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THE PROJECTIVE REPRESENTATIONS TYPES OF FINITE GROUPS OVER A RING OF FORMAL POWER SERIES

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Abstract

Let F be a field of characteristic $p > 0$, $S = F[[X]]$, S^* the unit group of S, and W a subgroup of S^*. We characterize finite groups depending on a projective (S, W)-representation type. We also give necessary and sufficient conditions for a finite group and its Sylow p-subgroups to be of the same projective (S, W)-representation type.

1. Introduction

Throughout this paper, we use the following notations: $p \geq 2$ is a prime; $\mathbb{N}$ is the set of all positive integers; F is a field of characteristic $p > 0$; $S = F[[X]]$ is the F-algebra of formal power series in the indeterminate X with coefficients in F; S^* is the unit group of S; W is a subgroup of S^*; $Z^2(G, W)$ is the group of all W-valued normalized 2-cocycles of the group G that acts trivially on W; G is a finite group of order $|G|$; e is the identity element of G; G' is the commutant of G; G_p is a Sylow p-subgroup of G; C_p is a Sylow p-subgroup of G'. We assume that $C_p \subset G_p$, hence $G'_p \subset C_p$.

Given a cocycle $\lambda\colon G \times G \to S^*$ in $Z^2(G, S^*)$, we denote by $S^\lambda G$ the twisted group ring of the group G over the ring S with the cocycle λ. An S-basis $\{u_g\colon g \in G\}$ of $S^\lambda G$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in G$ is called natural. If H is a subgroup of G, then the restriction of a cocycle $\lambda\colon G \times G \to S^*$ to $H \times H$ will also be denoted by λ. In this case $S^\lambda H$ is a subring of $S^\lambda G$. By an $S^\lambda G$-module we mean a finitely generated left $S^\lambda G$-module which is S-free, that is, an $S^\lambda G$-lattice (see [6, p. 140]). We denote by $[M]$ the isomorphism class of $S^\lambda G$-modules that contains an $S^\lambda G$-module M. Moreover, by $\mathrm{Ind}_d(S^\lambda G)$ we denote the set of all $[V]$, where V is an indecomposable $S^\lambda G$-module of S-rank d.

If M is an $S^\lambda G$-module, then we denote by M_H the module M viewed as an $S^\lambda H$-module. If N is an $S^\lambda H$-module, then $N^G = S^\lambda G \otimes_{S^\lambda H} N$ is the induced $S^\lambda G$-module.

If W is a subgroup of F^*, then $i_F(W)$ is the supremum of the set that consists of 0 and all positive integers m such that an F-algebra of the form

$$F[X]/(X^p - \alpha_1) \otimes_F \ldots \otimes_F F[X]/(X^p - \alpha_m)$$

is a field for some $\alpha_1, \ldots, \alpha_m \in W$.

In this paper, we continue the characterization of finite groups depending on a projective representation types as begun in [3], [4].

In Section 2, we present a number of propositions about the representations types of twisted group rings which are based on the results of Gaschütz [7] on relative projective and injective modules over group rings (see [5, pp. 426-430], [6, pp. 449-453]). In Section 3, we single out finite groups of every projective representation type in a sense of definitions in paper [3] (see also Section 3). We prove that if G is a finite group and $|C_p| > 2$, then G is of purely strongly unbounded projective (S, S^*)-representation type (Proposition 7). Assume that $p \neq 2$ and W is a subgroup of F^*. A group G is of purely strongly unbounded projective (S, W)-representation type if and only if $|C_p| \neq 1$ or G_p is a direct product of l cyclic subgroups, where $l \geq i_F(W) + 1$ (Theorem 1). We also establish that if $p = 2$ and $|C_2| \neq 2$, then G is of purely strongly unbounded projective (S, F^*)-representation type if and only if one of the following conditions holds: 1) $|C_2| > 2$; 2) G_2 is a direct product of l cyclic subgroups, where $l \geq i_F(F^*) + 2$; 3) G_2 is a direct product of $i_F(F^*) + 1$ cyclic subgroups whose orders are not equal to 2 (Theorem 2).

In Section 4, we characterize a finite group G such that G and G_p are of the same projective representation type over $S = F[[X]]$. If $C_p = G'_p$ or $|G'_p| > 2$, then the groups G and G_p are of the same projective (S, W)-representation type for any subgroup W of the group S^* (Proposition 10). Let $p \neq 2$, W be a subgroup of F^*, $|C_p| \neq 1$ and $|G'_p| = 1$. We prove that the groups G and G_p are of the same projective (S, W)-representation type if and only if G_p is a direct product of r cyclic subgroups, where $r \geq i_F(W) + 1$ (Proposition 11). Let $p = 2$, G be a finite group such that $|C_2| > 2$ and $|G'_2| = 1$. We establish that the groups G and G_2 are of the same projective

(S, F^*)-representation type if and only if one of the following conditions is satisfied:

(i) G_2 is a direct product of l cyclic subgroups, where $l \geq i_F(F^*) + 2$;

(ii) G_2 is a direct product of $i_F(F^*) + 1$ cyclic subgroups whose orders are not equal to 2 (Proposition 12).

We remark that our investigations were considerably stimulated by the well-known Brauer-Thrall conjectures for finite-dimensional algebras over an arbitrary field (see [1, p. 138] for a formulation of the conjectures).

2. The representation types of twisted group rings

Proposition 1. *Let G be a finite group, G_p a Sylow p-subgroup of G, $\lambda \in Z^2(G, S^*)$ and M an $S^\lambda G$-module. Then M is isomorphic to an $S^\lambda G$-component of $(M_{G_p})^G$.*

The proof of the Proposition 1 is similar to the proof of the analogous proposition for KG-modules, where K is a field of characteristic $p > 0$ (see [5, pp. 429-430]).

Proposition 2. *Let G be a finite group, H a subgroup of G, $\lambda \in Z^2(G, S^*)$ and W an $S^\lambda H$-module. Then W is isomorphic to an $S^\lambda H$-component of $(W^G)_H$.*

The proof of the Proposition 2 is the same as the proof of analogous proposition for KG-modules, where K is a field of characteristic $p > 0$ (see [5, p. 430]).

We recall that $S^\lambda G$ is of *finite* (resp. *infinite*) *representation type* if the set of all isomorphism classes of indecomposable $S^\lambda G$-modules is finite (resp. infinite). Let $\mathrm{D}(S^\lambda G)$ be the set of S-ranks of all indecomposable $S^\lambda G$-modules. If $\mathrm{D}(S^\lambda G)$ is finite (resp. infinite), then $S^\lambda G$ is of *bounded* (resp. *unbounded*) *representation type*. We say that $S^\lambda G$ is of SUR-type (*Strongly Unbounded Representation type*) if there exists a function $f_\lambda \colon \mathbb{N} \to \mathbb{N}$ such that $f_\lambda(n) \geq n$ and $\mathrm{Ind}_{f_\lambda(n)}(S^\lambda G)$ is an infinite set for every $n > 1$.

Proposition 3. *Let G be a finite group and $\lambda \in Z^2(G, S^*)$. Then $S^\lambda G$ is of finite (resp. infinite) representation type if and only if $S^\lambda G_p$ is of finite (resp. infinite) representation type.*

P r o o f. Apply Propositions 1 and 2.

Proposition 4. *Let G be a finite group and $\lambda \in Z^2(G, S^*)$. Then $S^\lambda G$ is of bounded (resp. unbounded) representation type if and only if $S^\lambda G_p$ is of bounded (resp. unbounded) representation type.*

P r o o f. Apply Propositions 1 and 2.

Proposition 5. *Let G be a finite group and $\lambda \in Z^2(G, S^*)$. Then $S^\lambda G$ is of* SUR*-type if and only if $S^\lambda G_p$ is of* SUR*-type*

P r o o f. Assume that $S^\lambda G$ is of SUR-type. Then there exists an infinite subset T of the set $\mathbb{N}$ such that $\mathrm{Ind}_n(S^\lambda G)$ is infinite for any $n \in T$. Let $[M] \in \mathrm{Ind}_n(S^\lambda G)$. In view of Proposition 1, M is isomorphic to an $S^\lambda G$-component of $(M_{G_p})^G$. Hence there is an indecomposable $S^\lambda G_p$-module W such that $W^G \cong M \oplus V$ for some $S^\lambda G$-module V and $n|G|^{-1} \leq d \leq n$, where d is the S-rank of W. It follows that there exists a natural number $d_\lambda(n)$ such that

$$n|G|^{-1} \leq d_\lambda(n) \leq n$$

and $\mathrm{Ind}_{d_\lambda(n)}(S^\lambda G_p)$ is an infinite set for every $n \in T$. Consequently $S^\lambda G_p$ is of SUR-type.

Conversely, let $S^\lambda G_p$ be of SUR-type. Suppose that $\mathrm{Ind}_n(S^\lambda G_p)$ is infinite for any $n \in \Omega$, where Ω is an infinite subset of $\mathbb{N}$. Let $[V] \in \mathrm{Ind}_n(S^\lambda G_p)$. By Proposition 2, V is isomorphic to $S^\lambda G_p$-component of $(V^G)_{G_p}$. It follows that there exists an indecomposable $S^\lambda G$-module M such that $n \leq \dim M \leq n \cdot |G|$ and $M_{G_p} \cong V \oplus W$ for some $S^\lambda G_p$-module W. The preceding arguments shows that there exists a function $f_\lambda \colon \Omega \to \mathbb{N}$ such that $n \leq f_\lambda(n) \leq n \cdot |G|$ and $\mathrm{Ind}_{f_\lambda(n)}(S^\lambda G)$ is an infinite set for every $n \in \Omega$. Therefore $S^\lambda G$ is of SUR-type.

□

Lemma 1 (see [3, pp. 277, 279])**.** *Let G_p be a finite p-group, $S = F[[X]]$ and $\lambda \in Z^2(G, F^*)$.*

(i) If $p \neq 2$ and the algebra $F^\lambda G$ is not semisimple, then the ring $S^\lambda G$ is of SUR*-type.*

(ii) If $p = 2$ and the algebra $F^\lambda G$ is not semisimple, then the set $\mathrm{Ind}_l(S^\lambda G)$ is infinite for some $l \leq |G|$.

Lemma 2 (see [3, p. 280])**.** *Let G_p be a finite p-group, $S = F[[X]]$ and $\lambda \in Z^2(G, S^*)$. Assume that G contains a subgroup H such that $|H| > 2$ and the restriction of λ to $H \times H$ is a coboundary. Then $S^\lambda G$ is of* SUR*-type.*

Let G_p be a finite p-group, W a subgroup of S^*, $\lambda \colon G_p \times G_p \to W$ a 2-cocycle. Denote by $\mathrm{Ker}(\lambda)$ the union of all cyclic subgroups $\langle g \rangle$ of G_p such that the restriction of λ to $\langle g \rangle \times \langle g \rangle$ is a W-valued coboundary. The set $\mathrm{Ker}(\lambda)$ is called the *kernel* of $\lambda \in Z^2(G_p, W)$ (see [3, p. 269]). We recall that $G'_p \subset \mathrm{Ker}(\lambda)$, $\mathrm{Ker}(\lambda)$ is a normal subgroup of G_p, and up to cohomology in $Z^2(G, W)$ $\lambda_{g,a} = \lambda_{a,g} = 1$ for all $g \in G$, $a \in \mathrm{Ker}(\lambda)$.

Let G be a finite group, G_p a Sylow p-subgroup of G, C_p a Sylow p-subgroup of G' and $C_p \subset G_p$. Assume that $\lambda \in Z^2(G, W)$ and μ is the restriction of λ to $G_p \times G_p$. Then $C_p \subset \mathrm{Ker}(\mu)$ [9, p. 42].

Suppose that G is a finite group and $p \mid |G'|$. The group G/G' is a direct product of its Sylow q-subgroups G_qG'/G', where G_q is a Sylow q-subgroup of G and q is a prime divisor of $|G : G'|$. The group G_p/C_p is isomorphic to the G_pG'/G'. Assume that $\varphi\colon G \to G/G'$ is the canonical homomorphism, $\psi\colon G/G' \to G_pG'/G'$ is a projector and $\chi\colon G_pG'/G' \to G_p/C_p$ is the isomorphism defined by $\chi(gG') = gC_p$ for every $g \in G_p$. Then

$$f := \chi\psi\varphi \tag{1}$$

is a homomorphism of G onto G_p/C_p. The restriction of f to G_p is the canonical homomorphism of G_p onto G_p/C_p.

Lemma 3. *Let $f\colon G \to H$ be the homomorphism (1), where $H = G_p/C_p$. If W is a subgroup of S^*, $\mu \in Z^2(H, W)$ and*

$$\lambda_{a,b} = \mu_{f(a),f(b)}$$

for all $a, b \in G$, then $\lambda\colon (a, b) \mapsto \lambda_{a,b}$ belongs to $Z^2(G, W)$ and $\lambda_{x,y} = \lambda_{y,x} = 1$ for all $x \in G_p$, $y \in C_p$. Moreover, if $\{u_g : g \in G\}$ is a natural S-basis of $S^\lambda C_p = SC_p$, then the set

$$V = \sum_{g \in C_p, g \neq e} S^\lambda G_p(u_g - u_e)$$

is an ideal of the ring $S^\lambda G_p$ and $S^\lambda G_p/V \cong S^\mu H$.

P r o o f. Direct calculation.

3. Projective representation types of finite groups

We recall from [2] that a *projective (S, W)-representation* of the group G of degree n is a mapping $\Gamma\colon G \to \mathrm{GL}(n, S)$ such that $\Gamma(e) = E$ and $\Gamma(a)\Gamma(b) = \lambda_{a,b}\Gamma(ab)$, where $\lambda_{a,b} \in W$ for all $a, b \in G$. It is easy to see that $\lambda\colon (a, b) \mapsto \lambda_{a,b}$ belongs to $Z^2(G, W)$. We also say that Γ is a *projective (S, W)-representation of G with cocycle* λ. Two projective (S, W)-representations Γ_1 and Γ_2 of G are called *equivalent* if there exists an invertible matrix C over S and elements $\alpha_g \in W$ ($g \in G$) such that $C^{-1}\Gamma_1(g)C = \alpha_g\Gamma_2(g)$ for every $g \in G$. If $W = S^*$, then Γ is called a *projective S-representation* of G. If $W = \{1\}$, then Γ is called a *linear S-representation* of G. By analogy with indecomposable projective S-representations of G (see [9, p. 108]), we can introduce the concept of an indecomposable projective (S, W)-representation of the group G.

Now we recall from [3] the concept of projective representation type of finite group. A group G is said to be of *finite projective* (S,W)*-representation type* if the number of (inequivalent) indecomposable projective (S,W)-representations of G with cocycle λ is finite for any $\lambda \in Z^2(G,W)$. Otherwise, G is of *infinite projective* (S,W)*-representation type*. We say that G is of *purely infinite projective* (S,W)*-representation type*, if the number of indecomposable projective (S,W)-representations of G with cocycle λ is infinite for any $\lambda \in Z^2(G,W)$. A group G is defined to be of *bounded projective* (S,W)*-representation type* if the set of degrees of all indecomposable projective (S,W)-representations of G with cocycle λ is finite for every $\lambda \in Z^2(G,W)$. Otherwise, G is said to be of *unbounded projective* (S,W)*-representation type*. We say that G is of *purely unbounded projective* (S,W)*-representation type* if the set of degrees of all indecomposable projective (S,W)-representations of G with cocycle λ is infinite for each $\lambda \in Z^2(G,W)$. A group G is of *strongly unbounded projective* (S,W)*-representation type* if for some cocycle $\lambda \in Z^2(G,W)$ there is a function $f_\lambda\colon \mathbb{N} \to \mathbb{N}$ such that $f_\lambda(n) \geq n$ and the number of indecomposable projective (S,W)-representations of G with cocycle λ and of degree $f_\lambda(n)$ is infinite for all $n > 1$. If there is such a function f_λ for every $\lambda \in Z^2(G,W)$, then G is said to be of *purely strongly unbounded projective* (S,W)*-representation type*.

Lemma 4 (see [3, p. 283])**.** *Let $S = F[[X]]$, W be a subgroup of F^*, G a finite p-group and G/G' a direct product of r cyclic subgroups, where $r \geq i_F(W)+1$ for $p > 2$ and $r \geq i_F(W)+2$ for $p = 2$. Then G is of purely strongly unbounded projective (S,W)-representation type.*

Lemma 5 (see [3, p. 283])**.** *Let G be a finite Abelian p-group and $S = F[[X]]$.*

(i) Assume that $W \subset F^*$ and $p \neq 2$. Then G is of purely strongly unbounded projective (S,W)-representation type if and only if G is a direct product of l cyclic subgroups, where $l \geq i_F(W)+1$.

(ii) Let $p = 2$. Then G is of purely strongly unbounded projective (S,F^*)-representation type if and only if one of the following conditions is satisfied: 1) G is a direct product of l cyclic subgroups, where $l \geq i_F(F^*)+2$; 2) G is a direct product of $i_F(F^*)+1$ cyclic subgroups whose orders are not equal to 2.

Proposition 6. *Let G be a finite group, $p \mid |G|$, $S = F[[X]]$ and W a subgroup of S^*.*

(i) A group G is of bounded projective (S,W)-representation type if and only if $p = 2$ and $|G_2| = 2$.

(ii) A group G is of unbounded projective (S,W)-representation type if and only if G is of strongly unbounded projective (S,W)-representation type.

P r o o f. (i) If G is of bounded projective (S, W)-representation type, then the group ring SG is of bounded representation type. It follows, by [8], that $p = 2$ and $|G_2| = 2$. Conversely, if $p = 2$ and $|G_2| = 2$ then, by Proposition 6 from [3], the group G_2 is of bounded projective (S, W)-representation type. In view of Proposition 4, G also is of bounded projective (S, W)-representation type.

(ii) If $|G_p| > 2$ then, by Theorem 1 from [3] and Proposition 5, the group ring SG is of strongly unbounded representation type. □

Proposition 7. *Let $S = F[[X]]$ and G be a finite group such that $|C_p| > 2$. Then G is of purely strongly unbounded projective (S, S^*)-representation type.*

P r o o f. Let $\lambda \in Z^2(G, S^*)$ and μ be the restriction of λ to $G_p \times G_p$. Since $C_p \subset \mathrm{Ker}(\mu)$, $S^\mu C_p$ is the group ring of C_p over S. By Lemma 2, the ring $S^\lambda G_p$ is of SUR-type. It follows from this and Proposition 5 that $S^\lambda G$ is of SUR-type for any $\lambda \in Z^2(G, S^*)$. Hence, G is of purely strongly unbounded projective (S, S^*)-representation type. □

Theorem 1. *Let $p \neq 2$ and W be a subgroup of F^*.*

(i) A group G is of purely strongly unbounded projective (S, W)-representation type if and only if $|C_p| \neq 1$ or G_p is a direct product of l cyclic subgroups, where $l \geq i_F(W) + 1$.

(ii) A group G is of purely strongly unbounded projective (S, W)-representation type if and only if G is of purely unbounded projective (S, W)-representation type.

P r o o f. (i) If $|C_p| \neq 1$ then, by Proposition 7, G is of purely strongly unbounded projective (S, W)-representation type. Let $|C_p| = 1$. In view of Lemma 3, for every cocycle $\mu \in Z^2(G_p, S^*)$ there exists a cocycle $\lambda \in Z^2(G, S^*)$ such that the restriction of λ to $G_p \times G_p$ is equal to μ. It follows from this and Proposition 5 that G is of purely strongly unbounded projective (S, W)-representation type if and only if G_p is of purely strongly unbounded projective (S, W)-representation type. Applying Lemma 5, we conclude that G is of purely strongly unbounded projective (S, W)-representation type if and only if G_p is a direct product of l cyclic subgroups, where $l \geq i_F(W) + 1$.

(ii) Let $|C_p| = 1$ and G_p be a direct product of r cyclic subgroups, where $r \leq i_F(W)$. Then there exists a cocycle $\mu \in Z^2(G_p, W)$ such that $F^\mu G_p$ is a field. Let $K = F^\mu G_p$. We have $S^\mu G_p \cong K[[X]]$. It follows that the ring $S^\mu G_p$ is of finite representation type. By Lemma 3, there is a cocycle $\lambda \in Z^2(G, W)$ such that the restriction of λ to $G_p \times G_p$ is equal to μ. In view of Proposition

3, the ring $S^\lambda G$ is of finite representation type. Hence, G is not of purely unbounded projective (S, W)-representation type. □

Theorem 2. *Let $S = F[[X]]$, where F is a field of characteristic 2.*

(i) Let W be a subgroup of F^* and G_2/C_2 a direct product of r cyclic subgroup, where $r \geq i_F(W) + 2$. Then G is of purely strongly unbounded projective (S, W)-representation type.

(ii) Let $|C_2| \neq 2$. A group G is of purely strongly unbounded projective (S, F^*)-representation type if and only if one of the following conditions is satisfied: 1) $|C_2| > 2$; 2) G_2 is a direct product of l cyclic subgroups, where $l \geq i_F(F^*) + 2$; 3) G_2 is a direct product of $i_F(F^*) + 1$ cyclic subgroups whose orders are not equal to 2.

(iii) Let $|C_2| \neq 2$. A group G is of purely strongly unbounded projective (S, F^*)-representation type if and only if G is of purely unbounded projective (S, F^*)-representation type.

P r o o f. (i) By Lemma 4, G_2 is of purely strongly unbounded projective (S, W)-representation type. It follows from this and Proposition 5 that G is of purely strongly unbounded projective (S, W)-representation type.

(ii) If $|C_2| > 2$ then, by Proposition 7, G is of purely strongly unbounded projective (S, F^*)-representation type. Let $|C_2| = 1$. In view of Lemma 3 and Proposition 5, G is of purely strongly unbounded projective (S, F^*)-representation type if and only if G_2 is of purely strongly unbounded projective (S, F^*)-representation type. Applying Lemma 5, we finish the proof.

(iii) Assume that $|C_2| = 1$. If G_2 is a direct product of r cyclic subgroup, where $r \leq i_F(F^*)$, then there exists a cocycle $\mu \in Z^2(G_2, F^*)$ such that $F^\mu G_2$ is a field. It follows that $S^\mu G_2$ is of finite representation type. By Lemma 3, there is a cocycle $\lambda \in Z^2(G, F^*)$ such that the restriction of λ to $G_2 \times G_2$ is equal to μ. In view of Proposition 3, the ring $S^\lambda G$ is of finite representation type.

Let $G_2 = H \times \langle a \rangle$, where $|a| = 2$ and H is a direct product of $i_F(F^*)$ cyclic subgroups. There exists a cocycle $\mu \in Z^2(G_2, F^*)$ such that $F^\mu G_2 = F^\mu H \otimes_F F\langle a \rangle$, where $F^\mu H$ is a field and $F\langle a \rangle$ is the group algebra of $\langle a \rangle$ over F. Let $K = F^\mu H$ and $R = K[[X]]$. Then $S^\mu G_2$ is the group ring $R\langle a \rangle$. It follows, by Proposition 6, that $S^\mu G_2$ is of bounded representation type. By Lemma 3, there is a cocycle $\lambda \in Z^2(G, F^*)$ such that the restriction of λ to $G_2 \times G_2$ is equal to μ. The ring $S^\lambda G$ is of bounded representation type, in view of Proposition 4. Hence, G is not of purely unbounded projective (S, F^*)-representation type. □

4. Isotypic conditions for groups G and G_p

In this Section, we assume that G is a finite group, $p \mid |G|$, G_p is a Sylow p-subgroup of G, C_p is a Sylow p-subgroup of G' and $C_p \subset G_p$.

Two groups are said to be $\mathrm{P}(S,W)\,\mathrm{R}$-*isotypic* if they are of the same projective (S,W)-representation type. From the above results, we will derive necessary and sufficient conditions for G and G_p to be $\mathrm{P}(S,W)\,\mathrm{R}$-isotypic.

Proposition 8. *Let $S = F[[X]]$ and W be a subgroup of F^*. The groups G and G_p are of purely infinite projective (S,W)-representation type if and only if one of the following conditions is satisfied: 1) $|G_p'| \neq 1$; 2) G_p is a direct product of l cyclic subgroups, where $l \geq i_F(W)+1$.*

P r o o f. Suppose that one of the conditions 1), 2) is satisfied. Then an algebra $F^\lambda G_p$ is not semisimple for any $\lambda \in Z^2(G_p,W)$. In view of Lemma 1, the ring $S^\lambda G_p$ is of infinite representation type. Applying Proposition 3, we conclude that $S^\mu G$ is of infinite representation type for each $\mu \in Z^2(G_p,W)$. Hence, the groups G and G_p are of purely infinite projective (S,W)-representation type.

Let G_p be a direct product of r cyclic subgroups, where $r \leq i_F(W)$. Then there is a cocycle $\mu \in Z^2(G_p,W)$ such that $F^\mu G_p$ is a field. Let $K = F^\mu G_p$. We have $S^\mu G_p \cong K[[X]]$, and so every indecomposable $S^\mu G_p$-module is isomorphic to $S^\mu G_p$. Since the ring $S^\mu G_p$ is of finite representation type, the group G_p is not of purely infinite projective (S,W)-representation type. □

Proposition 9. *Let $S = F[[X]]$ and W be a subgroup of S^*.*

(i) The groups G and G_p are of bounded projective (S,W)-representation type if and only if $p = 2$ and $|G_2| = 2$.

(ii) If the groups G and G_p are of unbounded projective (S,W)-representation type, then G and G_p are also of strongly unbounded projective (S,W)-representation type.

P r o o f. Apply Proposition 6.

Proposition 10. *Let G be a finite group and $S = F[[X]]$.*

(i) If $C_p = G_p'$ then the groups G and G_p are $\mathrm{P}(S,W)\,\mathrm{R}$-isotypic for any subgroup W of the group S^*.

(ii) If $|G_p'| > 2$ then G and G_p are of purely strongly unbounded projective (S,S^*)-representation type.

P r o o f. (i) If $C_p = G_p'$ then, by Lemma 3, for every $\mu \in Z^2(G_p,W)$ there exists a cocycle $\lambda \in Z^2(G,W)$ such that the restriction of λ to $G_p \times G_p$ is equal to μ. In view of Propositions 3-5, the rings $S^\lambda G$ and $S^\lambda G_p$ are of the same representation type. Hence the groups G and G_p are $\mathrm{P}(S,W)\,\mathrm{R}$-isotypic.

(ii) If $|G'_p| > 2$ then $|C_p| > 2$. By Proposition 7, G and G_p are of purely strongly unbounded projective (S, S^*)-representation type. □

Proposition 11. *Assume that G is a finite group, $p \neq 2$, $S = F[[X]]$ and W is a subgroup of F^*.*

(i) Let $|C_p| \neq 1$ and $|G'_p| = 1$. The groups G and G_p are $\mathrm{P}(S, W)\,\mathrm{R}$-isotypic if and only if G_p is a direct product of r cyclic subgroups, where $r \geq i_F(W) + 1$.

(ii) Let G_p be a direct product of r cyclic subgroups, where $r \geq i_F(W) + 1$. Then G and G_p are of purely strongly unbounded projective (S, W)-representation type.

P r o o f. (i) If $|C_p| \neq 1$ then $|C_p| > 2$. It follows, by Proposition 7, that G is of purely strongly unbounded projective (S, W)-representation type. In view of Theorem 1, the Abelian group G_p is of purely strongly unbounded projective (S, W)-representation type if and only if G_p is a direct product of r cyclic subgroups, where $r \geq i_F(W) + 1$.

(ii) Apply (i) and Proposition 5. □

Proposition 12. *Let $p = 2$, $S = F[[X]]$, G be a finite group such that $|C_2| > 2$ and $|G'_2| = 1$. The groups G and G_2 are $\mathrm{P}(S, F^*)\,\mathrm{R}$-isotypic if and only if one of the following conditions is satisfied:*

(i) G_2 is a direct product of l cyclic subgroups, where $l \geq i_F(F^*) + 2$;

(ii) G_2 is a direct product of $i_F(F^*) + 1$ cyclic subgroups whose orders are not equal to 2.

Moreover, if one of the conditions (i), (ii) is satisfied, then G and G_2 are of purely strongly unbounded projective (S, F^*)-representation type.

P r o o f. If $|C_2| > 2$ then, by Proposition 7, the group G is of purely strongly unbounded projective (S, F^*)-representation type. By Theorem 2, the group G_2 is of purely strongly unbounded projective (S, F^*)-representation type if and only if one of the conditions (i), (ii) is satisfied. □

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