

ON PREPONDERANTLY CONTINUOUS FUNCTIONS

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Abstract

In the present paper, a few different notions of preponderant continuity of a real function are discussed. We study the relationship between them and give some properties of preponderant continuity.

1. Preliminaries

Let \mathbb{R} , \mathbb{Q} , \mathbb{Q}^+ , \mathbb{Z} be the sets of real numbers, rational numbers, positive rational numbers and integer numbers, respectively. Next, let I denote a closed interval, U any open subset of \mathbb{R} and $\text{Int}(A)$ be an interior of a set $A \subset \mathbb{R}$ in the natural metric. Let λ stand for Lebesgue measure in \mathbb{R} . If $E \subset \mathbb{R}$ is a measurable set, we define the lower and upper densities of E at $x_0 \in \mathbb{R}$ by:

$$\underline{d}(E, x_0) = \liminf_{\lambda(I) \rightarrow 0, x_0 \in I} \frac{\lambda(I \cap E)}{\lambda(I)} \quad \text{and} \quad \bar{d}(E, x_0) = \limsup_{\lambda(I) \rightarrow 0, x_0 \in I} \frac{\lambda(I \cap E)}{\lambda(I)}.$$

If $\underline{d}(E, x_0) = \bar{d}(E, x_0)$, we denote this common value by $d(E, x_0)$ and call it the density of E at x_0 . In a similar way, we also define the one-sided lower and upper densities of the set E at the point x_0 : $\underline{d}^+(E, x_0)$, $\underline{d}^-(E, x_0)$, $\bar{d}^+(E, x_0)$ and $\bar{d}^-(E, x_0)$. It is easy to verify that $\underline{d}(E, x_0) = \min\{\underline{d}^+(E, x_0), \underline{d}^-(E, x_0)\}$ and $\bar{d}(E, x_0) = \max\{\bar{d}^+(E, x_0), \bar{d}^-(E, x_0)\}$.

Definition 1. [1, 3, 4] A point $x_0 \in \mathbb{R}$ is said to be the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ if $\underline{d}(E, x_0) > \frac{1}{2}$.

Definition 2. A point $x_0 \in \mathbb{R}$ is said to be the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ at right (at left, respectively) if $\underline{d}^+(E, x_0) > \frac{1}{2}$ ($\underline{d}^-(E, x_0) > \frac{1}{2}$, respectively).

Corollary 1. A point $x_0 \in \mathbb{R}$ is the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ iff it is the point of preponderant density in Denjoy sense of the measurable set E at right and at left.

Definition 3. [5] A point $x_0 \in \mathbb{R}$ is said to be the point of preponderant density in O'Malley sense of a measurable set $E \subset \mathbb{R}$ if there exists $\varepsilon > 0$ such that for every closed interval I satisfying conditions $x_0 \in I$ and $I \subset [x_0 - \varepsilon, x_0 + \varepsilon]$, the inequality $\frac{\lambda(E \cap I)}{\lambda(I)} > \frac{1}{2}$ holds.

Definition 4. A point $x_0 \in \mathbb{R}$ is said to be the point of preponderant density in O'Malley sense of a measurable set $E \subset \mathbb{R}$ at right (at left, respectively) if there exists $\varepsilon > 0$ such that for every $\delta < \varepsilon$ the inequality $\frac{\lambda(E \cap [x_0, x_0 + \delta])}{\delta} > \frac{1}{2}$ ($\frac{\lambda(E \cap [x_0 - \delta, x_0])}{\delta} > \frac{1}{2}$, respectively) holds.

Corollary 2. A point $x_0 \in \mathbb{R}$ is the point of preponderant density in O'Malley sense of a measurable set $E \subset \mathbb{R}$ iff it is the point of preponderant density in O'Malley sense of the measurable set E at right and at left.

Corollary 3. Let E be a measurable subset of \mathbb{R} and $x_0 \in \mathbb{R}$. If x_0 is the point of preponderant density in Denjoy sense of the set E , then x_0 is the point of preponderant density in O'Malley sense of the set E .

Definition 5. [1, 3, 4] A function $f: U \rightarrow \mathbb{R}$ is said to be preponderantly continuous in Denjoy sense at $x_0 \in U$ if there exists a measurable set $E \subset U$ containing x_0 such that $\underline{d}(E, x_0) > \frac{1}{2}$ and $f|_E$ is continuous at x_0 . A function $f: U \rightarrow \mathbb{R}$ is said to be preponderantly continuous in Denjoy sense if it is preponderantly continuous in Denjoy sense at each point $x_0 \in U$. The class of all functions which are preponderantly continuous in Denjoy sense will be denoted by \mathcal{PD} .

Definition 6. [6] A function $f: U \rightarrow \mathbb{R}$ is said to be preponderantly continuous in O'Malley sense at $x_0 \in U$ if there exists a measurable set $E \subset U$ containing x_0 such that x_0 is the point of preponderant density in O'Malley sense of the set E and $f|_E$ is continuous at x_0 . A function $f: U \rightarrow \mathbb{R}$ is said to be preponderantly continuous in O'Malley sense if it is preponderantly

continuous in O'Malley sense at each $x_0 \in U$. The class of all functions which are preponderantly continuous in O'Malley sense will be denoted by \mathcal{PO} .

Corollary 4.

$$\mathcal{PD} \subset \mathcal{PO}.$$

Grande [5] defined a property of real functions, called the A_1 property. Based on this result, we can define a similar property, which extends the notion of preponderant continuity.

Definition 7. A function $f: U \rightarrow \mathbb{R}$ is said to have A_1 property in Denjoy sense at $x_0 \in U$ if there exist measurable sets $E_1 \subset U$ and $E_2 \subset U$ containing x_0 such that x_0 is the point of preponderant density in Denjoy sense of both sets E_1 and E_2 , $f|_{E_1}$ is upper semicontinuous at x_0 and $f|_{E_2}$ is lower semicontinuous at x_0 . A function $f: U \rightarrow \mathbb{R}$ has A_1 property in Denjoy sense if it has A_1 property in Denjoy sense at each $x_0 \in U$. The class of all functions which have A_1 property in Denjoy sense will be denoted by \mathcal{GPD} .

Definition 8. A function $f: U \rightarrow \mathbb{R}$ is said to have A_1 property in O'Malley sense at $x_0 \in U$ if there exist measurable sets $E_1 \subset U$ and $E_2 \subset U$ containing x_0 such that x_0 is the point of preponderant density in O'Malley sense of both sets E_1 and E_2 , $f|_{E_1}$ is upper semicontinuous at x_0 and $f|_{E_2}$ is lower semicontinuous at x_0 . A function $f: U \rightarrow \mathbb{R}$ has A_1 property in O'Malley sense if it has A_1 property in O'Malley sense at every x_0 . The class of all functions which have A_1 property in O'Malley sense will be denoted by \mathcal{GPO} .

Corollary 5.

$$\mathcal{GPD} \subset \mathcal{GPO}.$$

In the sequel we will often consider "interval sets" at the point x_0 , that is sets of the particular form $E = \bigcup_{n=1}^{\infty} [a_n, b_n]$, where $a_{n+1} < b_{n+1} < a_n$ for every n and $x_0 = \lim_{n \rightarrow \infty} a_n$.

Proposition 1. Let $E = \bigcup_{n=1}^{\infty} [a_n, b_n]$, where $a_{n+1} < b_{n+1} < a_n$ for every n and $x_0 = \lim_{n \rightarrow \infty} a_n$. Then

$$\frac{\lambda([x_0, x] \cap E)}{\lambda([x_0, x])} \geq \frac{\lambda([x_0, a_n] \cap E)}{\lambda([x_0, a_n])} \quad \text{for } x \in [b_{n+1}, b_n]$$

and

$$\frac{\lambda([x_0, x] \cap E)}{\lambda([x_0, x])} \leq \frac{\lambda([x_0, b_{n+1}] \cap E)}{\lambda([x_0, b_{n+1}])} \quad \text{for } x \in [a_{n+1}, a_n].$$

Proof: It is easy to verify that the following inequality $\frac{a}{c} < \frac{a+b}{c+b}$ holds for positive reals $0 < c < a$ and $0 < b$. Moreover, if $x \in [b_{n+1}, a_n]$, then $\lambda([x_0, x] \cap E) = \lambda([x_0, b_{n+1}] \cap E) = \lambda([x_0, a_n] \cap E)$, and if $x \in [a_n, b_n]$, then $\lambda([x_0, x] \cap E) = \lambda([x_0, a_n] \cap E) + \lambda([a_n, x])$. Hence, for $x \in [b_{n+1}, a_n]$ we get

$$\begin{aligned} \frac{\lambda([x_0, b_{n+1}] \cap E)}{\lambda([x_0, b_{n+1}])} &\geq \frac{\lambda([x_0, b_{n+1}] \cap E)}{\lambda([x_0, x])} = \\ &= \frac{\lambda([x_0, x] \cap E)}{\lambda([x_0, x])} = \frac{\lambda([x_0, a_n] \cap E)}{\lambda([x_0, x])} \geq \frac{\lambda([x_0, a_n] \cap E)}{\lambda([x_0, a_n])}. \end{aligned}$$

Similarly, for each $x \in [a_{n+1}, b_{n+1}]$ it is true that

$$\frac{\lambda([x_0, b_{n+1}] \cap E)}{\lambda([x_0, b_{n+1}])} = \frac{\lambda([x_0, x] \cap E) + (b_{n+1} - x)}{\lambda([x_0, x]) + \lambda([x, b_{n+1}])} \geq \frac{\lambda([x_0, x] \cap E)}{\lambda([x_0, x])}$$

and for each $x \in [a_n, b_n]$ we get

$$\frac{\lambda([x_0, x] \cap E)}{\lambda([x_0, x])} = \frac{\lambda([x_0, a_n] \cap E) + (x - a_n)}{\lambda([x_0, a_n]) + \lambda([a_n, x])} \geq \frac{\lambda([x_0, a_n] \cap E)}{\lambda([x_0, a_n])}.$$

It implies required inequalities. \square

Corollary 6. Let $E = \bigcup_{n=1}^{\infty} [a_n, b_n]$, where $b_{n+1} < a_n < a_n$ for every n and $x_0 = \lim_{n \rightarrow \infty} a_n$. Then

$$[1] \quad \underline{d}^+(E, x_0) = \liminf_{n \rightarrow \infty} \frac{\lambda([x_0, a_n] \cap E)}{\lambda([x_0, a_n])} \quad \text{and} \quad \bar{d}^+(E, x_0) = \limsup_{n \rightarrow \infty} \frac{\lambda([x_0, b_n] \cap E)}{\lambda([x_0, b_n])}.$$

[2] The point x_0 is the point of preponderant density in Denjoy sense of the set E iff there exists $n_0 \in \mathbb{N}$ such that $\frac{\lambda(E \cap [x_0, a_n])}{\lambda([x_0, a_n])} > \frac{1}{2}$ for every $n \geq n_0$.

Similar formulas hold for sets of the form $E = \bigcup_{n=1}^{\infty} [c_n, d_n]$, where $d_n < c_{n+1} < d_{n+1}$ for every n and $x_0 = \lim_{n \rightarrow \infty} c_n$.

For a function $f: U \rightarrow \mathbb{R}$, $x_0 \in U$ and $\varepsilon > 0$ we will use the following notation

$$E_{f, x_0, \varepsilon} = \{x \in U : |f(x) - f(x_0)| < \varepsilon\}, \quad E_{f, x_0, \varepsilon}^+ = \{x \in U : f(x) < f(x_0) + \varepsilon\}$$

and $E_{f, x_0, \varepsilon}^- = \{x \in U : f(x) > f(x_0) - \varepsilon\}$.

2. Main results

Proposition 2. Let $f: U \rightarrow \mathbb{R}$ and $x_0 \in U$. Then

1. If f is preponderantly continuous in Denjoy sense at x_0 , then $\underline{d}(E_{f, x_0, \varepsilon}, x_0) > \frac{1}{2}$ for every $\varepsilon > 0$.

2. If f is preponderantly continuous in O'Malley sense at x_0 , then x_0 is the point of preponderant density in O'Malley sense of the set $E_{f,x_0,\varepsilon}$ for every $\varepsilon > 0$.
3. If f has A_1 property in Denjoy sense at x_0 , then $\underline{d}(E_{f,x_0,\varepsilon}^+, x_0) > \frac{1}{2}$ and $\underline{d}(E_{f,x_0,\varepsilon}^-, x_0) > \frac{1}{2}$ for every $\varepsilon > 0$.
4. If f has A_1 property in O'Malley sense at x_0 , then x_0 is the point of preponderant density in O'Malley sense of the sets $E_{f,x_0,\varepsilon}^+$ and $E_{f,x_0,\varepsilon}^-$ for every $\varepsilon > 0$.

Proof: 1) Let $E \subset U$ be a measurable set such that $x_0 \in E$, $\underline{d}(E, x_0) > \frac{1}{2}$ and $f|_E$ is continuous at x_0 . Fix $\varepsilon > 0$. There exists $\delta > 0$ such that $E \cap [x_0 - \delta, x_0 + \delta] \subset E_{f,x_0,\varepsilon}$. Hence

$$\underline{d}(E_{f,x_0,\varepsilon}, x_0) \geq \underline{d}(E, x_0) > \frac{1}{2}.$$

2) proof of this fact is similar to the proof of 1) and we omit it.

3) Let E_1, E_2 be measurable subsets of U such that $x_0 \in E_1 \cap E_2$, $\underline{d}(E_1, x_0) > \frac{1}{2}$, $\underline{d}(E_2, x_0) > \frac{1}{2}$, $f|_{E_1}$ is upper semicontinuous at x_0 and $f|_{E_2}$ is lower semicontinuous at x_0 . Fix $\varepsilon > 0$. There exists $\delta > 0$ such that $E_1 \cap [x_0 - \delta, x_0 + \delta] \subset E_{f,x_0,\varepsilon}^+$ and $E_2 \cap [x_0 - \delta, x_0 + \delta] \subset E_{f,x_0,\varepsilon}^-$. Hence

$$\underline{d}(E_{f,x_0,\varepsilon}^+, x_0) \geq \underline{d}(E_1, x_0) > \frac{1}{2}$$

and

$$\underline{d}(E_{f,x_0,\varepsilon}^-, x_0) \geq \underline{d}(E_2, x_0) > \frac{1}{2}.$$

4) proof of this fact is similar to the proof of 3) and we omit it. □

Example 1. We can construct a sequence of closed intervals $\{I_n = [a_n, b_n] : n \geq 1\}$ such that $0 < a_{n+1} < b_{n+1} < a_n$, $n \in \mathbb{N}$ and

$$d^+\left(\bigcup_{n=1}^{\infty} I_{3n}, 0\right) = d^+\left(\bigcup_{n=0}^{\infty} I_{3n+1}, 0\right) = d^+\left(\bigcup_{n=0}^{\infty} I_{3n+2}, 0\right) = \frac{1}{3}.$$

Define a function $f: \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \bigcup_{n=1}^{\infty} I_{3n}, \\ 0 & \text{if } x \in \bigcup_{n=0}^{\infty} I_{3n+1} \cup (-\infty, 0] \cup [b_1, \infty), \\ -1 & \text{if } x \in \bigcup_{n=0}^{\infty} I_{3n+2}, \\ \text{linear on the intervals of the set } [0, a_1) \setminus \bigcup_{n=1}^{\infty} I_n. \end{cases}$$

The function f is continuous at every point except the point 0. Let

$$E_1 = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} I_{3n+1} \cup \bigcup_{n=1}^{\infty} I_{3n}, \quad E_2 = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} I_{3n+1} \cup \bigcup_{n=1}^{\infty} I_{3n+2}.$$

Then $\underline{d}(E_1, 0) = \frac{2}{3}$, $\underline{d}(E_2, 0) = \frac{2}{3}$, $f|_{E_1}$ is lower semi-continuous and $f|_{E_2}$ is upper semi-continuous. Hence $f \in \mathcal{GPO}$. On the other hand,

$$\underline{d}^+(\{x: |f(x)| < 1\}, 0) \leq \underline{d}^+\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty} (I_{3n} \cup I_{3n+2}), 0\right) = \frac{1}{3} < \frac{1}{2}.$$

Thus $f \notin \mathcal{PO}$.

Corollary 7.

$$\mathcal{GPO} \not\subset \mathcal{PO} \quad \text{and} \quad \mathcal{GPD} \not\subset \mathcal{PD}.$$

Example 2. Let $I_n = [3^{-n} - 8^{-n}, 2 \cdot 3^{-n} + 8^{-n}]$, $J_n = [2 \cdot 3^{-n} + 6^{-n}, 3^{-n+1} - 6^{-n}]$ for $n = 1, 2, \dots$. For every $n \geq 1$ we have

$$\begin{aligned} \frac{\lambda(\bigcup_{i=1}^{\infty} I_i \cap [0, 3^{-n} - 8^{-n}])}{3^{-n} - 8^{-n}} &= \frac{\lambda(\bigcup_{i=n+1}^{\infty} I_i \cap [0, 3^{-i} - 8^{-i}])}{3^{-n} - 8^{-n}} = \\ &= \frac{\sum_{i=n+1}^{\infty} (3^{-i} + 2 \cdot 8^{-i})}{3^{-n} - 8^{-n}} = \frac{\frac{3}{2}3^{-n-1} + \frac{16}{7}8^{-n-1}}{3^{-n} - 8^{-n}} = \frac{1}{2} \cdot \frac{3^{-n} + \frac{4}{7}8^{-n}}{3^{-n} - 8^{-n}} > \frac{1}{2}. \end{aligned}$$

By Corollary 6, 0 is the point of preponderant density of the set $(-\infty, 0] \cup \bigcup_{n=1}^{\infty} I_n$. On the other hand, $\bigcup_{n=1}^{\infty} I_n \cap \bigcup_{n=1}^{\infty} J_n = \emptyset$ and

$$\begin{aligned} \frac{\lambda(\bigcup_{i=1}^{\infty} J_i \cap [0, 3^{-n+1} - 6^{-n}])}{3^{-n+1} - 6^{-n}} &= \frac{\lambda(\bigcup_{i=n}^{\infty} J_i \cap [0, 3^{-n+1} - 6^{-n}])}{3^{-n+1} - 6^{-n}} = \\ &= \frac{\sum_{i=n}^{\infty} (3^{-i} - 2 \cdot 6^{-i})}{3^{-n+1} - 6^{-n}} = \frac{\frac{3}{2}3^{-n} - \frac{12}{5}6^{-n}}{3^{-n+1} - 6^{-n}} = \frac{1}{2} \cdot \frac{3^{-n+1} - \frac{24}{5}6^{-n}}{3^{-n+1} - 6^{-n}} \end{aligned}$$

for every $n \geq 1$. Hence $\bar{d}^+(\bigcup_{n=1}^{\infty} J_n, 0) = \frac{1}{2}$. Define a function $f: \mathbb{R} \rightarrow \mathbb{R}$,

$$f(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0] \cup [1, \infty) \cup \bigcup_{n=1}^{\infty} I_n, \\ 1 & \text{for } x \in \bigcup_{n=1}^{\infty} J_n, \\ \text{linear on the intervals } [2 \cdot 3^{-n} + 8^{-n}, 2 \cdot 3^{-n} + 6^{-n}] \\ \text{and } [3^{-n} - 6^{-n}, 3^{-n} - 8^{-n}], & n = 1, 2, \dots \end{cases}$$

Certainly, f is continuous at every point except the point 0. Since $f(x) = 0$ for each $x \in (-\infty, 0] \cup \bigcup_{n=1}^{\infty} I_n$, we conclude that f is preponderantly

continuous in O'Malley sense at 0. Hence $f \in \mathcal{PO} \cap \mathcal{GPO}$. On the other hand, $\bigcup_{n=1}^{\infty} J_n \subset \{x \in \mathbb{R} : f(x) > \frac{1}{2}\}$. Thus

$$\underline{d}(\{x \in \mathbb{R} : f(x) < f(0) + \frac{1}{2}\}, 0) \leq 1 - \bar{d}\left(\bigcup_{n=1}^{\infty} J_n, 0\right) = \frac{1}{2}.$$

Therefore, f does not have the A_1 property in Denjoy sense at 0 and $f \notin \mathcal{PD} \cup \mathcal{GPD}$.

Corollary 8.

$$\mathcal{PO} \not\subset \mathcal{PD} \quad \text{and} \quad \mathcal{GPO} \not\subset \mathcal{GPD}.$$

All proven inclusions between defined classes of functions are presented in the following diagram:

$$\begin{array}{ccc} \mathcal{PD} & \longrightarrow & \mathcal{PO} \\ \downarrow & & \downarrow \\ \mathcal{GPD} & \longrightarrow & \mathcal{GPO} \end{array}$$

We proved that no other inclusion holds.

Theorem 1. *If $f \in \mathcal{GPO}$ then f is the Baire class 1 function.*

Proof: (This proof is based on the proof of Theorem 1 in [5].)

Suppose that there exists $f \in \mathcal{GPO}$ which is not the Baire class 1 function. Then we can find a perfect set P such that $f|_P$ is discontinuous at every point. Let $\omega(f, P, x)$ be an oscillation of $f|_P$ at x and let

$$P_n = \{x \in P : \omega(f, P, x) \geq \frac{1}{n}\}$$

for $n \geq 1$. Then every set P_n is closed and $P = \bigcup_{n=1}^{\infty} P_n$. Since P is a complete space, there exist $k \in \mathbb{N}$ and a closed non-degenerate interval I_1 such that $\emptyset \neq P \cap \text{Int}(I_1)$ and $P_k \cap I_1 = P \cap I_2$. Next, let

$$P_k^m = \{x \in P \cap I_1 : f(x) \in [\frac{m}{3k}, \frac{m+1}{3k}]\}$$

for $m \in \mathbb{Z}$. Again, using completeness of the set $P \cap I_1$ we can find $l \in \mathbb{N}$ and a closed non-degenerate interval $I_2 \subset I_1$ for which P_k^l is a dense subset of $P \cap \text{Int}(I_2) \neq \emptyset$. Since $\omega(f, P, x) \geq \frac{1}{k}$ for each $x \in P_k^l \subset P_k$, we get that

$$\{x \in P \cap I_2 : f(x) \leq \frac{l-1}{3k}\} \cup \{x \in P \cap I_2 : f(x) \geq \frac{l+2}{3k}\}$$

is a dense subset of $P \cap I_2$. Hence there exists a closed non-degenerate interval $I_3 \subset I_2$ such that one of the components of this sum is dense in $P \cap \text{Int}(I_3) \neq \emptyset$. Since

$$P_k^l \cap I_3 \subset \{x \in P \cap I_3 : f(x) \leq \frac{l+1}{3k}\} \quad \text{and} \quad P_k^l \cap I_3 \subset \{x \in P \cap I_3 : f(x) \geq \frac{l}{3k}\},$$

we get that the sets $\{x \in P \cap I_3 : f(x) \leq \frac{l+1}{3k}\}$ and $\{x \in P \cap I_3 : f(x) \geq \frac{l}{3k}\}$ are dense in $P \cap I_3$.

Thus we have proved that there exist a perfect set $F = P \cap I_3$ and $\alpha, \beta \in \mathbb{R}$, ($\alpha = \frac{l-1}{3k}$ and $\beta = \frac{l}{3k}$ or $\alpha = \frac{l+1}{3k}$ and $\beta = \frac{l+2}{3k}$), $\alpha < \beta$ such that the sets $\{x : f(x) \geq \beta\}$ and $\{x : f(x) \leq \alpha\}$ are dense in F .

Let

$$B_1 = F \cap \{x : f(x) \leq \alpha\}, \quad B_2 = F \cap \{x : f(x) \geq \beta\},$$

$$B_3 = F \cap \left\{x : f(x) \leq \alpha + \frac{2(\beta-\alpha)}{3}\right\} \quad \text{and} \quad B_4 = F \cap \left\{x : f(x) \geq \alpha + \frac{\beta-\alpha}{3}\right\}.$$

Now, we use the assumption that $f \in \mathcal{GPO}$. Let $x \in B_i$, $i = 1, 2, 3, 4$ and let E_1, E_2 be the sets satisfying the conditions of Definition 8. Then there exists $r > 0$ such that

$$[x-r, x+r] \cap E_1 \subset B_i \quad \text{if} \quad i = 1, 3 \quad \text{and} \quad [x-r, x+r] \cap E_2 \subset B_i \quad \text{if} \quad i = 2, 4.$$

Thus for each $x \in B_i$, $i = 1, 2, 3, 4$ we can find a positive rational number $r_i(x)$ for which, if $0 < \delta < r_i(x)$, then $\lambda(B_i \cap [x, x + \delta]) > \frac{1}{2}\delta$ and $\lambda(B_i \cap [x - \delta, x]) > \frac{1}{2}\delta$. Let

$$D_i^r = \{x \in B_i : r_i(x) = r\} \quad \text{for} \quad i = 3, 4 \quad \text{and} \quad r \in \mathbb{Q}^+.$$

Since $B_3 \cup B_4 = F$, $B_3 = \bigcup_{r \in \mathbb{Q}^+} D_3^r$ and $B_4 = \bigcup_{r \in \mathbb{Q}^+} D_4^r$, we get that there exist $q \in \mathbb{Q}^+$, $j \in \{3, 4\}$ and a closed interval I_4 such that the set D_j^q is dense in $F \cap \text{Int}(I_4) \neq \emptyset$. Without loss of generality, we can assume that $j = 3$. Let $y \in B_2 \cap I_4$. Since the set B_2 is dense in F , such a point exists. Let $r = \min\{r_2(y), q\}$ and $z \in D_3^q \cap I_4 \cap [y - r, y + r]$. Then $\lambda(B_2 \cap [y, z]) > \frac{1}{2}\lambda([y, z])$, because $|z - y| < r_2(y)$, and $\lambda(B_3 \cap [y, z]) > \frac{1}{2}\lambda([y, z])$, because $|z - y| < r_3(z) = q$. Since $\lambda(B_2 \cap [y, z]) + \lambda(B_3 \cap [y, z]) > \lambda([y, z])$, we get that $B_2 \cap B_3 \neq \emptyset$. This contradicts the fact that $B_2 \cap B_3 = \emptyset$. The proof is complete. \square

Corollary 9.

$$\mathcal{PO} \subset \mathcal{B}_1, \quad \mathcal{GPD} \subset \mathcal{B}_1 \quad \text{and} \quad \mathcal{PD} \subset \mathcal{B}_1.$$

Now, we would like to show that inverse of the Proposition 2 is true. We start from the preponderant continuity and A_1 condition in Denjoy sense.

Lemma 1. *If F is a measurable subset of \mathbb{R} and $\underline{d}^+(F, x_0) = q > 0$, then:*

$$\forall n \in \mathbb{N} \exists \varepsilon > 0 \forall 0 < a < b < \varepsilon \quad \frac{\lambda\left(\left[x_0 + \frac{a}{2n}, x_0 + b\right] \cap F\right)}{\lambda([x_0, x_0 + b])} > q - \frac{1}{n}.$$

Proof: Take $n \in \mathbb{N}$. Since $\underline{d}^+(F, x_0) = q$, there exists $\varepsilon > 0$ such that

$$\frac{\lambda([x_0, x_0 + c] \cap F)}{\lambda([x_0, x_0 + c])} > q - \frac{1}{4n}$$

for each $0 < c < \varepsilon$. If $0 < a < b < \varepsilon$, then

$$\begin{aligned} \lambda\left(\left[x_0 + \frac{a}{2n}, x_0 + b\right] \cap F\right) &= \lambda([x_0, x_0 + b] \cap F) - \lambda\left(\left[x_0, x_0 + \frac{a}{2n}\right] \cap F\right) \geq \\ &\geq b\left(q - \frac{1}{4n}\right) - \frac{a}{2n} > bq - \frac{3b}{4n}. \end{aligned}$$

Therefore $\frac{\lambda\left(\left[x_0 + \frac{a}{2n}, x_0 + b\right] \cap F\right)}{\lambda([x_0, x_0 + b])} \geq \frac{bq - \frac{3b}{4n}}{b} > q - \frac{1}{n}$. Thus the proof is completed. \square

An analogous lemma holds for a left-sided neighbourhood of x .

Lemma 2. *Let $\{E_n : n \geq 1\}$ be a descending family of measurable sets, $x_0 \in \bigcap_{n=1}^{\infty} E_n$ and $\underline{d}(E_n, x_0) \geq q > 0$ for $n \in \mathbb{N}$. Then there exists a measurable set E such that $\underline{d}(E, x_0) \geq q$, $x_0 \in E$ and for every $n \in \mathbb{N}$ one can find $\delta_n > 0$ for which $E \cap [x_0 - \delta_n, x_0 + \delta_n] \subset E_n$.*

Proof: By assumptions, $\underline{d}^+(E_n, x_0) \geq q > 0$ for every $n \in \mathbb{N}$. Let ε_n fulfil Lemma 1 for $F = E_n$, $n \in \mathbb{N}$. By recursion, one can construct a sequence $\{a_n\}_{n \in \mathbb{N}}$ of positive reals such that $a_1 < \varepsilon_1$ and $a_{n+1} < \min\{\varepsilon_{n+1}, \frac{a_n}{2n}\}$ for $n \geq 1$. Let

$$H = \bigcup_{n=1}^{\infty} (E_n \cap [x_0 + a_{n+2}, x_0 + a_{n+1}]).$$

We shall show that $\underline{d}^+(H, x_0) \geq q$. Fix $n \in \mathbb{N}$ and take $c \in [a_{n+1}, a_n]$. Then

$$H \cap [x_0 + a_{n+2}, x_0 + c] \supset E_n \cap [x_0 + a_{n+2}, x_0 + c]$$

and applying Lemma 1, we get

$$\lambda\left(E_n \cap \left[x_0 + \frac{a_{n+1}}{2n}, x_0 + c\right]\right) > c\left(q - \frac{1}{n}\right).$$

Since $\frac{a_{n+1}}{2n} > a_{n+2}$, we get

$$\begin{aligned} \lambda(H \cap [x_0, x_0 + c]) &> \lambda\left(H \cap \left[x_0 + \frac{a_{n+1}}{2n}, x_0 + c\right]\right) \geq \\ &\geq \lambda\left(E_n \cap \left[x_0 + \frac{a_{n+1}}{2n}, x_0 + c\right]\right) > c \cdot \left(q - \frac{1}{n}\right). \end{aligned}$$

It implies that $\underline{d}^+(H, x_0) \geq q$. Exactly from the definition of the set H we get $H \cap [x_0, x_0 + a_{n+1}] \subset E_n$.

In a similar way, we can construct a measurable set $G \subset (-\infty, x_0)$ such that $\underline{d}^-(G, x_0) \geq q$ and for every $n \in \mathbb{N}$ there is $\delta_n > 0$ for which $G \cap [x_0 - \delta_n, x_0] \subset E_n$. The set $E = H \cup G \cup \{x_0\}$ has all required properties. \square

Using the last Lemma we can easily prove the following Theorem.

Theorem 2. (i) A measurable function $f: U \rightarrow \mathbb{R}$ is preponderantly continuous in Denjoy sense at $x_0 \in U$ iff $\lim_{n \rightarrow \infty} \underline{d}\left(E_{f, x_0, \frac{1}{n}}, x_0\right) > \frac{1}{2}$,

(ii) A measurable function $f: U \rightarrow \mathbb{R}$ has A_1 property in Denjoy sense at $x_0 \in U$ iff

$$\lim_{n \rightarrow \infty} \underline{d}\left(E_{f, x_0, \frac{1}{n}}^+, x_0\right) > \frac{1}{2} \quad \text{and} \quad \lim_{n \rightarrow \infty} \underline{d}\left(E_{f, x_0, \frac{1}{n}}^-, x_0\right) > \frac{1}{2}.$$

Proof: (i) If f is preponderantly continuous in Denjoy sense at x_0 , then there exists a measurable set E containing x_0 such that $f|_E$ is continuous at x_0 and $\underline{d}(E, x_0) = q > \frac{1}{2}$. Then we get

$$\lim_{n \rightarrow \infty} \underline{d}\left(E_{f, x_0, \frac{1}{n}}, x_0\right) \geq \underline{d}(E, x_0) = q > \frac{1}{2}.$$

Now, suppose that $\lim_{n \rightarrow \infty} \underline{d}\left(E_{f, x_0, \frac{1}{n}}, x_0\right) = q > \frac{1}{2}$. Then $\underline{d}\left(E_{f, x_0, \frac{1}{n}}, x_0\right) \geq q > \frac{1}{2}$ for every $n \in \mathbb{N}$ and $x_0 \in \bigcap_{n=1}^{\infty} E_{f, x_0, \frac{1}{n}}$. Applying Lemma 2 with $E_n = E_{f, x_0, \frac{1}{n}}$ for $n \in \mathbb{N}$, one can construct a measurable set E such that $x_0 \in E$, $\underline{d}(E, x_0) > \frac{1}{2}$ and for every n there exists $\delta_n > 0$ for which $E \cap [x_0 - \delta_n, x_0 + \delta_n] \subset E_n$. The last condition implies that $f|_E$ is continuous at x_0 . Hence f is preponderantly continuous in Denjoy sense at x_0 . (ii) the proof is similar and we omit it. \square

Now, we show equivalent conditions for preponderant continuity and A_1 property in O'Malley sense.

Lemma 3. If $F \subset \mathbb{R}$ is a measurable set and x_0 is a point of preponderant density in O'Malley sense of the set F , then:

$$\exists_{\varepsilon > 0} \forall_{0 < b < a < \varepsilon} \exists_{0 < c < b} \forall_{b < \eta < a} \frac{\lambda([x_0 + c, x_0 + \eta] \cap F)}{\lambda([x_0, x_0 + \eta])} > \frac{1}{2}.$$

Proof: There exists $\varepsilon > 0$ such that $\lambda([x_0, x_0 + a] \cap F) > \frac{1}{2} \cdot a$ for any $0 < a \leq \varepsilon$. Let us take any $a, b \in U$, $0 < b < a < \varepsilon$ and define a function $f: [b, a] \rightarrow \mathbb{R}$,

$$f(t) = \lambda([x_0, x_0 + t] \cap F) - \frac{t}{2}.$$

Since the function f is continuous and $f(t) > 0$ for any $t \in [b, a]$, we conclude that $c = \frac{1}{2} \cdot \min\{f(t) : t \in [b, a]\} > 0$. Let $b < \eta < a$. Then $\lambda(F \cap [x_0, x_0 + \eta]) = f(\eta) + \frac{\eta}{2}$ and therefore

$$\begin{aligned} \lambda(F \cap [x_0 + c, x_0 + \eta]) &= \lambda(F \cap [x_0, x_0 + \eta]) - \lambda(F \cap [x_0, x_0 + c]) \geq \\ &\geq f(\eta) + \frac{\eta}{2} - c \geq 2c + \frac{\eta}{2} - c > \frac{\eta}{2}. \end{aligned}$$

Thus the proof is completed. \square

An analogous lemma holds for a left-sided neighbourhood of x_0 .

Lemma 4. *Let $\{E_n : n \geq 1\}$ be a descending family of measurable subsets of the real line, $x_0 \in \bigcap_{n=1}^{\infty} E_n$ and x_0 is a point of preponderant density in O'Malley sense of every set E_n for $n \in \mathbb{N}$. Then there exists a measurable set E such that x_0 is the point of preponderant density in O'Malley sense of the set E , $x_0 \in E$ and for every $n \in \mathbb{N}$ we can find $\delta_n > 0$ for which $E \cap [x_0 - \delta_n, x_0 + \delta_n] \subset E_n$.*

Proof: Let ε_n fulfil Lemma 3 with $F = E_n$ for $n \geq 1$. We may assume that the sequence $\{\varepsilon_n : n \geq 1\}$ is strictly decreasing. We shall construct a decreasing sequence of positive reals $\{a_n : n \geq 1\}$. Let a_1 be any real number belonging to an open interval $(0, \varepsilon_1)$ and let a_2 be any real number belonging to an open interval $(0, a_1)$. If numbers a_1, \dots, a_n , $0 < a_n < \dots < a_1$ are chosen and c fulfils Lemma 3 with $F = E_{n-1}$, $\varepsilon = \varepsilon_{n-1}$, $a = a_{n-1}$ and $b = a_n$, then we take $a_{n+1} = \min\{c, \frac{\varepsilon_{n+1}}{2}\}$. Let

$$H = \bigcup_{n=1}^{\infty} (E_n \cap [x_0 + a_{n+2}, x_0 + a_{n+1}]).$$

It follows directly from the definition of the set H that $H \cap [x_0, x_0 + a_{n+1}] \subset E_n$ and

$$E_n \cap [x_0 + a_{n+2}, x_0 + z] \subset H \cap [x_0, x_0 + z] \subset E_n$$

for any $n \in \mathbb{N}$ and $z \in [a_{n+2}, a_{n+1}]$. Hence, if $z \in [a_{n+2}, a_{n+1}]$, then applying Lemma 3, we get

$$\lambda(H \cap [x_0, x_0 + z]) \geq \lambda(E_n \cap [x_0 + a_{n+2}, x_0 + z]) > \frac{z}{2}.$$

In a similar way, we can construct a measurable set $G \subset (-\infty, x_0)$ such that x_0 is the point of preponderant density in O'Malley's sense of the set G and for every $n \in \mathbb{N}$ we can find $\delta_n > 0$ for which $G \cap [x_0 - \delta_n, x_0] \subset E_n$. The set $E = H \cup G \cup \{x_0\}$ has all required properties. \square

Using the last Lemma we can prove the following Theorem.

Theorem 3. (i) A measurable function $f: U \rightarrow \mathbb{R}$ is preponderantly continuous in O'Malley sense at $x_0 \in U$ iff x_0 is the point of preponderant density in O'Malley sense of the set $E_{f,x_0,\varepsilon}$ for each $\varepsilon > 0$,

(ii) A measurable function $f: U \rightarrow \mathbb{R}$ has A_1 property in O'Malley sense at a point $x_0 \in U$ iff x_0 is a point of preponderant density in O'Malley sense of both sets $E_{f,x_0,\varepsilon}^+$ and $E_{f,x_0,\varepsilon}^-$ for each $\varepsilon > 0$.

Proof: (i) If f is preponderantly continuous in O'Malley sense at x_0 , then there exists a measurable set E containing x_0 such that $f|_E$ is continuous at x_0 and x_0 is the point of preponderant density of the set E . Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$E \cap [x_0 - \delta, x_0 + \delta] \subset \{y: |f(x) - f(y)| < \varepsilon\} \cap [x_0 - \delta, x_0 + \delta].$$

Hence x_0 is the point of preponderant density of every set $E_{f,x_0,\varepsilon}$ for $\varepsilon > 0$.

Now, suppose that x_0 is the point of preponderant density of every set $E_{f,x_0,\varepsilon}$ for each $\varepsilon > 0$. Applying Lemma 4 with $E_n = E_{f,x_0,\frac{1}{n}}$ for $n \geq 1$, we can construct a measurable set E such that $x_0 \in E$, x_0 is the point of preponderant density of the set E and for every $n \in \mathbb{N}$ there exists $\delta_n > 0$ for which $E \cap [x_0 - \delta_n, x_0 + \delta_n] \subset E_n$. The last condition implies that $f|_E$ is continuous at x_0 . Thus f is preponderantly continuous in O'Malley sense at x_0 .

(ii) the proof is analogous and we omit it. □

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