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ON CLUSTER SETS OF CONNECTED FUNCTIONS

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Abstract
In the article we discuss some properties of a few classes of connected functions and cluster sets of functions from those classes.

1. Preliminaries

We shall consider some properties of functions defined in a topological space X with values in a topological space Y.

Definition 1. *We shall say that a function* $f\colon X \longrightarrow Y$ *has the Darboux property if the image of connected subset of* X *is connected.*

The set of all functions which have the Darboux property will be denoted by $\mathscr{D}$.

Definition 2. *We shall say that a function* $f\colon X \longrightarrow Y$ *has the local Darboux property if for each point* x *in* X *and every neighbourhood* U *of* x *there exists a connected neighbourhood* V *of* x *such that* $V \subset U$ *and* $f(V)$ *is a connected subset of* Y.

The set of all functions which have the local Darboux property will be denoted by $\mathscr{D}_l$.

Definition 3. *We shall say that a function* $f : X \longrightarrow Y$ *is connected if its graph is a connected subset of* $X \times Y$.

The set of all functions which are connected will be denoted by $\mathscr{C}$.

Definition 4. *We shall say that a function $f : X \longrightarrow Y$ is strongly connected if $f|E$ is a connected set for each connected subset E of X.*

The set of all functions which are strongly connected will be denoted by $\mathscr{C}_s$.

Definition 5. *We shall say that a function $f : X \longrightarrow Y$ is locally strongly connected if for each x in X and its open neighbourhood U there exists open and connected neighbourhood E of x such that $E \subset U$ and $f|E$ is a connected set in the space $X \times Y$.*

The set of all functions which are strongly connected will be denoted by $\mathscr{C}_{ls}$.

By a subgraph of a function $f\colon X \longrightarrow \mathbb{R}$ we mean the set

$$\{(x, y) \in X \times \mathbb{R} : y < f(x)\},$$

which is denoted by $f(-)$.

By an overgraph of a function $f\colon X \longrightarrow \mathbb{R}$ we mean the set

$$\{(x, y) \in X \times \mathbb{R} : y > f(x)\},$$

which is denoted by $f(+)$.

We shall make no distinction between a function and its graph.

Definition 6. *We shall say that a function $f\colon X \longrightarrow \mathbb{R}$ cuts continuum if*

$$f \cap M \neq \emptyset$$

for each continuum M for which $M \cap f(+) \neq \emptyset$ and $M \cap f(-) \neq \emptyset$.

The definitions 3, 4, 5 and 6 define the same class of functions when X and Y are equal to $\mathbb{R}$ with natural topology.

Similarly, definitions 1 and 2 define the same class of functions when X and Y are equal to $\mathbb{R}$ with natural topology.

All the notions undefined in the article are taken from books [1] and [7].

2. First properties

In the article we shall discuss some properties of those classes and give some sufficient conditions for the space X in which real functions defined in X form the same class.

Immediately form the definitions the next properties follow:

Property 1. *Each continuous function is strongly connected and has the Darboux property.*

Property 2. *Each continuous function defined in a connected space is connected.*

Property 3. *Each continuous function defined in a locally connected space is locally strongly connected and has the local Darboux property.*

Theorem 1. *For every topological spaces X and Y*

$$\mathscr{C}_s \subset \mathscr{D}.$$

Theorem 2. *If a topological spaces X is connected and Y is an arbitrary topological space, then*

$$\mathscr{C}_s \subset \mathscr{C}.$$

This theorem can be completed to get sufficient condition for a space X to be connected.

Theorem 3. *If a topological space Y has at least two elements and for topological space X*

$$\mathscr{C}_s \subset \mathscr{C},$$

then X is connected.

Theorem 4. *If a topological spaces X is locally connected and Y is an arbitrary topological space, then*

$$\mathscr{C}_s \subset \mathscr{C}_{ls}, \quad \mathscr{C}_s \subset \mathscr{D}_l, \quad \mathscr{C}_{ls} \subset \mathscr{D}_l, \quad \mathscr{D} \subset \mathscr{D}_l.$$

Theorem 5. *If a topological spaces X is connected and locally connected and Y is an arbitrary topological space, then*

$$\mathscr{C}_{ls} \subset \mathscr{C}.$$

At the end of this part, let us consider real functions defined in a topological space X.

The following lemma need not any proof.

Lemma 1. *If a function $f\colon X \longrightarrow \mathbb{R}$ cuts continuum and $E \subset X$, then $f|E$ cuts continuum as well.*

Lemma 2. *Let $\phi\colon Y \longrightarrow X$ be a homeomorphism of the topological space Y onto X. If a function $f\colon X \longrightarrow \mathbb{R}$ cuts continuum, then the function $f \circ \phi$ also cuts continuum.*

P r o o f. Let K be a continuum which has points in common with overgraph and undergraph of the function $f \circ \phi$. It means that there are two points (y_1, r_1) and (y_2, r_2) in K such that

$$r_1 < f \circ \phi(y_1) \quad \textit{and} \quad r_2 > f \circ \phi(y_2).$$

Let now $x_1 = \phi(y_1)$ and $x_2 = \phi(y_2)$ and

$$K^* = \left\{(x, r) \in X \times \mathbb{R} \,:\, (\phi^{-1}(x), r) \in K\right\}.$$

The set K^* is also a continuum (in the space $X \times \mathbb{R}$) which has common points with undergraph and overgraph of the function f. Then there exists a point $(x_0, f(x_0))$ in K^* so $(\phi^{-1}(x_0), f(x_0)) \in K$.

In this way, we have proved that the function $f \circ \phi$ cuts continuum. □

Property 4. *If X is an arcwise connected topological space, then every function $f\colon X \longrightarrow \mathbb{R}$ which cuts continuum has a connected graph.*

P r o o f. Let $(x_1, f(x_1))$ and $(x_2, f(x_2))$ be two points of the graph of function f. Since X is arcwise connected, then there is an arc L whose ends are x_1 and x_2. Let $\phi\colon [0,1] \longrightarrow L$ be a homeomorphism of $[0,1]$ onto L. The function $f|L$ cuts continuum, then also $f|L \circ \phi$ cuts continuum. It follows from the article [2] that a function $f|L \circ \phi$ has connected graph. Hence the function $f|L$ has connected graph.

In this way, we have proved that every two points of the graph of f can be joined by a connected set (contained in this graph); thus the graph of f is connected. □

Property 5. *Let X be a locally connected and locally arcwise connected topological space. If $f\colon X \longrightarrow \mathbb{R}$ cuts continuum, then it is locally strongly connected function.*

P r o o f. Let x_0 be any point of the space X and U be its open neighbourhood. Let V be an open and connected neighbourhood of x_0 contained in U. Let $(x_1, f(x_1))$ and $(x_2, f(x_2))$ be two arbitrary points of the graph of the function $f|V$. There exists an arc L contained in V such that x_1 and x_2 are its ends. The function $f|L$ cuts continuum, hence it has a connected graph. It follows, then, that $f|L$ has a connected graph. Therefore, $f|V$ is also a connected set, which completes the proof. □

As a consequence of this property, we can state:

Property 6. *Let X be a locally connected and locally arcwise connected topological space. If a function $f\colon X \longrightarrow \mathbb{R}$ cuts continuum, then it has the local Darboux property.*

Property 7. *Let X be a locally compact and locally connected topological space. If a function $f\colon X \longrightarrow \mathbb{R}$ cuts continuum, then it has the local Darboux property.*

P r o o f. Let x_0 be any point of X and U its open connected neighbourhood. Let a_1 and a_2 be two arbitrary points from $f(U)$ such that $a_1 < a_2$. Let $d \in (a_1, a_2)$. There are points x_1 and x_2 such that

$$x_1 \in U, \quad x_2 \in U, \quad a_1 = f(x_1), \quad a_2 = f(x_2).$$

Let x be an arbitrary point from the set U. For each poit x from U there exists an open and connected set U_x such that $x \in U_x$, $U_x \subset U$ and $\overline{U_x}$ is compact. From local connectedness of the space X, we can find a connected open set V_x such that

$$x \in V_x, \quad V_x \subset U_x.$$

Hence

$$U = \bigcup_{x\in U} V_x.$$

Since U is an open set, then there exists a finite sequence $(t_1, \dots, t_n)$ of points from U such that

$$x_1 \in V_{t_1}, \quad x_2 \in V_{t_n}$$

and

$$V_{t_i} \cap V_{t_j} \neq \emptyset \iff |t_i - t_j| \leq 1.$$

The sets $\overline{V_{t_i}}$ are connected and compact, then the set K, where

$$K = \bigcup_{i=1}^{n} \overline{V_{t_i}} \times \{d\},$$

is a continuum which has common points with overgraph and undergraph of the function f. Then there exists a common point of K and f. Hence $d \in f(U)$, which completes the proof. □

3. Cluster sets

By $\mathscr{B}_x$ we shall denote the class of all open neighbourhood of the point x from X. Let $f : X \longrightarrow Y$ be an arbitrary function.

Definition 7. *The set $C(f,x)$ defined by (see [5])*

$$C(f,x) = \bigcap_{U\in\mathscr{B}_x} \overline{f(U)} \tag{1}$$

is called the cluster set *of the function f at the point x from X.*

Definition 8. *The set $L(f,x)$ defined by (see [5])*

$$L(f,x) = \bigcap_{U\in\mathscr{B}_x} \overline{f(U\setminus\{x\})} \tag{2}$$

is called the set of limit points *of the function f at the point x from X.*

Definition 9. *The set $R(f,x)$ defined by (see [5])*

$$R(f,x) = \bigcap_{U\in\mathscr{B}_x} f(U) \tag{3}$$

is called the range set *of the function f at the point x from X.*

If X is dense in itself topological space, then the difference between $C(f,x)$ and $L(f,x)$ is given by the next formula

$$C(f,x) = L(f,x)\cup\{f(x)\}.$$

It is quite easy to see that $y_0\in C(f,x_0)$ if and only if there exists a net (MS-sequence, generalized sequence) $\{x_\sigma\}_{\sigma\in\Sigma}$ such that $x_\sigma\in X$ if $\sigma\in\Sigma$ and

$$x_\sigma\to x_0 \quad\text{and}\quad f(x_\sigma)\to y_0.$$

Moreover, $y_0\in L(f,x_0)$ if and only if there exists a net (MS-sequence, generalized sequence) $\{x_\sigma\}_{\sigma\in\Sigma}$ such that $x_\sigma\in X\setminus\{x_0\}$ if $\sigma\in\Sigma$ and

$$x_\sigma\to x_0 \quad\text{and}\quad f(x_\sigma)\to y_0,$$

and $y_0\in R(f,x_0)$ if and only if there exists a net $\{x_\sigma\}_{\sigma\in\Sigma}$ such that

$$x_\sigma\to x_0,\quad f(x_\sigma)=y_0 \quad\text{and}\quad x_\sigma\in X\setminus\{x_0\},\quad\text{if}\quad \sigma\in\Sigma.$$

Theorem 6. *Let X be a locally connected topological space such that x_0 is an accumulation point of X. If a function $f\colon X\longrightarrow\mathbb{R}$ has the local Darboux property, then*

$$f(x_0)\in L(f,x_0).$$

P r o o f. Let $x_0 \in X$.
If

$$f(x_0) \in \overline{f(U \setminus \{x_0\}},$$

then there is nothing to prove.

Now let us suppose that

$$f(x_0) \notin \overline{f(U \setminus \{x_0\}}.$$

There exists a local basis $\mathscr{B}_{x_0}$ of X at x_0 consisting of open and connected sets. The set $\mathscr{B}_{x_0} \times \mathbb{N}_+$ is directed by relation

$$(U, n) \prec (V, m) \iff (V \subset U \land n \leq m).$$

Definition 8 implies (in view of [5]) that the set $L(f, x_0)$ is nonempty. There exists y_0 in $L(f, x_0)$ different from $f(x_0)$. Let us suppose that $y_0 < f(x_0)$ and assume that $d = f(x_0) - y_0$. For each U from $\mathscr{B}_{x_0}$, we have:

$$y_0 \in f(U)$$

and the set $f(U)$ is connected containing $f(x_0)$ and, moreover, for every positive integer n there exists a point $x_{(U,n)}$ such that

$$x_{(U,n)} \in U \setminus \{x_0\}, \quad f\left(x_{(U,n)}\right) = f(x_0) - \frac{d}{n}.$$

In that way we have defined a net $\left\{x_{(U,n)}\right\}_{(U,n)\in\mathscr{B}_{x_0}\times\mathbb{N}_+}$ which is convergent to x_0 and such that $\left\{f\left(x_{(U,n)}\right)\right\}_{(U,n)\in\mathscr{B}_{x_0}\times\mathbb{N}_+}$ is a net convergent to $f(x_0)$, which ends the proof. □

As a simple consequence of this theorem, one can get:

Corollary 1. *Let X be a locally connected topological space such that x_0 is an accumulation point of X. If a function $f\colon X \longrightarrow \mathbb{R}$ has the local Darboux property, then*

$$C(f, x_0) = L(f, x_0).$$

Theorem 7. *Let X be a locally connected and dense in itself topological space. If $f\colon X \longrightarrow \mathbb{R}$ has the local Darboux property, then the set $L(f, x)$ is connected for every x in X.*

P r o o f. Let $x_0 \in X$ and suppose that there are two points y_1 and y_2 such that

$$y_1 \in L(f, x_0), \quad y_2 \in L(f, x_0) \quad \text{and} \quad y_1 < y_2.$$

Let c be any number lying between y_1 and y_2. For each U from a local basis $\mathscr{B}_{x_0}$ of X at x_0 consisting of open and connected sets,

$$y_1 \in \overline{f(U)} \quad \text{and} \quad y_2 \in \overline{f(U)}.$$

Since the sets $\overline{f(U)}$ are connected (i.e. intervals) for each U from $\mathscr{B}$, then $c \in \overline{f(U)}$ as well. Hence

$$c \in L(f, x_0),$$

which completes the proof. □

Applying the notion of a range set of a function and the proof of the previous theorem, we can obtain:

Corollary 2. *Let X be a locally connected and dense in itself topological space. If $f\colon X \longrightarrow \mathbb{R}$ has the local Darboux property and $y_1 \in L(f, x)$, $y_2 \in L(f, x)$, where $y_1 < y_2$, then*

$$(y_1, y_2) \subset R(f, x)$$

for each point x from the space X.

Corollary 3. *Let X be a locally connected and dense in itself topological space. If $f\colon X \longrightarrow \mathbb{R}$ has the local Darboux property, then the set $L(f, x)$ is closed, connected set containing $f(x)$ for each point x from the space X.*

Applying first properties of discussed classes of functions, we can obtain the following corollaries.

Corollary 4. *Let X be a locally connected and dense in itself topological space. If $f\colon X \longrightarrow \mathbb{R}$ is strongly connected, then the set $L(f, x)$ is a closed, connected set containing $f(x)$ for each point x from the space X.*

Corollary 5. *Let X be a locally connected, locally compact and dense in itself topological space. If $f\colon X \longrightarrow \mathbb{R}$ cuts continuum, then the set $L(f, x)$ is a closed, connected set containing $f(x)$ for each point x from the space X.*

Corollary 6. *Let X be a locally connected, locally arcwise connected and dense in itself topological space. If $f\colon X \longrightarrow \mathbb{R}$ cuts continuum, then the set $L(f, x)$ is a closed, connected set containing $f(x)$ for each point x from the space X.*

Theorem 8. *If X is a locally connected, dense in itself topological space and $f\colon X \longrightarrow \mathbb{R}$ has the local Darboux property, then*

$$\operatorname{Int} L(f,x) \subset R(f,x)$$

for each point x from the space X.

P r o o f. Of course, $f(x) \in R(f,x)$ for each point x if $\mathscr{B}_x$ is a local base for x consisting of open and connected sets.

If $y \in \operatorname{Int} L(f,x)$, then there are points u and v such that

$$u < y < v, \quad u \in L(f,x) \quad \text{and} \quad v \in L(f,x).$$

Let U be an arbitrary set from $\mathscr{B}_x$. Then

$$u \in \overline{f(U)} \quad \text{and} \quad v \in \overline{f(U)}.$$

Since $f(U)$ is a connected set (it is an interval), then $\overline{f(U)}$ is also an interval and there are points u_1 and v_1 such that

$$u < u_1 < y < v_1 < v \quad \text{and} \quad u_1 \in f(U) \quad \text{and} \quad u_2 \in f(U).$$

From connectivity of the set $f(U)$, we can infer that $y \in f(U)$.

Since U is arbitrary set from the base $\mathscr{B}_x$, then $y \in R(f,x)$. $\square$

In view of connections among connected classes of functions, we can get the following corollaries:

Corollary 7. *If X is a locally connected, dense in itself topological space and $f\colon X \longrightarrow \mathbb{R}$ has the Darboux property, then*

$$\operatorname{Int} L(f,x) \subset R(f,x)$$

for each point x from the space X.

Corollary 8. *If X is a locally connected, dense in itself topological space and $f\colon X \longrightarrow \mathbb{R}$ is strongly connected, then*

$$\operatorname{Int} L(f,x) \subset R(f,x)$$

for each point x from the space X.

Corollary 9. *If X is a locally connected, dense in itself topological space and $f\colon X \longrightarrow \mathbb{R}$ is locally strongly connected, then*

$$\operatorname{Int} L(f,x) \subset R(f,x)$$

for each point x from the space X.

Corollary 10. *If X is a locally connected, locally compact dense in itself topological space and $f\colon X \longrightarrow \mathbb{R}$ cuts continuum, then*

$$\operatorname{Int} L(f,x) \subset R(f,x)$$

for each point x from the space X.

Corollary 11. *If X is a locally connected, locally arcwise connected and dense in itself topological space and $f\colon X \longrightarrow \mathbb{R}$ cuts continuum, then*

$$\operatorname{Int} L(f,x) \subset R(f,x)$$

for each point x from the space X.

Gillespie [3] gave a sufficient condition for a real function of a real variable to have the Darboux property. Here we can formulate the analogous theorem for function defined in a topological space.

Theorem 9. *Let a topological space X be locally connected. If for a function $f\colon X \longrightarrow \mathbb{R}$ the sets $C(f,x)$ are connected and*

$$C(f,x) \subset R(f,x)$$

for all x, then $f(U)$ is connected for each connected subset U of X.

P r o o f. Let us suppose that there exists an open and connected set U such that $f(U)$ is not connected. Then there are points y_1, y_2 and y such that

$$y_1 < y < y_2, \quad y_1 \in f(U), \quad y_2 \in f(U) \quad \text{and} \quad y \notin f(U).$$

Let

$$A = U \cap f^{-1}\left((-\infty, y)\right) \quad \text{and} \quad B = U \cap f^{-1}\left((y, \infty)\right).$$

The sets A and B are complements (with respect to U), then $\operatorname{Fr} A \neq \emptyset$ (with respect to U). The sets A and B are not separated, then there exists a point x_0 belonging to $\operatorname{Fr} A$. There exist two nets $(\alpha_\sigma)_{\sigma\in\Sigma}$ and $(\beta_\lambda)_{\lambda\in\Lambda}$ such that

$$\alpha_\sigma \in A, \quad \beta_\lambda \in B, \quad \alpha_\sigma \to x_0 \quad \text{and} \quad \beta_\lambda \to x_0.$$

The nets $\left(f\left(\alpha_\sigma\right)\right)_{\sigma\in\Sigma}$ and $\left(f\left(\beta_\lambda\right)\right)_{\lambda\in\Lambda}$ have their elements in the intervals $(-\infty, y)$ and (y, ∞), respectively. There exist two subnets $(f(\alpha'_\sigma))_{\sigma'\in\Sigma'}$ and $\left(f(\beta'_\lambda)\right)_{\lambda'\in\Lambda'}$ converging to some points y_1 and y_2, where

$$y_1 \in [-\infty, y] \quad \text{and} \quad y_2 \in [y, \infty].$$

In each case

$$y_1 \leq y \leq y_2.$$

Thus

$$y_1 \in L(f, x_0) \quad \text{and} \quad y_2 \in L(f, x_0).$$

From the assumptions we infer that $y \in L(f, x)$ and hence $y \in R(f, x)$. Therefore, $y \in f(U)$ which contradicts the assumptions.

Immediately from this theorem, one can get the following corollary:

Corollary 12. *Let X be a locally connected topological space. If for a function $f\colon X \longrightarrow \mathbb{R}$ the sets $C(f, x)$ are connected and*

$$C(f, x) \subset R(f, x)$$

for all x, then f has the local Darboux property.

In view of properties of connected functions and examples from the article [6], it is impossible to give similar characterization for all remained classes of functions.

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