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SOME REMARKS ON MAPPINGS THAT PRESERVE UNIT DISTANCE

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Abstract
In the present paper we investigate some properties on isometric mappings between Euclidean spaces. In addition, non-isometric distance one preserving mappings are also considered.

1. Introduction

We begin with the definition of an isometry: Let X, Y be two metric spaces and d_1, d_2 the distances in X and Y, respectively. A mapping $f : X \to Y$ is defined as an *isometry* if

$$d_2(f(x), f(y)) = d_1(x, y)$$

for all elements x, y of X. If f is surjective, then the inverse mapping $f^{-1} : Y \to X$ is also an isometry. Two metric spaces X and Y are defined to be *isometric* if there exists an isometry of X onto Y. Thus, it follows that an isometry is an isomorphism for the metric space structures.

Consider the following condition for a mapping $f : X \to Y$ which is referred to as distance one preserving property (DOPP) (cf. [2], [6]).

$$\textit{Let } x, y \in X \textit{ with } d_1(x, y) = 1. \textit{ Then } d_2(f(x), f(y)) = 1. \qquad \text{(DOPP)}$$

In 1970, A.D. Alexandrov posed the following problem: *Under what condition is a map of a metric space into itself preserving unit distance an isometry ?*

In the present work, we discuss some properties and research problems concerning unit distance preserving mappings.

2. Isometries between Euclidean spaces

Let E^n denote the n-dimensional real Euclidean space. Beckman and Quarles [1] proved that if $E^n \to E^n$ for $2 \leq n < \infty$ satisfies the condition (DOPP), then f is an isometry[1].

This property does not hold for E^1, the Euclidean line. A simple counterexample is following:

Let $f : E^1 \to E^1$ be defined by

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ an integer point,} \\ x & \text{otherwise.} \end{cases}$$

This mapping satisfies the condition (DOPP) but is not an isometry. However, the mapping is not continuous. This property does not also hold for E^∞, a Hilbert space (cf. [9] for counterexample).

It is interesting to examine what happens when the mapping is required to be continuous. In E^1 the transformation

$$f : x \to [x] + \{x\}^2$$

(where $[x]$ denotes the integer part of x and $\{x\} = x - [x]$) is continuous and satisfies the condition (DOPP) but is not an isometry. It is not known yet what happens in E^∞ with the additional condition of continuity of the mapping. Rassias [8] conjectured that such a mapping, satisfying the condition (DOPP), must be an isometry.

3. Non-isometric distance one preserving mappings

If $f : E^n \to E^m$ preserves some distance, it follows that $n \leq m$. This is true because E_m has equilateral n-dimensional simplices if and only if $n \leq m$. It remains to examine the case when $1 < n < m < \infty$.

In the following we explain a method allowing us to construct examples and to prove that for each n there exists an m and a mapping $f : E^n \to E^m$

[1] For non-Euclidean spaces the Beckman-Quarles property has been derived by the Russian school, especially by A.V. Kuz'minykh [4] (see also [7]).

which is distance one preserving but not an isometry. The following examples illustrate the case of a mapping $f : E^2 \to E^8$ (cf. [6]). For this purpose, consider partitioning the plane E^2 into squares of unit diagonal as follows (*Figure 1*):

	9	7	8	9	
2	3	1	2	3	1
5	6	4	5	6	4
8	9	7	8	9	7
2	3	1	2	3	1
	6	4	5	6	

Figure 1

Each square contains the bottom edge, the left edge and the bottom left corner but none of the other corners. Next label the nine vertices of the regular simplex in E^8 with the edge length 1 and map each square labelled i to the ith vertex. This mapping satisfies the condition (DOPP) but is not an isometry.
Using hexagons instead of squares, we can construct such mapping from $E^2 \to E^6$ (*Figure 2*).

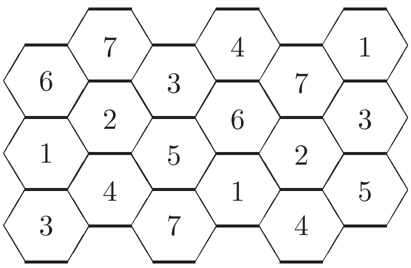

Figure 2

In this case, there exist sets $C_1, C_2, \ldots, C_7$ such that

(a) $E^2 = C_1 \cup C_2 \cup \ldots \cup C_7,\ C_i \cap C_j = \emptyset,\ i \neq j;$

(b) The ends of each segment of unit length in E^2 belong to distinct sets C_i.

Let now $v_1, v_2, \dots, v_7$ be the vertices of a regular unit simplex in E^6. Put $f(x) = v_i$ if $x \in C_i,\ i = 1, 2, \dots, 7$, then mapping $f : E^2 \to E^6$ satisfies the condition (DOPP) but is not an isometry.

Remark. A partition of the type (b) will be called colouring of the space by the colours $C_1, C_2, \dots$.

This idea can be easily extended to dimension $n > 2$ (cf. [3]). For this purpose, let ϱ satisfy the following condition

$$\frac{1}{[\sqrt{n}\,] + 1} < \varrho < \frac{1}{\sqrt{n}}.$$

Take a cube with an edge length $([\sqrt{n}\,] + 1)\varrho$ in E^n. We will call it "big" cube. Divide it into $([\sqrt{n}\,]+2)^n$ "small" cubes with an edge length ϱ. Colour the interior of each small cube by one of $([\sqrt{n}\,] + 2)^n$ colours, using distinct colours for distinct cubes. Paving E^n by the translates of the big coloured cube, one coloures E^n. Colouring the faces of the small cubes (in E^n) is non-essential as long as the colour is passed from one of the adjacent small cubes. From the condition (a) above, it follows that no unit segment can have its ends in one small cube or in distinct small cubes of the same colour. As above, it results in a non-isometric distance one preserving mapping $f : E^n \to E^{([\sqrt{n}]+2)^n - 1}$.

Theorem ([6]). *For any integer $n \geq 1$, there exists an integer n_m such that $N \geq n_m$ implies that there exists a mapping $f : E^n \to E^N$ which satisfies the condition (DOPP) but is not an isometry.*

Some further questions can be asked. For instance: What is the minimum m, $2 \leq n < m$, for which a non-isometric distance one preserving mapping $E^n \to E^m$ exists? Also, it is still an open problem whether or not there is a distance one preserving mapping $f : E^2 \to E^3$ which is not an isometry (cf. [5]).

4. Summary

The theory of isometry had its beginning in the important paper published in 1932 by S. Mazur and S. Ulam who proved that every isometry of a normed real vector space onto another normed real vector space is a linear mapping up to translation. The general problem of isometry in metric spaces is considered in this paper. We present various results concerning the problem of distance one preserving mappings.

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