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COMPARISON OF ψ-SPARSE TOPOLOGIES

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Abstract
The paper includes a necessary condition and sufficient conditions under which two ψ-sparse topologies generated by two functions ψ_1 and ψ_2 are equal. Additionally we proved that the intersection of all ψ-sparse topologies is equal to the Hashimoto topology.

1. Introduction

We shall use the following notations: $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{N}$ the set of all positive integers, m^* the outer Lebesgue measure, $\mathcal{L}$ the σ-algebra of Lebesgue measurable sets, m the Lebesgue measure and $\mathcal{C}$ the family of all continuous, nondecreasing functions $\psi : (0,\infty) \to (0,1)$ such that $\lim_{x\to 0^+} \psi(x) = 0$.

For any $\psi \in \mathcal{C}$, $E \subset \mathbb{R}$ and $x \in \mathbb{R}$, we let

$$\underline{d}(E,x) = \liminf_{h\to 0^+} \frac{m^*(E \cap [x-h, x+h])}{2h}$$

and

$$\overline{d}(E,x) = \limsup_{h\to 0^+} \frac{m^*(E \cap [x-h, x+h])}{2h}$$

as the lower and upper outer density of a set E at a point x, respectively.

Analogously, let

$$\psi - \underline{d}(E,x) = \liminf_{h\to 0^+} \frac{m^*(E\cap[x-h,x+h])}{2h\psi(2h)}$$

and

$$\psi - \overline{d}(E,x) = \limsup_{h\to 0^+} \frac{m^*(E\cap[x-h,x+h])}{2h\psi(2h)}$$

denote the lower and upper outer ψ-density of a set E at a point x, respectively.

Definition 1. [1] *We say that $x \in \mathbb{R}$ is a density point of a set $E \in \mathcal{L}$ if $\underline{d}(E,x) = 1$. We say that $x \in \mathbb{R}$ is a dispersion point of a set $E \in \mathcal{L}$ if x is the ψ-density point of the set $\mathbb{R} \setminus E$.*

Set, for each $E \in \mathcal{L}$,

$$\Phi(E) = \{x \in \mathbb{R} : x \text{ is a density point of } E\}.$$

Then the family $d = \{E \in \mathcal{L} : \ E \subset \Phi(E)\}$ is a topology on the real line called the density topology [1].

Definition 2. [4] *Let $\psi \in \mathcal{C}$. We say that $x \in \mathbb{R}$ is a ψ-dispersion point of a set $E \in \mathcal{L}$ if $\psi - \overline{d}(E,x) = 0$. We say that $x \in \mathbb{R}$ is a ψ-density point of a set $E \in \mathcal{L}$ if x is the ψ-dispersion point of the set $\mathbb{R} \setminus E$.*

For any $\psi \in \mathcal{C}$ and $E \in \mathcal{L}$, let

$$\Phi_\psi(E) = \{x \in \mathbb{R} : \ x \text{ is a } \psi - \text{density point of } E\}$$

and

$$\mathcal{T}_\psi = \{E \in \mathcal{L} : \ E \subset \Phi_\psi(E)\}.$$

Theorem 1. [4] *Let $\psi \in \mathcal{C}$. Then $\mathcal{T}_\psi$ is a topology on the real line, stronger than the Euclidean topology and weaker than the density topology d.*

Definition 3. [3] *We say that a set E is sparse at a point $x \in \mathbb{R}$ on the right if there exists, for every $\varepsilon > 0$, $\delta > 0$ such that every interval $(a,b) \subset (x, x+\delta)$, with $m^*((x,a)) < \delta m^*((x,b))$, contains at least one point y such that $m^*(E \cap (x,y)) < \varepsilon m^*((x,y))$.*

The family of sets sparse at x on the right is denoted by $\mathcal{S}(x+)$, and E is said to be sparse at x if $E \in \mathcal{S}(x) = \mathcal{S}(x+) \cap \mathcal{S}(x-)$.
($\mathcal{S}(x-)$ denotes, by convention, the family of sets sparse at x on the left.)

Let $\mathcal{S}_0(x) = \{E \subset \mathbb{R} : \ \overline{d}(E,x) = 0\}$. Then by [3], for each $x \in \mathbb{R}$ $S_0(x) \subset S(x)$.

Theorem 2. [3] *Let $x \in \mathbb{R}$ and $E \subset \mathbb{R}$. The following conditions are equivalent:*

(i) $E \in \mathcal{S}(x)$,

(ii) *for each $F \subset \mathbb{R}$, if $\underline{d}(F, x) = 0$, then $\underline{d}(E \cup F, x) = 0$.*

Definition 4. [2] *Let $\psi \in \mathcal{C}$. We say that a set E is ψ-sparse at a point $x \in \mathbb{R}$ if for each $F \subset \mathbb{R}$ the following holds:*

$$\text{if } \psi - \underline{d}(F, x) = 0, \quad \text{then } \psi - \underline{d}(E \cup F, x) = 0.$$

For each $x \in \mathbb{R}$, we denote by $\psi - \mathcal{S}(x)$ the family of all sets which are ψ-sparse at x. Put for each $x \in \mathbb{R}$, $\psi - \mathcal{S}_0(x) = \{E \subset \mathbb{R} : \ \psi - \overline{d}(E, x) = 0\}$.

Theorem 3. [2] *We assume that $\psi \in \mathcal{C}$ and $g(x) = 2x\psi(2x)$ for $x \in (0, 1]$. Let $E \subset \mathbb{R}$ and let A be a measurable cover of E. Then the following conditions are equivalent:*

(i) $E \in \psi - \mathcal{S}(0)$.

(ii) *for each $\varepsilon \in (0, 1)$, there exists $\delta \in (0, 1)$ such that, for each interval $[a, b] \subset (0, \delta)$, if $g(a) < \delta g(x - \frac{\varepsilon}{2} g(x))$ for each $x \in [b, 1]$, then there exists $y \in (a, b)$ such that $m^*(E \cap (-y, y)) < \varepsilon g(y)$.*

(iii) $A \in \psi - \mathcal{S}(0)$.

Let $\psi \in \mathcal{C}$. For $E \in \mathcal{L}$, put

$$\Gamma_\psi(E) = \{x \in \mathbb{R} : \ x \text{ is a } \psi - \text{sparse point of } \mathbb{R} \setminus E\}.$$

Theorem 4. [2] *Let $\psi \in \mathcal{C}$ and*

$$\tau_\psi = \{E \in \mathcal{L} : \ E \subset \Gamma_\psi(E)\}.$$

Then τ_ψ is a topology on the real line, stronger than the ψ-density topology $\mathcal{T}_\psi$ and weaker than the density topology d.

2. Comparison of ψ-sparse topologies

It is easy to see the following:

Theorem 5. *Let $\psi \in \mathcal{C}$, $E \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then a set $E \in \psi - \mathcal{S}(x)$ if and only if the set $\{y - x : \ y \in E\} \in \psi - \mathcal{S}(0)$.*

Lemma 1. *Let $\psi_1 \in \mathcal{C}$ and $\psi_2 \in \mathcal{C}$. If $\psi_1 - \mathcal{S}(0) = \psi_2 - \mathcal{S}(0)$, then $\tau_{\psi_1} = \tau_{\psi_2}$.*

Lemma 2. *Let $\psi_1 \in \mathcal{C}$ and $\psi_2 \in \mathcal{C}$. If for each $E \subset \mathbb{R}$,*

$$\psi_1 - \underline{d}(E, 0) = 0 \text{ if and only if } \psi_2 - \underline{d}(E, 0) = 0,$$

then $\psi_1 - \mathcal{S}(0) = \psi_2 - \mathcal{S}(0)$.

Definition 5. *We say that two functions $\psi_1 \in \mathcal{C}$ and $\psi_2 \in \mathcal{C}$ are equivalent if and only if there exist positive numbers α, β and δ such that for each $x \in (0, \delta)$*

$$\alpha < \frac{\psi_1(x)}{\psi_2(x)} < \beta.$$

Clearly, two functions $\psi_1 \in \mathcal{C}$ and $\psi_2 \in \mathcal{C}$ are equivalent if and only if

$$\limsup_{x \to 0^+} \frac{\psi_1(x)}{\psi_2(x)} < \infty$$

and

$$\liminf_{x \to 0^+} \frac{\psi_1(x)}{\psi_2(x)} > 0.$$

Lemma 3. *Let $\psi_1 \in \mathcal{C}$ and $\psi_2 \in \mathcal{C}$. If the functions ψ_1 and ψ_2 are equivalent, then*

$$\psi_1 - \underline{d}(E, 0) = 0 \text{ if and only if } \psi_2 - \underline{d}(E, 0) = 0$$

for each $E \subset \mathbb{R}$.

P r o o f. Assume that $\liminf_{x \to 0^+} \frac{m^*(E \cap [-x,x])}{2x\psi_1(2x)} = 0$. By the equivalence of the functions ψ_1 and ψ_2, there exist positive real numbers $\delta > 0$ and $\beta > 0$ such that $\frac{\psi_1(x)}{\psi_2(x)} < \beta$ for each $x \in (0, \delta)$. Thus,

$$\begin{aligned} 0 \leq \frac{m^*(E \cap [-x, x])}{2x\psi_2(2x)} \cdot \frac{\psi_1(2x)}{\psi_1(2x)} &= \frac{m^*(E \cap [-x, x])}{2x\psi_1(2x)} \cdot \frac{\psi_1(2x)}{\psi_2(2x)} \\ &< \beta \cdot \frac{m^*(E \cap [-x, x])}{2x\psi_1(2x)} \end{aligned}$$

for each $x \in (0, \delta)$. Therefore,

$$\liminf_{x \to 0^+} \frac{m^*(E \cap [-x, x])}{2x\psi_2(2x)} \leq \beta \cdot \liminf_{x \to 0^+} \frac{m^*(E \cap [-x, x])}{2x\psi_1(2x)} = 0.$$

The rest of the proof runs as before.

By the above lemmas we have the following.

Theorem 6. *Let $\psi_1 \in \mathcal{C}$ and $\psi_2 \in \mathcal{C}$. If the functions ψ_1 and ψ_2 are equivalent, then $\tau_{\psi_1} = \tau_{\psi_2}$.*

It appears that equivalence of ψ_1 and ψ_2 is a sufficient condition for the equality $\tau_{\psi_1} = \tau_{\psi_2}$, but not necessary. To prove this fact we need the following lemma.

Lemma 4. *Let $\{(a_n, b_n)\}_{n\in\mathbf{N}}$ be a sequence of intervals such that $\lim_{n\to\infty} b_n = 0$ and $0 < b_{n+1} < a_n < 1$ for each $n \in \mathbb{N}$. Assume that $H = \bigcup_{n=1}^{\infty}(a_n, b_n)$ and $g_1 : [0,1] \to [0,1]$ and $g_2 : [0,1] \to [0,1]$ are two increasing, continuous functions such that*

(I) $g_1(0) = g_2(0) = 0$,

(II) *if $x \notin H$, then $g_1(x) = g_2(x)$, and if $x \in H$, then $g_1(x) < g_2(x)$, for each $x \in [0,1]$,*

(III) $b_n - a_n \le \frac{1}{n} g_1(b_n)$ *for each $n \in \mathbb{N}$.*

Then $\liminf_{x\to 0^+} \frac{m^*(E\cap[-x,x])}{g_1(x)} = 0$ if and only if $\liminf_{x\to 0^+} \frac{m^*(E\cap[-x,x])}{g_2(x)} = 0$, for each $E \subset \mathbb{R}$.

P r o o f. By the condition (II), we have that $\frac{m^*(E\cap[-x,x])}{g_2(x)} \le \frac{m^*(E\cap[-x,x])}{g_1(x)}$ for each $x \in (0,1]$. Thus if $\liminf_{x\to 0^+} \frac{m^*(E\cap[-x,x])}{g_1(x)} = 0$, then

$$\liminf_{x\to 0^+} \frac{m^*(E\cap[-x,x])}{g_2(x)} \le \liminf_{x\to 0^+} \frac{m^*(E\cap[-x,x])}{g_1(x)} = 0.$$

Now we assume that $\liminf_{x\to 0^+} \frac{m^*(E\cap[-x,x])}{g_2(x)} = 0$. Then there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset (0,1)$ such that $\lim_{k\to\infty} x_k = 0$ and

$$\lim_{k\to\infty} \frac{m^*(E\cap[-x_k,x_k])}{g_2(x_k)} = 0. \tag{1}$$

If there exists a subsequence $\{x_{k_m}\}_{m\in\mathbb{N}}$ such that $x_{k_m} \notin H$ for each $m \in \mathbb{N}$, then by (II), $\frac{m^*(E\cap[-x_{k_m},x_{k_m}])}{g_1(x_{k_m})} = \frac{m^*(E\cap[-x_{k_m},x_{k_m}])}{g_2(x_{k_m})}$ for each $m \in \mathbb{N}$. Therefore, by the above and by (1),

$$\lim_{m\to\infty} \frac{m^*(E\cap[-x_{k_m},x_{k_m}])}{g_1(x_{k_m})} = \lim_{k\to\infty} \frac{m^*(E\cap[-x_k,x_k])}{g_2(x_k)} = 0$$

and $\liminf_{x\to 0^+} \frac{m^*(E\cap[-x,x])}{g_1(x)} = 0$.

Assume that there exists $k_0 \in \mathbb{N}$ such that $x_k \in H$, for each $k > k_0$. Put $k > k_0$. By the definition of the set H, there exists $n_k \in \mathbb{N}$ such that $x_k \in (a_{n_k}, b_{n_k})$. Then

$$g_1(b_{n_k}) = g_2(b_{n_k}) \geq g_2(x_k) \tag{2}$$

and

$$m^*(E \cap [-b_{n_k}, b_{n_k}]) \leq m^*(E \cap [-x_k, x_k]) + m^*(E \cap ([-b_{n_k}, -x_k] \cup [x_k, b_{n_k}])).$$

Thus from (2) we have $\frac{m^*(E\cap[-x_k,x_k])}{g_1(b_{n_k})} \leq \frac{m^*(E\cap[-x_k,x_k])}{g_2(x_k)}$, and by (III),

$$m^*(E \cap ([-b_{n_k}, -x_k] \cup [x_k, b_{n_k}]) \leq 2(b_{n_k} - a_{n_k}) \leq \frac{2}{n_k} g_1(b_{n_k}).$$

Hence

$$\frac{m^*(E \cap [-b_{n_k}, b_{n_k}]}{g_1(b_{n_k})} \leq \frac{m^*(E \cap [-x_k, x_k])}{g_2(x_k)} + \frac{2}{n_k}.$$

Observe that if $\lim_{k\to\infty} x_k = 0$, then $\lim_{k\to\infty} b_{n_k} = 0$ and $\lim_{k\to\infty} n_k = \infty$. Therefore by (1),

$$\liminf_{x\to 0^+} \frac{m^*(E \cap [-x, x])}{g_1(x)} \leq \liminf_{k\to\infty} \frac{m^*(E \cap [-b_{n_k}, b_{n_k}]}{g_1(b_{n_k})} = 0.$$

Theorem 7. *There exist two functions $\psi_1 \in \mathcal{C}$ i $\psi_2 \in \mathcal{C}$ such that*

$$\liminf_{x\to 0^+} \frac{\psi_1(x)}{\psi_2(x)} = 0,$$

$$0 < \limsup_{x\to 0^+} \frac{\psi_1(x)}{\psi_2(x)} < \infty$$

for which $\tau_{\psi_1} = \tau_{\psi_2}$.

P r o o f. Let $a_n = \frac{1}{n+2}$, $b_n = a_n + \frac{2}{n} a_n \frac{1}{(n+2)!}$ and $c_n = \frac{a_n+b_n}{2}$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} b_n = 0$ and $b_{n+1} < a_n < b_n < 2b_n < 1$ for all n. Let $\psi_1 \in \mathcal{C}$ and $\psi_2 \in \mathcal{C}$ such that

$$\psi_1(t) = \begin{cases} \frac{1}{2!} & \text{for} \quad t \in [2b_1, \infty), \\ \frac{1}{(n+2)!} & \text{for} \quad t \in [2b_{n+1}, 2c_n] \text{ and } n \geq 1, \\ \text{linear} & \text{for the remaining} \quad t \in (0, \infty), \end{cases}$$

$$\psi_2(t) = \begin{cases} \frac{1}{2!} & \text{for} \quad t \in [2c_1, \infty), \\ \frac{1}{(n+2)!} & \text{for} \quad t \in [2c_{n+1}, 2a_n] \text{ and } n \geq 1, \\ \text{linear} & \text{for the remaining} \quad t \in (0, \infty). \end{cases}$$

Therefore for each $n \in \mathbb{N}$, if $t \in (2a_n, 2b_n)$, then

$$\psi_1(t) < \psi_2(t), \tag{3}$$

and if $t \in [2b_{n+1}, 2a_n] \cup [2b_1, \infty)$, then

$$\psi_1(t) = \psi_2(t). \tag{4}$$

Hence

$$\limsup_{t \to 0^+} \frac{\psi_1(t)}{\psi_2(t)} \leq 1 < \infty,$$

$$\limsup_{t \to 0^+} \frac{\psi_1(t)}{\psi_2(t)} \geq \limsup_{n \to \infty} \frac{\psi_1(2a_n)}{\psi_2(2a_n)} = 1 > 0$$

and

$$\liminf_{t \to 0^+} \frac{\psi_1(t)}{\psi_2(t)} \leq \liminf_{n \to \infty} \frac{\psi_1(2c_n)}{\psi_2(2c_n)} = \lim_{n \to \infty} \frac{\frac{1}{(n+2)!}}{\frac{1}{(n+1)!}} = \lim_{n \to \infty} \frac{1}{n+2} = 0.$$

Let $H = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ and

$$g_1(x) = \begin{cases} 2x\psi_1(2x) & \text{for} \quad x \in (0,1], \\ 0 & \text{for} \quad x = 0, \end{cases}$$

$$g_2(x) = \begin{cases} 2x\psi_2(2x) & \text{for} \quad x \in (0,1], \\ 0 & \text{for} \quad x = 0. \end{cases}$$

Then the functions g_1 i g_2 are continuous and increasing. Observe that if $x \in H$, then there exists $n \in \mathbb{N}$ such that $x \in (a_n, b_n)$, so by (3),

$$g_1(x) = 2x\psi_1(2x) < 2x\psi_2(2x) = g_2(x),$$

and if $x \notin H$, then $x \notin (a_n, b_n)$ for $n \in \mathbb{N}$, thus by (4),

$$g_1(x) = 2x\psi_1(2x) = 2x\psi_2(2x) = g_2(x).$$

Additionally, by the definition of the numbers a_n and b_n,

$$b_n - a_n = \frac{2}{n} a_n \frac{1}{(n+2)!} = \frac{1}{n} 2a_n \psi_1(2a_n) = \frac{1}{n} g_1(a_n) < \frac{1}{n} g_1(b_n)$$

for $n \in \mathbb{N}$. Therefore, by lemma 4 we have that

$$\liminf_{x \to 0^+} \frac{m^*(E \cap [-x, x])}{g_1(x)} = 0$$

if and only if

$$\liminf_{x\to 0^+} \frac{m^*(E\cap[-x,x])}{g_2(x)} = 0$$

for each set $E \subset \mathbb{R}$. Hence, by lemmas 2 and 1 we obtain that $\tau_{\psi_1} = \tau_{\psi_2}$.

It is easy to prove the following lemma.

Lemma 5. *Let $s : [0,1] \to [0,1]$ be a continuous increasing function such that $s(x) < x$ for $x \in (0,1]$ and $s(0) = 0$. If $h(x) = x - s(x)$ and*

$$p(x) = \min\{t \in [x,1] : h(t) = \min\{h(z) : z \in [x,1]\}\}$$

for each $x \in (0,1]$, then $\lim_{x\to 0^+} p(x) = 0$.

Let $\psi_1 \in \mathcal{C}$ and $\psi_2 \in \mathcal{C}$. Set

$$g_1(x) = \begin{cases} 2x\psi_1(2x) & \text{for} \quad x \in (0,1], \\ 0 & \text{for} \quad x = 0 \end{cases}$$

and

$$g_2(x) = \begin{cases} 2x\psi_2(2x) & \text{for} \quad x \in (0,1], \\ 0 & \text{for} \quad x = 0. \end{cases}$$

Put $h_k^j(x) = x - \frac{1}{2k} g_j(x)$ and

$$p_k^j(x) = \min\{t \in [x,1] : \ h_k^j(t) = \min\{h_k^j(z) : z \in [x,1]\}\}$$

for $k \in \mathbb{N}$, $j \in \{1,2\}$ and $x \in (0,1]$.

Lemma 6. *Let $k \in \mathbb{N}$, $k \geq 2$ and $d \in (0,1)$. If $g_1(p_1^1(d)) < \frac{1}{2k} g_2(p_1^1(d))$, then $0 < h_k^2(p_k^2(x)) < p_1^1(d) - g_1(p_1^1(d))$ for $0 < x < p_1^1(d)$.*

P r o o f. By $k \geq 2$ and by the definition of g_2, we have that $g_2(x) < 2x$ and $h_k^2(x) = x - \frac{1}{2k} g_2(x) > 0$ for $x \in (0,1]$. Therefore,

$$h_k^2(p_k^2(x)) = \min\{h_k^2(z) : z \in [x,1]\} > 0$$

for each $x \in (0,1]$.

Let $d \in (0,1)$ and $x \in (0, p_1^1(d))$. Then $h_k^2(p_k^2(x)) = \min\{h_k^2(z) : z \in [x,1]\}$ and

$$h_k^2(p_k^2(p_1^1(d))) = \min\{h_k^2(z) : z \in [p_1^1(d),1]\} \leq h_k^2(p_1^1(d)).$$

Therefore,

$$h_k^2(p_k^2(x)) \leq h_k^2(p_k^2(p_1^1(d))) \leq h_k^2(p_1^1(d)) = p_1^1(d) - \frac{1}{2k}g_2(p_1^1(d)). \quad (5)$$

By the assumption, $g_1(p_1^1(d)) < \frac{1}{2k}g_2(p_1^1(d))$, so

$$p_1^1(d) - \frac{1}{2k}g_2(p_1^1(d)) < p_1^1(d) - g_1(p_1^1(d)). \quad (6)$$

Thus by (5) and (6) we have that $h_k^2(p_k^2(x)) < p_1^1(d) - g_1(p_1^1(d))$.

Theorem 8. *If*

$$\lim_{x \to 0^+} \frac{\psi_1(x)}{\psi_2(x)} = 0,$$

then there exists a set $E \in \tau_{\psi_2} \setminus \tau_{\psi_1}$.

P r o o f. By the assumption, there exists a decreasing sequence of positive numbers $\{\gamma_k\}_{k \in \mathbb{N}}$ such that $\lim\limits_{k \to \infty} \gamma_k = 0$ and

$$g_1(x) < \frac{1}{2k^2}g_2(x) \quad (7)$$

for any $k \in \mathbb{N}$ and $x \in (0, \gamma_k)$. Let $k \in \mathbb{N}$. Suppose that we have also chosen the intervals $[a_1, b_1], \ldots, [a_k, b_k]$ and $[c_1, d_1], \ldots, [c_k, d_k]$ such that $d_1 \leq p_1^1(d_1) < \gamma_1$ and for each $i \in \{2, \ldots, k\}$:

(I) $d_i \leq p_1^1(d_i) < \min\{\gamma_i, \frac{1}{2i}g_2(a_{i-1}), \frac{1}{2i}g_2(c_{i-1})\}$,

(II) $c_i = p_1^1(d_i) - g_1(p_1^1(d_i))$,

(III) $\frac{1}{2k}g_2(b_i) = g_1(h_1^1(p_1^1(d_i))$,

(IV) $a_i = b_i - \frac{1}{2k}g_2(b_i)$,

(V) $0 < b_i < p_1^1(d_i) - \frac{1}{2}g_1(p_1^1(d_i)) < d_i$.

Let

$$\alpha < \min\left\{\gamma_{k+1}, \frac{1}{2(k+1)}g_2(a_k), \frac{1}{2(k+1)}g_2(c_k)\right\}. \quad (8)$$

By lemma 5, there exists $d_{k+1} \in (0, \alpha)$ such that $d_{k+1} \leq p_1^1(d_{k+1}) < \alpha$. Moreover, by lemma 6 we have that $p_1^1(d_{k+1}) - g_1(p_1^1(d_{k+1})) > 0$. Put $c_{k+1} = p_1^1(d_{k+1}) - g_1(p_1^1(d_{k+1}))$ and $w_{k+1} = h_1^1(p_1^1(d_{k+1})) = p_1^1(d_{k+1}) - \frac{1}{2}g_1(p_1^1(d_{k+1}))$. Then

$$0 < c_{k+1} < w_{k+1} = p_1^1(d_{k+1}) - \frac{1}{2}g_1(p_1^1(d_{k+1})) \leq d_{k+1} - \frac{1}{2}g_1(d_{k+1}) < d_{k+1},$$

so by $d_{k+1} \in (0, \gamma_{k+1})$ and by (7), we obtain

$$0 < g_1(w_{k+1}) < \frac{1}{2(k+1)^2} g_2(w_{k+1}) < \frac{1}{2(k+1)} g_2(w_{k+1}).$$

Therefore, there exists $b_{k+1} \in (0, w_{k+1})$ such that $\frac{1}{2(k+1)} g_2(b_{k+1}) = g_1(w_{k+1})$. Set $a_{k+1} = b_{k+1} - \frac{1}{2(k+1)} g_2(b_{k+1})$. Then $0 < a_{k+1} < b_{k+1} < w_{k+1} < d_{k+1}$, $0 < c_{k+1} < w_{k+1} < d_{k+1}$ and by (8), $d_{k+1} < \min\{a_k, c_k\}$.
Put $E_1 = \bigcup_{k=1}^{\infty} [a_k, b_k]$ and $E_2 = \bigcup_{k=1}^{\infty} [c_k, d_k]$. We shall prove that $E_1 \in \psi_2 - \mathcal{S}(0)$. By (V), (I) and (IV), we observe that

$$\begin{aligned} m(E_1 \cap [-t, t]) &= m(E_1 \cap [0, t]) \leq b_{k+2} + b_{k+1} - a_{k+1} \\ &< \frac{1}{2(k+2)} g_2(a_{k+1}) + \frac{1}{2(k+1)} g_2(b_{k+1}) \\ &\leq \frac{1}{k+1} g_2(b_{k+1}) \leq \frac{1}{k+1} g_2(t) \end{aligned} \tag{9}$$

for any $k \in \mathbb{N}$ and $t \in [b_{k+1}, a_k]$.

Let $k \in \mathbb{N}$, $k > 1$ and let $\delta = \min\left\{\gamma_k, \frac{1}{2k} g_2(a_{k-1}), \frac{1}{2k} g_2(c_{k-1})\right\}$. We consider an interval $[a, b] \subset (0, \delta)$ such that $g_2(a) < \delta g_2\left(p_k^2(b) - \frac{1}{2k} g_2(p_k^2(b))\right)$. If $b \in (b_m, a_{m-1}]$ for some $m \geq k$, then by (9),

$$m(E_1 \cap [-t, t]) < \frac{1}{m} g_2(t) \leq \frac{1}{k} g_2(t)$$

for $t \in (a, b) \cap [b_m, a_{m-1}]$. Assume that $b \in (a_m, b_m]$ for some $m \geq k$. The function $h_k^2 \circ p_k^2$ is nondecreasing, therefore

$$\begin{aligned} p_k^2(b) - \frac{1}{2k} g_2(p_k^2(b)) &\leq p_k^2(b_m) - \frac{1}{2k} g_2(p_k^2(b_m)) \leq b_m - \frac{1}{2k} g_2(b_m) \\ &\leq b_m - \frac{1}{2m} g_2(b_m) = a_m \end{aligned}$$

and

$$g_2(a) \leq \delta g_2\left(p_k^2(b) - \frac{1}{2k} g_2(p_k^2(b))\right) < g_2\left(p_k^2(b) - \frac{1}{2k} g_2(p_k^2(b))\right).$$

Hence, $a < p_k^2(b) - \frac{1}{2k} g_2(p_k^2(b)) \leq a_m$. Put $t = a_m$. Then $t \in (a, b)$ and by (9), we have that

$$m(E_1 \cap [-t, t]) < \frac{1}{m+1} g_2(t) < \frac{1}{k} g_2(t).$$

Thus, by theorem 3 we know that $E_1 \in \psi_2 - \mathcal{S}(0)$.

Now we shall prove that $E_2 \in \psi_2 - \mathcal{S}(0)$. By (I), (II) and (7), we observe that

$$\begin{aligned} m(E_2 \cap [-t,t]) &= m(E_2 \cap [0,t]) \le d_{k+2} + d_{k+1} - c_{k+1} \\ &< \frac{1}{2(k+2)} g_2(c_{k+1}) + g_1(p_1^1(d_{k+1})) \\ &< \frac{1}{2(k+1)} g_2(c_{k+1}) + \frac{1}{2(k+1)^2} g_2(p_1^1(d_{k+1})) \\ &< \frac{1}{k+1} g_2(t) \end{aligned} \tag{10}$$

for any $k \in \mathbb{N}$ and $t \in [p_1^1(d_{k+1}), c_k]$.

Let $k \in \mathbb{N}$, $k > 1$ and let $\delta = \min\left\{\gamma_k, \frac{1}{2k} g_2(a_{k-1}), \frac{1}{2k} g_2(c_{k-1})\right\}$. We consider an interval $[a,b] \subset (0,\delta)$ such that $g_2(a) < \delta g_2\left(p_k^2(b) - \frac{1}{2k} g_2(p_k^2(b))\right)$. If $b \in (p_1^1(d_m), c_{m-1}]$ for some $m \ge k$, then by (10),

$$m(E_2 \cap [-t,t]) < \frac{1}{m} g_2(t) \le \frac{1}{k} g_2(t)$$

for all $t \in (a,b) \cap [p_1^1(d_m), c_{m-1}]$. Assume that $b \in (c_m, p_1^1(d_m)]$ for some $m \ge k$. Then by (4) we have $g_1(p_1^1(d_m)) < \frac{1}{2k} g_2(p_1^1(d_m))$. Therefore by the above and by lemma 6,

$$p_k^2(b) - \frac{1}{2k} g_2(p_k^2(b)) \le p_1^1(d_m) - g_1(p_1^1(d_m)) = c_m.$$

Moreover,

$$g_2(a) \le \delta g_2 \left(p_k^2(b) - \frac{1}{2k} g_2(p_k^2(b)) \right) < g_2 \left(p_k^2(b) - \frac{1}{2k} g_2(p_k^2(b)) \right),$$

so $a < p_k^2(b) - \frac{1}{2k} g_2(p_k^2(b)) \le c_m$. Put $t = c_m$. Then $t \in (a,b)$ and

$$m(E_2 \cap [-t,t]) < \frac{1}{m+1} g_2(t) < \frac{1}{k} g_2(t).$$

Thus, by theorem 3 we know that $E_2 \in \psi_2 - \mathcal{S}(0)$.

Put $E = (\mathbb{R} \setminus E_1) \cap (\mathbb{R} \setminus E_2)$. By the above and by the definition of τ_{ψ_2}-topology, we know that the set $E \in \tau_{\psi_2}$.

We shall show that $E \notin \tau_{\psi_1}$. It suffices to prove that $\mathbb{R} \setminus E \notin \psi_1 - \mathcal{S}(0)$. Put $H = \mathbb{R} \setminus E = E_1 \cup E_2$. Let $k \in \mathbb{N} \setminus \{1\}$. We consider an interval $[b_k, d_k]$. Then by (I), (7) and by lemma 6, we have

$$0 < p_1^1(d_k) - g_1(p_1^1(d_k) < p_1^1(d_k) - \frac{1}{2} g_1(p_1^1(d_k)$$

and $d_k < \frac{1}{k}$. Thus by (7) and by (III), we obtain

$$\begin{aligned} g_1(b_k) &< \frac{1}{2k^2} g_2(b_k) = \frac{1}{k} g_1\left(p_1^1(d_k) - \frac{1}{2} g_1(p_1^1(d_k))\right) \\ &< \frac{1}{k} g_1\left(p_1^1(d_k) - \frac{1}{4} g_1(p_1^1(d_k))\right). \end{aligned}$$

We shall show that $m(H \cap [-y, y]) > \frac{1}{2} g_1(y)$ for all $y \in (b_k, d_k)$. Let $y \in (b_k, d_k)$. If $y \in (b_k, p_1^1(d_k) - \frac{1}{2} g_1(p_1^1(d_k))]$, then by (IV) and (III),

$$m(H \cap [-y, y]) > b_k - a_k = \frac{1}{2k} g_2(b_k) = g_1\left(p_1^1(d_k) - \frac{1}{2} g_1(p_1^1(d_k))\right) \geq g_1(y).$$

If $y \in (p_1^1(d_k) - \frac{1}{2} g_1(p_1^1(d_k)), d_k)$, then by (II) we have

$$m(H \cap [-y, y]) > p_1^1(d_k) - \frac{1}{2} g_1(p_1^1(d_k)) - c_k = \frac{1}{2} g_1(p_1^1(d_k)) > \frac{1}{2} g_1(y).$$

Therefore by theorem 3, we know that $\mathbb{R} \setminus E \notin \psi_1 - \mathcal{S}(0)$.

Corollary 1. *If $\tau_{\psi_1} = \tau_{\psi_2}$, then $\limsup_{x \to 0^+} \frac{\psi_1(x)}{\psi_2(x)} > 0$ and $\limsup_{x \to 0^+} \frac{\psi_2(x)}{\psi_1(x)} > 0$.*

Set

$$A_k^+ = \left\{ x \in (0, 1) : g_1(x) < \frac{1}{k} g_2(x) \right\},$$

$$B_k^+ = \left\{ x \in (0, 1) : g_2(x) < \frac{1}{k} g_1(x) \right\}$$

and $A_k = A_k^+ \cup (-A_k^+)$ i $B_k = B_k^+ \cup (-B_k^+)$ for $k \in \mathbb{N}$.

Lemma 7. *Let $k \in \mathbb{N}$. Assume that $(\mathbb{R} \setminus A_k^+) \cap \left(0, \frac{1}{n}\right) \neq \emptyset$ for all $n \in \mathbb{N}$. If $E \subset \mathbb{R}$ satisfies condition $\liminf_{x \to 0^+} \frac{m^*(E \cap [-x, x])}{g_2(x)} = 0$, then there exists a sequence $\{y_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} y_n = 0$ and*

$$\limsup_{n \to \infty} \frac{m^*(E \cap [-y_n, y_n])}{g_1(y_n)} \leq \limsup_{n \to \infty} \frac{m^*(A_k \cap [-y_n, y_n])}{g_1(y_n)}.$$

P r o o f. If $\liminf_{x \to 0^+} \frac{m^*(E \cap [-x, x])}{g_2(x)} = 0$, then there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} x_n = 0$ and

$$\lim_{n \to \infty} \frac{m^*(E \cap [-x_n, x_n])}{g_2(x_n)} = 0. \tag{11}$$

Consider two cases:

1. There exists a subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$ such that $x_{n_m} \notin A_k^+$ for each $m \in \mathbb{N}$.

Then $g_1(x_{n_m}) \geq \frac{1}{k} g_2(x_{n_m})$ and $\frac{m^*(E\cap[-x_{n_m},x_{n_m}])}{g_1(x_{n_m})} \leq \frac{km^*(E\cap[-x_{n_m},x_{n_m}])}{g_2(x_{n_m})}$ for each $m \in \mathbb{N}$. Thus by (7),

$$\limsup_{m\to\infty} \frac{m^*(E\cap[-x_{n_m},x_{n_m}])}{g_1(x_{n_m})} \leq \lim_{m\to\infty} \frac{km^*(E\cap[-x_{n_m},x_{n_m}])}{g_2(x_{n_m})} = 0.$$

Put $y_m = x_{n_m}$ for each $m \in \mathbb{N}$. Then $\{y_m\}_{m\in\mathbb{N}} \subset (0,1)$, $\lim_{m\to\infty} y_m = 0$ and

$$\limsup_{m\to\infty} \frac{m^*(E\cap[-y_m,y_m])}{g_1(y_m)} = 0 \leq \limsup_{m\to\infty} \frac{m^*(A_k\cap[-y_m,y_m])}{g_1(y_n)}.$$

2. There exists $n_0 \in \mathbb{N}$ such that $x_n \in A_k^+$ for each $n > n_0$.

Set $n > n_0$. By the continuity of the functions g_1 and g_2, we know that the set A_k^+ is open. Therefore, there exists a component interval (a_n, b_n) of the set A_k^+ such that $x_n \in (a_n, b_n)$. Then

$$g_1(b_n) = \frac{1}{k} g_2(b_n) \geq \frac{1}{k} g_2(x_n) \tag{12}$$

and

$$m^*(E\cap[-b_n,b_n]) \leq m^*(E\cap[-x_n,x_n]) + m^*(E\cap([-b_n,-x_n]\cup[x_n,b_n])).$$

Moreover by (12), $\frac{m^*(E\cap[-x_n,x_n])}{g_1(b_n)} \leq k\frac{m^*(E\cap[-x_n,x_n])}{g_2(x_n)}$ and

$$m^*(E\cap([-b_n,-x_n]\cup[x_n,b_n])) \leq m^*(A_k\cap[-b_n,b_n]).$$

Thus,

$$\frac{m^*(E\cap[-b_n,b_n])}{g_1(b_n)} \leq \frac{km^*(E\cap[-x_n,x_n])}{g_2(x_n)} + \frac{m^*(A_k\cap[-b_n,b_n])}{g_1(b_n)}.$$

By assumption, $(\mathbb{R} \setminus A_k^+) \cap \left(0, \frac{1}{m}\right) \neq \emptyset$ for each $m \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = 0$, therefore there exist subsequences $\{b_{n_m}\}_{m\in\mathbb{N}}$ and $\{x_{n_m}\}_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty} b_{n_m} = 0$ and $x_{n_m} \in (a_{n_m}, b_{n_m})$ for all m. Hence by the above and by (11), we have that

$$\limsup_{m\to\infty} \frac{m^*(E\cap[-b_{n_m},b_{n_m}])}{g_1(b_{n_m})} \leq \limsup_{m\to\infty} \frac{m^*(A_k\cap[-b_{n_m},b_{n_m}])}{g_1(b_{n_m})}.$$

Lemma 8. *If* $\liminf_{x\to 0^+} \frac{m^*(A_k\cap[-x,x])}{g_1(x)} < \infty$, *then* $(\mathbb{R}\setminus A_k^+)\cap\left(0,\frac{1}{m}\right) \neq \emptyset$ *for each* $m\in\mathbb{N}$.

P r o o f. By $\liminf_{x\to 0^+} \frac{m^*(A_k\cap[-x,x])}{g_1(x)} < \infty$, we have that there exist $a<\infty$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \frac{m^*(A_k\cap[-x_n,x_n])}{g_1(x_n)} = a$. Suppose that there exists $m\in\mathbb{N}$ such that $(\mathbb{R}\setminus A_k^+)\cap\left(0,\frac{1}{m}\right) = \emptyset$. Then there exists $n_0\in\mathbb{N}$ such that

$$[-x_n,x_n]\subset\left(-\frac{1}{m},\frac{1}{m}\right)\subset A_k$$

for each $n>n_0$. Hence

$$\lim_{n\to\infty}\frac{m^*(A_k\cap[-x_n,x_n])}{g_1(x_n)} = \lim_{n\to\infty}\frac{2x_n}{2x_n\psi_1(2x_n)} = \infty > a,$$

a contradiction.

Let

$$\varepsilon_k = \limsup_{x\to 0^+}\frac{m^*(A_k\cap[-x,x])}{g_1(x)}$$

and

$$\eta_k = \limsup_{x\to 0^+}\frac{m^*(B_k\cap[-x,x])}{g_2(x)}$$

for each $k\in\mathbb{N}$.

Theorem 9. *Let* $E\subset\mathbb{R}$ *such that* $\liminf_{x\to 0^+}\frac{m^*(E\cap[-x,x])}{g_2(x)} = 0$. *If* $\lim_{k\to\infty}\varepsilon_k = 0$, *then* $\liminf_{x\to 0^+}\frac{m^*(E\cap[-x,x])}{g_1(x)} = 0$.

Proof. We may assume that all $\varepsilon_k<\infty$. Then $\liminf_{x\to 0^+}\frac{m^*(A_k\cap[-x,x])}{g_1(x)} \le \varepsilon_k < \infty$ for each $k\in\mathbb{N}$. Thus by lemma 8, $(\mathbb{R}\setminus A_k^+)\cap\left(0,\frac{1}{m}\right)\neq\emptyset$ for any $k\in\mathbb{N}$ and $m\in\mathbb{N}$.

We shall show that for each $k\in\mathbb{N}$ there exists $z_k\in\left(0,\frac{1}{k}\right)$ such that

$$m^*(E\cap[-z_k,z_k]) < \left(\varepsilon_k+\frac{1}{k}\right)g_1(z_k).$$

Let $k\in\mathbb{N}$. By lemma 7, we have that there exists a sequence $\{y_n^k\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} y_n^k = 0$ and

$$\limsup_{n\to\infty}\frac{m^*(E\cap[-y_n^k,y_n^k])}{g_1(y_n^k)} \le \limsup_{n\to\infty}\frac{m^*(A_k\cap[-y_n^k,y_n^k])}{g_1(y_n^k)}.$$

Moreover,

$$\limsup_{n\to\infty}\frac{m^*(A_k\cap[-y_n^k,y_n^k])}{g_1(y_n^k)}\le\limsup_{x\to0^+}\frac{m^*(A_k\cap[-x,x])}{g_1(x)}=\varepsilon_k<\varepsilon_k+\frac{1}{k}.$$

Hence,

$$\limsup_{n\to\infty}\frac{m^*(E\cap[-y_n^k,y_n^k])}{g_1(y_n^k)}<\varepsilon_k+\frac{1}{k}.$$

Therefore, there exists $n_0\in\mathbb{N}$ such that

$$m^*(E\cap[-y_n^k,y_n^k])<\left(\varepsilon_k+\frac{1}{k}\right)g_1(y_n^k)$$

for $n>n_0$. We chose $n>n_0$ such that $y_n^k\in\left(0,\frac{1}{k}\right)$ and put $z_k=y_n^k$. Then

$$m^*(E\cap[-z_k,z_k])<\left(\varepsilon_k+\frac{1}{k}\right)g_1(z_k).$$

Analogously we can prove the following theorem.

Theorem 10. *Let $E\subset\mathbb{R}$ such that $\liminf_{x\to0^+}\frac{m^*(E\cap[-x,x])}{g_1(x)}=0$. If $\lim_{k\to\infty}\eta_k=0$, then $\liminf_{x\to0^+}\frac{m^*(E\cap[-x,x])}{g_2(x)}=0$.*

By theorems 9 and 10 and by lemmas 1 and 2, we have the following:

Theorem 11. *If $\lim_{k\to\infty}\eta_k=0=\lim_{k\to\infty}\varepsilon_k$, then $\tau_{\psi_1}=\tau_{\psi_2}$.*

Let

$$\mathcal{O}^*=\{U\setminus Z : U \text{ is an open set in the Euclidean topology and } m(Z)=0\}.$$

($\mathcal{O}^*$ is the so-called Hashimoto topology considered for the σ-ideal of sets of measure zero.)

Theorem 12. $\bigcap_{\psi\in\mathcal{C}}\tau_\psi=\mathcal{O}^*$.

P r o o f. By theorem 4, we have that $\mathcal{T}_\psi\subset\tau_\psi$ for all $\psi\in\mathcal{C}$. Thus, $\bigcap_{\psi\in\mathcal{C}}\mathcal{T}_\psi\subset\bigcap_{\psi\in\mathcal{C}}\tau_\psi$. Moreover, by theorem 2.11 [4], $\bigcap_{\psi\in\mathcal{C}}\mathcal{T}_\psi=\mathcal{O}^*$. Hence $\mathcal{O}^*\subset\bigcap_{\psi\in\mathcal{C}}\tau_\psi$.

Suppose now that there exists a set $A\in\bigcap_{\psi\in\mathcal{C}}\tau_\psi\setminus\mathcal{O}^*$. Then there exists $x\in A$ such that $m((\mathbb{R}\setminus A)\cap[x-t,x+t])>0$ for each $t>0$. Set

$E=\{y-x : y\in\mathbb{R}\setminus A\}$ and $f(t)=\frac{m(E\cap[-t,t])}{2t}$, for each $t>0$. Then the function f is continuous and $f(t)>0$ for each $t>0$. Moreover, by theorem 4, the set $A\in d$. Therefore, 0 is a point of dispersion of the set E and $\lim_{t\to 0^+} f(t)=0$.

Let

$$q(x)=\begin{cases} f\left(\frac{1}{2}x\right) & \text{for} \quad x>0,\\ 0 & \text{for} \quad x=0.\end{cases}$$

Set $x_0=2$ and $x_n=\max\left\{x\in[0,x_{n-1}] : \ q(x)=\frac{1}{2}q(x_{n-1})\right\}$ for each $n\in\mathbb{N}$. It is easy to see that $q(x)\geq q(x_n)$ for any $n\in\mathbb{N}$ and $x\in[x_n,x_{n-1}]$, and $\lim_{n\to\infty} x_n=0$. Let $\psi\in\mathcal{C}$ such that

$$\psi(x)=\begin{cases} \frac{1}{2}q(2) & \text{for} \quad x\geq 2,\\ \frac{1}{2}q(x_{n-1})) & \text{for} \quad x\in[\frac{1}{2}(x_n+x_{n-1}),x_{n-1}] \text{ and } n\in\mathbb{N},\\ \text{linear} & \text{for} \quad \text{the remaining } x\in(0,2).\end{cases}$$

Then $\psi(2x)\leq q(2x)=f(x)$ for $x\in(0,1]$. Thus,

$$\frac{m(E\cap[-t,t])}{2t\psi(2t)}\geq 1$$

for each $t\in(0,1]$ and the set $E\notin\psi-S(0)$. Therefore, $A\notin\tau_\psi$, a contradiction.

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