that the discussed events come from different probabilistic spaces being models of specific Markov chains.

Introduction. Let us consider a random board game g_{x-y}. The board consists of two circles: o_v and o_w. At the beginning, there are x coins inside the circle o_v and y coins inside the circle o_w. Let us assume that $x + y = 3$ and $y \neq 3$. There are two players, G_a and G_b, in the game. They take turns and toss 3 coins placed on the game board. The coins that show heads stay in the circle they were originally placed in. The coins that show tails change their circle. If all the coins end up in the circle o_w after the toss, the player who tossed them wins. Let us assume that the player G_a takes the first run (see [2]).

Later, in the article we will answer the question: which of the games g_{3-0}, g_{2-1} and g_{1-2} is the best (gives the best chance to win) for the player G_a and which is the best for the player G_b.

We will mark the experiment conducted in the game as δ_{x-y}. Let A_{x-y} means that the player G_a wins and B_{x-y} means that the player G_b wins.

The random experiment δ_{x-y} is conducted in several phases. Each single phase consists of a coins toss and placing them in the circles. The experiment status after the nth phase is a pair (v_n, w_n), where v_n means the number of coins placed in the circle o_v and w_n means the number of coins placed in the circle o_w after this phase. As $v_n + w_n = 3$, the experiment status after the nth phase is defined by the number v_n. The possibilities here make a set $S = \{0, 1, 2, 3\}$. We can interpret them as the graph loops. The beginning of the game becomes the start loop and the experiment status before the game (the 0 stage) becomes the edge loop (see [1]). Let us mark the probability of the experiment going from the j to the k status as p_{jk}. If $p_{jk} > 0$, we connect the j and k loop points on the graph with a line. Then we write the number p_{jk} next to the line. This way we get a stochastic graph and a game board simultaneously.

I. Let us start with the game g_{3-0}. The graph and the stages of constructing it are shown in Figure 1.

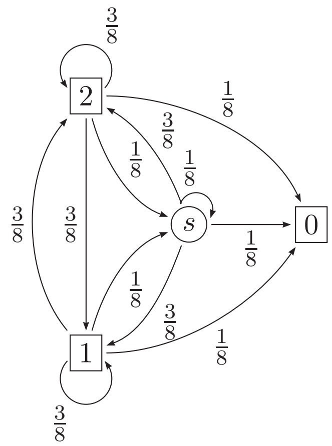

Figure 1.

This graph is a particularly useful tool of argumentation while calculating the probability of a certain player victory in the game.

We call the phases 0, 1 and 2 the inner ones. We can notice that once the experiment δ_{3-0} gets to one of the inner stages, the next toss will lead it either to the 0 state with the probability $\frac{1}{8}$ or to another inner state with the probability $\frac{7}{8}$. These symmetries prove that the graph from Figure 1 reduces to that from Figure 2.

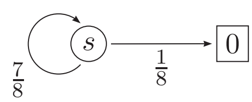

Figure 2.

The course of the game and its result can be registered if we include the time which it takes. Figure 3 shows the graph of the experiment δ_{3-0} after this modification.

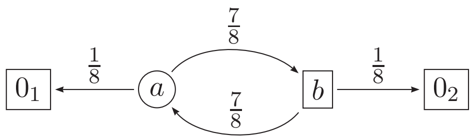

Figure 3.

The player G_a can win only if the experiment after an even toss takes the $\boxed{0}$ stage of the graph from Figure 2 or, which is really the same, the $\boxed{0_1}$ stage of the graph from Figure 3. The player G_b can win if the experiment after an odd toss takes the $\boxed{0}$ stage of the graph from Figure 2 or, which is really the same, the $\boxed{0_2}$ stage of the graph from Figure 3. From the above interpretations we can see that:

1) In the case of the graph from Figure 2 we get

$$P(A_{3-0})=\frac{1}{8}+\left(\frac{7}{8}\right)^2\cdot\frac{1}{8}+\left(\frac{7}{8}\right)^4\cdot\frac{1}{8}+\ldots=\frac{\frac{1}{8}}{1-(\frac{7}{8})^2}=\frac{8}{15} \quad \text{and} \quad P(B_{3-0})=\frac{7}{15}.$$

2) Let $P(A_{3-0}) = x$ and $P(B_{3-0}) = y = 1 - x$. We know from the graph shown in Figure 3 that

$$x=\frac{1}{8}+x\cdot\left(\frac{7}{8}\right)^2, \quad \text{so} \quad x=\frac{8}{15} \quad \text{and} \quad y=\frac{7}{15}.$$

So the player who starts the game has a better chance to win it.

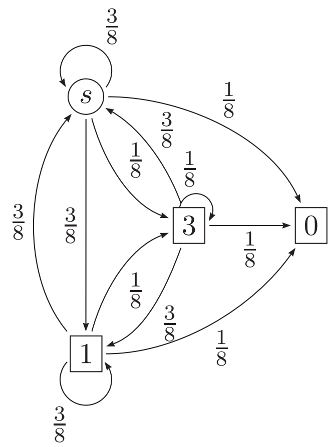

Figure 4.

II. Let us consider the game g_{2-1}. The δ_{2-1} experiment status at the beginning of the game is 2. Figure 4 shows the stochastic graph of the experiment δ_{2-1}.

Just like in the previous game g_{3-0}, once the experiment δ_{2-1} gets to one of the inner stages, the next coins toss leads it either to the 0 stage with the probability $\frac{1}{8}$ (the player who started the game wins it) or to another inner stage with the probability $\frac{7}{8}$. So we can see that the graph of this experiment is isomorphic to the one from Figure 2 (and, considering the time, to the graph from Figure 3), so

$$P(A_{2-1}) = \frac{8}{15} \quad \text{and} \quad P(B_{2-1}) = \frac{7}{15}.$$

III. It is easy to see that when we consider the game g_{1-2} we get a graph (as its board) isomorphic to the graphs of the games g_{3-0} and g_{2-1}. Hence,

$$P(A_{1-2}) = \frac{8}{15} \quad \text{and} \quad P(B_{1-2}) = \frac{7}{15}.$$

Finally we get:

$$P(A_{3-0}) = P(A_{2-1}) = P(A_{1-2}) = \frac{8}{15}$$

and

$$P(B_{3-0}) = P(B_{2-1}) = P(B_{1-2}) = \frac{7}{15}.$$

Summary. The final conclusion of our deliberation is surprising: the players chance to win does not depend on the experiment status at the beginning of the game, the player who starts the game wins it with the probability $\frac{8}{15}$ and the player who takes the second turn tossing the coins wins it with the probability $\frac{7}{15}$.

References

[1] A. Płocki. *Stochastyka dla nauczyciela.* Wydawnictwo Naukowe NOVUM, Płock 2010.

[2] A. Engel, T. Varga, W. Walser. *Strategia czy przypadek? Gry kombinatoryczne i probabilistyczne.* WSiP, Warszawa 1979.