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PRINCIPLE OF TRIPLE ENTRY IN THE THEORY OF ACCOUNTING

Introduction

For a long time (L. Pacioli [7]) the principle of double entry has practically dominated in the theory of accouting, i.e.: it has been commonly and entirely applied.

But now, more and more often there appears the necessity of numerical recording of an event also in the third place besides recording it in two accounts on opposite sides.

The purpose of this work is the presentation of uniform theory of triple entry where all three elements of this entry are equivalent.

The consideration will be founded on basical assumptions /axioms/ – similarly as in refs. [2] and [3], definitions of notions and theorems concerning certain facts will generalize – in certain meaning – the theory of double entry. The analogies to ref. [3] will be stressed and denotations – where it is possible – will be identical.

Due to the above the new concept will be contained in the well known form.

The formalization of fundamentals of the theory of triple entry can facilitate the preparation of a computer program and application of computer for coding and analyzing the data, i.e.: receiving the different information of economical character.

It is expected that the shortening of algebraic operations performed in practice can be obtained, following the synthetic presentation of the theory that can create real advantages.

In this work the following denotation will be used:

Q – empty set,

E – set of real numbers,

N – set of natural numbers,

$N(s) = \{1,2,\ldots,s\}$ for $s \in N$,

$A^2 = A \times A$,

$A^2_+ = \{(a,a) : a \in A\}$,

$A^2_- = A^2 - A^2_+$,

$A^3 = A \times A \times A$,

$A^3_+ = \{(a,b,c) : a,b,c \in A \wedge (a=b \vee b=c \vee c=a)\}$,

$A^3_- = A^3 - A^3_+$,

$l(A)$ – number of elements of finite set A,

$T(x) = \{t : t \in T,\ t \leq x \in E$,

$N_o = N \cup \{0\}$.

We introduce now theoretical fundamentals of the account system.

Axiom 1. Let the set

$$A = \{a_1, a_2, \ldots, a_m\}$$

of m elements ($3 \leq m \in N$) be given.

Definition 1. Every element $a \in A$ is called „an account" and the set A – „a system of accounts"

Axiom 2. Let the set

$$T = \{t_1, t_2, \ldots, t_n\} E,\ (n \in N)$$

is such that $\alpha = t_1 < t_2 < \ldots < t_n = \beta$, $\alpha, \beta \in E$ and $\alpha < \beta$.

Definition 2. The real numbers $t_1, t_2, \ldots, t_n$ are called distinguished time instances and the number $t_s(s \in N(n))$ is specially called the s-th distinguished time instance.
Every number $x \in <\alpha, \beta>$ is called time instance.

Axiom 3. Let the mapping

$$\sigma : T \rightarrow B,$$

where $B = A^3 - A^3_+$ be given.

Definition 3. We define the sets:

$$B^* = \sigma(T),$$

$$Z = T \circ B^* = \{(t,q) : t \in T,\ q \in B^*,\ q = \sigma(t)\}.$$

Every element of the set Z is called a partial event.

It is seen that set Z of all partial events is a subset of the set $T \times B$ because $T \circ B^* \subset T \times B^*$ and $B^* \subset B$.

The partial event:

$$z=(t,q) \text{ where } q=\sigma(t)=(a,b,c),\ a,b,c \in A,$$

(a,b,c – different accounts) will be denoted by:

$$(t,(a,b,c)),$$

and even by:

$$(t,a,b,c).$$

It does not cause misunderstanding because between the pairs (t,/a,b,c)) and quadruplets (t,a,b,c) the ideal correspondence occurs.

From definition 3 it results – for such denotation – that partial event is an orderly quadruplet of elements composed of distinguished time instances $t \in T$ and three different accounts $a,b,c \in A$ the orderly triplet $(a,b,c) \in B^*$ of which is the image of the number t for the mapping $\sigma:(a,b,c)=\sigma(t)$.

Then there exists the equality:

$$l(Z)=n$$

because $l(T)=n$.

Definition 4. If $z_1, z_2 \in Z$ and $z_1=(t_i,\sigma(t_i))$, $z_2=(t_j,\sigma(t_j))$, $(i,j) \in N^2_{\cdot}(m)$, $i<j$ we will say that the event z_1 is earlier than the event z_2 and we will write $z_1<z_2$ (see[3], def.3).

One can then prove:

Theorem 1. For every pair of different partial events $(z,u) \in Z^2_{\cdot}$ only one of the following relations holds:

$$z=u,\ z<u,\ u<z.$$

We mention here that denotation $z=u$ means that there exists such $i \in N(n)$ for which $z=(t_i,\sigma(t_i))$ and $u=(t_i,\sigma(t_i))$, i.e.: that the partial events z, u are identical.

The existence of equivalent mapping

$$\tau: Z \rightarrow N(n)$$

such that $\tau(t_s,q)=s$, $(s \in N(n))$
enables us to numerate the partial events by the numbers 1,2,...,n.

Definition 5. The mapping (1) is called chronology of partial events of the set Z. One can prove (see [5]) the theorem of transitivity of relation <: to be earlier event".

Definition 6. In the set Z of partial events the relation of similarity $\sim$ is defined in the following way:
if $z=(t_i,\sigma(t_i))$, $u=(t_j,\sigma(t_j))$, $(i,j)\in N^2(m)$
then

$$z\sim u\Leftrightarrow\sigma(t_i)=\sigma(t_j).$$

It is immediately seen that the relation of similarity $\sim$ of partial events is equivalent ($z\sim z$), symmetrical ($z\sim u\Rightarrow u\sim z$) and transitive ($z\sim w\wedge w\sim u\Rightarrow z\sim$) in the set Z, and therefore it divides the set Z in separable classes, so called abstraction classes.
Every class consists of only such pairs (t,q) (i.e.: all quadruplets (t,a,b,c)) that have the same second element of the pair (or in the quadruplet: the second, third and fourth elements).
Therefore, for the established $q\in B^*$ the following denotation can be introduced:

$$\{(t,q):t\in T,\ q=\sigma(t)\}=Z_q. \tag{2}$$

Here, Z_q is the class of equivalence determined by an orderly triplet $q=(a,b,c)$ of different accounts $a,b,c\in A$ such that $(a,b,c)=\sigma(t)$ for certain $t\in T$ (or for more values of $t\in T$).

Definition 7. Every class Z_q defined by equality (2) for established $q=B^*$ is called an event (or: q – event).
One can notice that the class Z_q univocally points the triplet $q\in B^*$ and vice versa.
Because of $B^*=\sigma(T)$ and $T=\sigma^{-1}(B^*)$
it means that

$$l(B^*)=l(W)$$

where

$$W=\{Z_q:q\in B^*\}, \tag{3}$$

namely, W is the set of all events.
From the above it results that identification of the class Z_q with the triplet q and the set W with the set B^* is permissible.
For such agreement, from definition 7 it results that the event is every element of the set $B^*=\sigma(T)$, i.e.: the orderly triplet $q=(a,b,c)\in B^*$ of different accounts for which there exists the distinguished time instance $t\in T$ (or more of such instances), so that $q=\sigma(t)$.

Axiom 4. The mapping

$$\varphi:Z\to U\subset N$$

is given.

For $z=(t_s,a_i,a_j,a_k)\in Z$ where $a_i,a_j,a_k\in A$, $(i,j,k)\in N^3_-(m)$, $s\in N(n)$ and

$(a_i,a_j,a_k)=\sigma(t_s)$

we introduced denotation

$$\varphi(z)=u^s_{ijk}.$$

Natural numbers u^s_{ijk} create „the matrix" with four indices of dimensions n,m^3 (it means: n,m,m,m) but it is „an incomplete" matrix because the triplet (i,j,k) is the only image of the number s.
The complete matrix M of dimensions n,m^3 has $n\cdot m^3$ elements and „an incomplete" matrix M^*, considered here, has elements where $s=n=l(Z)$.

3. Partial book – keeping event, book – keeping event, flow, three sides of account, traffic, balancing.

Definition 8. The partial book – keeping event is every pair

$$(z,\varphi),\ z\in Z.$$

Every natural number in the form

$$\varphi(z)=u^s_{ijk}\in U \text{ for } z\in Z$$

is called a book – keeping event of the type (s,i,j,k) and this notation is called left for the account $a_i\in A$, central – for the account $a_j\in A$ and right for the account $a_k\in A$.

Definition 9. The mapping

$$\Phi: B^*\to V\subset N$$

is defined as the following:

for $t\in T$ and $q=(a_i,a_j,a_k)=\sigma(t)\in B^*$, $(i,j,k)\in N^3_-(m)$
we have

$$\Phi(q)=\sum_{t\in T}\varphi((t,q))=v_{ijk}\in V. \qquad (4)$$

Definition 10. The book – keeping event is every couple:

$$(q,\Phi),\ q\in B^*.$$

Every number in the form $\Phi(q)=v_{ijk}$ for $q\in B^*$ is called the flow of the type (i,j,k) – left for the account $a_i\in A$, central – for the account $a_j\in A$ and right for the account $a_k\in A$.

Theorem 2.

$$v_{ijk}=\sum_{t_s\in T}u^s_{ijk},\ (s\in N(n),\ (i,j,k)\in N^3_-(m)).$$

It means: the flow of the type (i,j,k) is the sum of all book – keeping entries of the type (s,i,j,k) calculated with respect to all $s \in N(n)$ such that $\sigma(t_s) \doteq (a_i, a_j, a_k)$.

Proof. From equality (4) we have

$$v_{ijk} = \Phi((a_i, a_j, a_k)) = \sum_{t \in T} \varphi((t, (a_i, a_j, a_k))) =$$

$$= \sum_{t_s \in T} \varphi((t_s, (a_i, a_j, a_k))) = \sum_{t_s \in T} u^s_{ijk}.$$

Definition 11. For established $i \in N(m)$ the set of all left (central or right) entries of the account is called left (central, right) side of the account $a_i \in A$. The set of all left (central, right) flows of the account $a_i \in A$ is called left (central, right) side of flows of that account.

If L_i, M_i, P_i denote the left, central or right side of the account $a_i \in A$ and L^i, M^i, P^i – the left, central and right side of flows of the account $a \in A$, respectively, then:

$$L_i = \{u^s_{ijk} : s \in N(n)\ ,\ (j,k) \in N^2_-(m)\},$$

$$M_i = \{u^s_{jik} : s \in N(n),\ (j,k) \in N^2_-(m)\},$$

$$P_i = \{u^s_{jki} : s \in N(n),\ (j,k) \in N^2_-(m)\}$$

and

$$L^i = \{v_{ijk} : (j,k) \in N^2_-(m)\},$$

$$M^i = \{v_{jik : (j,k) \in N^2_-}(m)\},$$

$$P^i = \{v_{jki} : (j,k) \in N^2_-(m)\}.$$

It can happen that one of the listed sets $L_i, M_i, P_i, L^i, M^i, P^i$ is empty, or some of them are empty.

We then say that the corresponding side of account or account flows is empty, e.g.: when $M^i = Q$ we say that the central side of flows of the account $a_i \in A$ is empty.

Definition 12. The traffic of the left (central, right) side of the established account $a_i \in A$ is a sum of all left (central, right) entries of this account.

If a side of the account is empty (the set of entries is empty) the traffic will be defined as zero.

If λ_i, μ_i, and π_i denote the traffic of the left, central and right side of the account $a_i \in A$, respectively, we have:

$$\lambda_i = \sum_{t_s \in T} \sum_{j \in N(m)} \sum_{k \in N(m)} u^s_{ijk},$$

$$\mu_i = \sum_{t_s \in T} \sum_{j \in N(m)} \sum_{k \in N(m)} u^s_{jik},$$

$$\Pi_i = \sum_{t_s \in T} \sum_{j \in N(m)} \sum_{k \in N(m)} u^s_{jki}.$$

Definition 13. The number

$$l_i = \omega_i - \lambda_i \quad (m_i = \omega_i - \mu_i, \; p_i = \omega_i - \Pi_i),$$

where

$$\omega_i = \max(\lambda_i, \mu_i, \Pi_i)$$

is called left (central, right) balance of the account $a_i \in A$.
It is clearly seen that from among three balances l_i, m_i, p_i of the account $a_i \in A$ at least one of them is equal to zero and none is negative, it means:

$$l_i, m_i, p_i \geqslant 0$$

$$l_i \cdot m_i \cdot p_i = 0$$

Using the symbol of mapping φ (compare axiom 4) the traffic of sides of the established account $a_i \in A$ can be expressed as the following:

$$\lambda_i = \sum_{t \in T} \sum_{b \in A} \sum_{c \in A} \varphi((t, (a_i, b, c))),$$

$$\mu_i = \sum_{t \in T} \sum_{b \in A} \sum_{c \in A} \varphi((t, (b, a_i, c))),$$

$$\Pi_i = \sum_{t \in T} \sum_{b \in A} \sum_{c \in A} \varphi((t, (b, c, a_i))),$$

where the summation is performed for all elements of the sets T and A for which $\varphi(z)$ is defined.
The traffic of the empty side is obviously equal to zero.
One can notice that for the established account $a_i \in A$ the sum of traffic and balance is the same for all three sides of the account,namely:

$$\lambda_i + l_i = \mu_i + m_i = \Pi_i + p_i = \omega_i.$$

It makes possible to close the account with the same number on each side.

Definition 14. The traffic oA of the system A of accounts is defined as follows:

$$oA = \sum_{q \in B^*} \Phi(q).$$

Theorem 3. The sum of traffics of the left (central, right) side of all accounts of the set A is equal to the traffic of the system A:

$$\sum_{i\in N(m)} \nu_i = oA,$$

where one should substitute λ, μ or Π for operator ν.

Proof. On the basis of equality (4) and definition 14 we have, for $(a,b,c) \in B^*$

$$\sum_{i\in N(m)} \lambda_i = \sum_{i\in N(m)} \sum_{t\in T} \sum_{b\in A} \sum_{c\in A} \varphi((t,(a_i,b,c)))=$$

$$= \sum_{i\in N(m)} \sum_{b,c\in A} \Big(\sum_{t\in T} \varphi((t,(a_i,b,c))) \Big) =$$

$$= \sum_{i\in N(m)} \sum_{b,c\in A} \Phi((a_i,b,c)) = \sum_{q\in B^*} \Phi(q) = oA.$$

For operators μ, Π the consideration is performed in the same way.

Definition 15. For every established element $(x,a)\in I\times A$ we define the following functions of natural values:

$$f(x,a) = \sum_{t\in T(x)} \sum_{b,c\in A} \varphi((t,a,b,c)),$$

$$g(x,a) = \sum_{t\in T(x)} \sum_{b,c\in A} \varphi((t,b,a,c)),$$

$$h(x,a) = \sum_{t\in T(x)} \sum_{b,c\in A} \varphi((t,b,c,a)),$$

$$\omega(x,a) = \max(f(x,a),\ g(x,a),\ h(x,a)),$$

$$\gamma(x,a) = \omega(x,a) - f(x,a),$$

$$\psi(x,a) = \omega(x,a) - g(x,a),$$

$$\chi(x,a) = \omega(x,a) - h(x,a),$$

$$\Omega(x,A) = \sum_{q\in B} \sum_{t\in T(x)} \varphi((t,q)).$$

These functions are called as follows: state of traffic of the left, central and right side of account, maximal function, state of balances of the left, central and right side of account $a \in A$ and state of traffic of the system A at the time instance $x \in <\alpha, \beta>$.

Theorem 4. For every time instance $x \in I$ there occurs equality:

$$\sum_{a \in A} \delta(x, a) = \Omega(x, A),$$

where instead of δ we put f, g or h.

Proof.
$$\sum_{a \in A} f(x, a) = \sum_{a \in A} \sum_{t \in T(x)} \sum_{b, c \in A} \varphi((t, a, b, c)) =$$

$$= \sum_{a, b, c \in A} \sum_{t \in T(x)} \varphi((t, a, b, c)) =$$

$$= \sum_{q \in B^*} \sum_{t \in T(x)} \varphi((t, q)) = \Omega(x, A).$$

For g and h the consideration is performed in the same way as for f.
Theorem 4 is the consequence of theorem 3 that can be now obtained from theorem 4 putting $x = \beta$.
Because the functions $f, g, h, \omega, \gamma, \psi, \chi$ are defined by the product $I \times A$, then for the established account $a \in A$ we have:

$$f(x, a) = F(x),$$

$$g(x, a) = G(x),$$

$$h(x, a) = H(x),$$

$$\omega(x, a) = K(x),$$

$$\gamma(x, a) = L(x),$$

$$\psi(x, a) = P(x),$$

$$\chi(x, a) = Q(x).$$

The functions F, G, H, K, L, P, Q that are mapping the range I to the set $N^* \subset N_o$ are constant within the ranges and in the points of jumps (which number is not larger than n) they are continuous with respect to the right hand side term (see [3], def. 13).
Similar behaviour exhibits the function $\Gamma(x) = \Omega(x, A)$.

From definitions 12 – 15 it results that for the given account $a_i \in A$ the traffic (balances) is the state of traffic (state of balances) in the time instant $x=\beta$:

$$\lambda_i = f(\beta, a_i),$$

$$\mu_i = g(\beta, a_i),$$

$$\Pi_i = h(\beta, a_i),$$

$$\omega_i = \omega(\beta, a_i),$$

$$l_i = \gamma(\beta, a_i),$$

$$m_i = \psi(\beta, a_i),$$

$$p_i = \chi(\beta, a_i).$$

Also:

$$oA = \Omega(\beta, A).$$

Definition 16. The orderly triplet (P, Q, R) of the following sets is called the balance of traffic of the system A at the time instance $x \in I$:

$$P = \{f(x, a) : a \in A\},$$

$$Q = \{g(x, a) : a \in A\},$$

$$R = \{h(x, a) : a \in A\}.$$

The final balance of traffic of the system A is the balance of traffic at the time instance $x=\beta$, i.e.: the orderly triplet (P_*, Q_*, R_*) of the sets

$$P_* = \{\lambda_i : i \in N(m)\},$$

$$Q_* = \{\mu_i : i \in N(m)\},$$

$$R_* = \{\Pi_i : i \in N(m)\}.$$

We obviously have $K_* = \{k(\beta, a) : a \in A\}$, where instead of the pair (K, k) one can substitute the pair (P, f) or the pair (Q, g) as well as the pair (R, h). From theorem 4 it immediately results:

Theorem 5. For every time instance $x \in I$ the balance equation holds:

$$\sum_{a \in A} f(x, a) = \sum_{a \in A} g(x, a) = \sum_{a \in A} h(x, a).$$

One can observe thet the balance equation is a consequence of triple entry: every entry exists on the left, central and right side of three different accounts of the system A.

Definition 17. The balance of balances of the system A at the time instance $x \in I$ we call the orderly triplet (S^1, S^2, S^3) of sets:

$$S^1 = \{\gamma(x,a) : a \in A\},$$

$$S^2 = \{\psi(x,a) : a \in A\},$$

$$S^3 = \{\chi(x,a) : a \in A\}.$$

The final balance of balances we call the balance of balances at a time instance $x = \beta$, i.e.: the orderly triplet (S^1_*, S^2_*, S^3_*) of sets:

$$S^1_* = \{l_i : i \in N(m)\},$$

$$S^2_* = \{m_i : i \in N(m)\},$$

$$S^3_* = \{p_i : i \in N(m)\}.$$

Theorem 6. For every time instance $x \in I$ the balance equation of balances holds:

$$\sum_{a \in A} \gamma(x,a) = \sum_{a \in A} \psi(x,a) = \sum_{a \in A} \chi(x,a).$$

Proof. On the basis of definition 15 and theorem 4 we have:

$$\sum_{a \in A} \gamma(x,a) = \sum_{a \in A} (\omega(x,a) - f(x,a)) = \sum_{a \in A} \omega(x,a) - \Omega(x,A).$$

Also

$$\sum_{a \in A} \psi(x,a) = \sum_{a \in A} \chi(x,a) = \sum_{a \in A} \omega(x,a) - \Omega(x,A).$$

Theorem 6 results from the fact that $\gamma = \omega - f$, $\psi = \omega - g$ and $\chi = \omega - h$. Similarly as in ref. [3] in our consideration we introduce the stiffening by assumption that not all relations between accounts are permissible.

Let

$$(i^1_1, i^1_2, \ldots, i^1_{j_1}),$$

....

$$(i^l_1, i^l_2, \ldots, i^l_{j_l}), \; j_1 + j_2 + \ldots + j_l \leq m$$

are subseries with no common elements of the series:

$$(1, 2, \ldots, m).$$

If we assume

$$((a^u_{i_k}, a^v_{i_1}, a^w_{i_m}) = Б(t), t \in T) \Rightarrow (u-v)(v-w)(w-u) \neq 0,$$

we will get the stiffening of the system as in ref. [3].

In this work we used theoretical fundamentals of accouting ([1], [4], [10], [11]), the attempts of its axiomatization ([6], [8], [9] and [2], [3]) and other sources ([5], [7], [11]) connected with the subject.
We assume that the subject (enterprise) acts within the determined closed period of time $<\alpha,\beta>$.
The initial entries that are called the opening balance are considered similarly as the latter ones: they are classed into the set of partial events.
In this paper we only present the formal attempt to the problem but it is possible to fill them up with economical content.
Additional information can also make it possible.
In such attempt as the presented one one can not decide on the number of accounts and whether all relations betwen them are permissible.
The further works will connect in the greater extent the theory and practice and will also contain the theoretical generalization.

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STRESZCZENIE

Przedmiotem pracy jest zasada zapisu potrójnego w teorii rachunkowości, której podstawy ujmuje się w sposób dedukcyjny, oparty na aksjomatach.
Wymieniona zasada polega na tym, że liczbowy obraz zdarzenia gospodarczego wpisuje się trzykrotnie: na trzech różnych stronach (lewej, środkowej i prawej) trzech różnych kont rozpatrywanego systemu.
Zdarzenie cząstkowe określa się jako uporządkowaną czwórkę (t,a,b,c), złożoną z chwili t oraz jej obrazu (a,b,c).

Zapis księgowy jest to para (z, φ), gdzie z jest zdarzeniem cząstkowym zaś φ – odwzorowaniem tego z na liczbę naturalną.
Zdarzenie jest to trójka uporządkowana $q=(a,b,c)$ kont, a zdarzenie księgowe określa się jako parę (q, Φ), gdzie Φ odwzorowuje q na liczbę rzeczywistą.
Bilans określa się jako uporządkowaną trójkę (P,Q,R) zbiorów.
Wprowadza się relację: „być zdarzeniem cząstkowym wcześniejszym" i pokazuje, że jest ona tranzytywna, formułuje się i dowodzi: związek między zapisami księgowymi i strumieniami, równanie bilansowe, równanie bilansowe sald oraz przedstawia się bilansowanie w dowolnej chwili rozpatrywanego przedziału czasu.