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Scientific Issues, Mathematics XIV, Częstochowa 2009

ANALYSIS OF FUNDAMENTAL SOLUTIONS TO FRACTIONAL DIFFUSION-WAVE EQUATION IN POLAR COORDINATES

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Abstract
The diffusion-wave equation is a mathematical model of a wide range of important physical phenomena. The first and second Cauchy problems and the source problem for the diffusion-wave equation are considered in polar coordinates. The Caputo fractional derivative is used. The Laplace and Hankel transforms are employed. The results are illustrated graphically.

1. Introduction

In recent years considerable interest has been shown in time-fractional diffusion-wave equation which is a mathematical model of a wide range of important physical phenomena [1–5].

The fundamental solution for the fractional diffusion-wave equation in one Cartesian space-dimension was obtained by Mainardi [6] using the Laplace transform. Wyss [7] obtained the solution of the Cauchy problem in terms of H-functions using the Mellin transform. Schneider and Wyss [8] converted the diffusion-wave equation with appropriate initial conditions into the integro-differential equation and found the corresponding Green functions in terms of Fox functions. Hanyga [9] studied Green's functions and propagator functions in one, two and three dimensions. This paper completes the results obtained in [10, 11]. The Laplace and Hankel transforms are employed to reduce the considered equation to an ordinary algebraic equation. The inverse Laplace transform is expressed in terms of Mittag-Leffler type functions. Inversion of Hankel transform leads to integral representation of the solution.

2. The first Cauchy problem

Consider the Cauchy problem for the diffusion-wave equation with the Caputo time-fractional derivative [12] and the delta-function initial value of a sought-for function:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \quad 0 < \alpha \leqslant 2, \tag{1}$$

$$t = 0: \quad u = p\,\frac{\delta_+(r)}{2\pi r}, \quad 0 < \alpha \leqslant 2, \tag{2}$$

$$t = 0: \quad \frac{\partial u}{\partial t} = 0, \quad 1 < \alpha \leqslant 2. \tag{3}$$

As usually, we impose the zero condition at at infinity: $\lim_{r\to\infty} u(r,t) = 0$.

Using the Laplace transform with respect to time t and the Hankel transform with respect to the spatial coordinate r, we obtain

$$u^* = \frac{p}{2\pi}\,\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}, \tag{4}$$

where the asterisk denotes the transforms, or, after invertion of integral transforms,

$$u = \frac{p}{2\pi}\int_0^\infty E_\alpha(-a\xi^2 t^\alpha)\,J_0(r\xi)\,\xi\,d\xi. \tag{5}$$

The similarity variable $\bar{r}$, new integration variable η and nondimensional solution $\bar{u}$ are defined as

$$\bar{r} = \frac{r}{\sqrt{a}t^{\alpha/2}}, \qquad \eta = \sqrt{a}t^{\alpha/2}\xi, \qquad \bar{u} = \frac{at^\alpha u}{p}. \tag{6}$$

Hence,

$$\bar{u} = \frac{1}{2\pi}\int_0^\infty E_\alpha(-\eta^2)\,J_0(\bar{r}\eta)\,\eta\,d\eta. \tag{7}$$

Now we consider several particular cases of the obtained solution.

For the Helmholtz equation ($\alpha \to 0$) we have

$$\bar{u} = \frac{1}{2\pi}\,K_0(\bar{r}). \tag{8}$$

Here $K_0(r)$ is the modified Bessel function of the second kind.

In the case of the classical diffusion equation ($\alpha = 1$) we get

$$\bar{u} = \frac{1}{4\pi}\,\mathrm{e}^{-\bar{r}/4}. \tag{9}$$

For the wave equation ($\alpha = 2$) we obtain

$$u = \frac{p}{2\pi\sqrt{a}} \frac{\partial}{\partial t} \frac{\theta(\sqrt{a}t - r)}{\sqrt{at^2 - r^2}}, \tag{10}$$

where $\theta(x)$ is the Heaviside step function (see also [13]).

The subdiffusion with $\alpha = 1/2$ leads to

$$\bar{u} = \frac{1}{4\pi^{3/2}} \int_0^\infty \frac{1}{y} \exp\left(-y^2 - \frac{\bar{r}^2}{8y}\right) dy. \tag{11}$$

Now we analyze the behavior of the solution at the origin. As we have

$$L^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}\right\} \sim \frac{1}{\Gamma(1-\alpha)a\xi^2 t^\alpha} \qquad \text{for } \xi \to \infty, \quad 0 < \alpha < 2, \tag{12}$$

only the fundamental solution to the classical diffusion equation ($\alpha = 1$) has no singularity at the origin. To investigate the type of singularity we rewrite Eq. (7) in the following form

$$\bar{u} = \frac{1}{2\pi} \int_0^\infty \left[E_\alpha(-\eta^2) - \frac{1}{\Gamma(1-\alpha)(1+\eta^2)}\right] J_0(\bar{r}\eta)\, \eta\, d\eta$$

$$+\frac{1}{2\pi\Gamma(1-\alpha)} \int_0^\infty \frac{1}{1+\eta^2} J_0(\bar{r}\eta)\, \eta\, d\eta. \tag{13}$$

The first integral in (13) has no singularity at the origin, while the second one can be calculated analytically and yields the logarithmic singularity at the origin

$$\bar{u} \sim \frac{1}{2\pi\Gamma(1-\alpha)} K_0(\bar{r}), \qquad 0 < \alpha < 2 \tag{14}$$

or

$$\bar{u} \sim -\frac{1}{2\pi\Gamma(1-\alpha)} \ln \bar{r}. \tag{15}$$

Comparison of (14) and (8) allows us to substitute the condition $0 < \alpha < 2$ by $0 \leqslant \alpha < 2$.

Equation (15) rewritten in terms of (dimensional) solution u (see (6))

$$u \sim -\frac{p}{2\pi a t^\alpha \Gamma(1-\alpha)} \ln r \tag{16}$$

is consistent with the behavior of the solution for small r obtained in [14].

Dependence of nondimensional solution $\bar{u}$ on nondimensional distance is shown in Fig. 1.

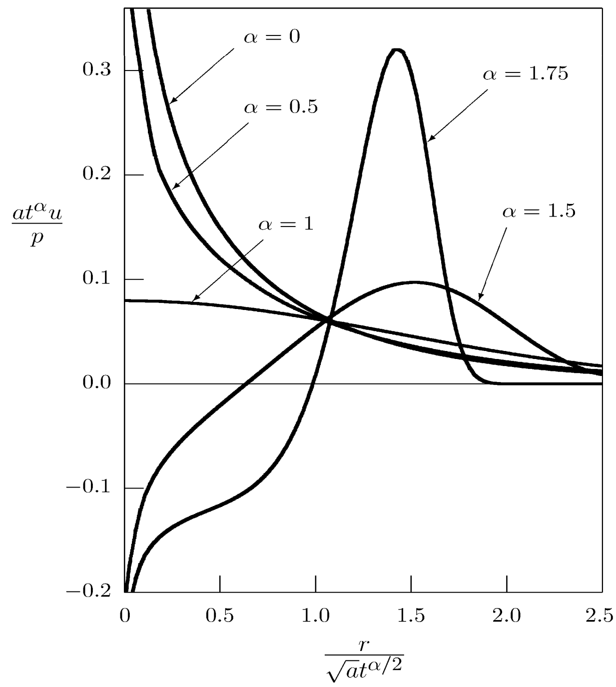

Fig. 1. Dependence of solution on the similarity variable (the first Cauchy problem with the delta pulse initial condition for the function).

3. The second Cauchy problem

Consider the Cauchy problem for the diffusion-wave equation with the delta-function initial value of the time-derivative of a sought-for function:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r}\right), \quad 1 < \alpha \leqslant 2, \tag{17}$$

$$t = 0: \quad u = 0, \quad 1 < \alpha \leqslant 2, \tag{18}$$

$$t = 0: \quad \frac{\partial u}{\partial t} = w\,\frac{\delta_{+}(r)}{2\pi r}, \quad 1 < \alpha \leqslant 2. \tag{19}$$

The solution reads

$$u^{*} = \frac{w}{2\pi}\,\frac{s^{\alpha-2}}{s^{\alpha} + a\xi^{2}} \tag{20}$$

and

$$\bar{u} = \frac{1}{2\pi}\int_0^{\infty} E_{\alpha}(-\eta^2)\, J_0(\bar{r}\eta)\,\eta\, d\eta. \tag{21}$$

where $\bar{u} = at^{\alpha-1}u/w$.

In the case of the wave equation ($\alpha = 2$)

$$\bar{u} = \begin{cases} \dfrac{1}{2\pi\sqrt{1-\bar{r}^2}}, & 0 < \bar{r} < 1, \\ 0, & 1 < \bar{r} < \infty. \end{cases} \tag{22}$$

To study the behavior of solution at the origin we observe that for large values of ξ

$$L^{-1}\left\{\frac{s^{\alpha-2}}{s^\alpha + a\xi^2}\right\} \sim \frac{1}{\Gamma(2-\alpha)a\xi^2 t^{\alpha-1}} \qquad \text{for } \xi \to \infty, \quad 1 < \alpha < 2. \tag{23}$$

Computations similar to those carried out above lead to

$$\bar{u} \sim \frac{1}{2\pi\Gamma(2-\alpha)}\, K_0(\bar{r}), \qquad 1 < \alpha < 2 \tag{24}$$

or

$$\bar{u} \sim -\frac{1}{2\pi\Gamma(2-\alpha)} \ln \bar{r}, \qquad 1 < \alpha < 2. \tag{25}$$

Hence, in the case of the second Cauchy problem with delta pulse initial condition the solution also has the logarithmic singularity at origin.

Dependence of nondimensional solution $\bar{u}$ on the similarity variable (nondimensional distance) is depicted in Fig. 2.

4. The source problem

Next we analyze the source problem with zero initial conditions

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + q\,\delta_+(t)\,\frac{\delta_+(r)}{2\pi r}, \quad 0 < \alpha \leqslant 2, \tag{26}$$

$$t = 0: \quad u = 0, \quad 0 < \alpha \leqslant 2, \tag{27}$$

$$t = 0: \quad \frac{\partial u}{\partial t} = 0, \quad 1 < \alpha \leqslant 2. \tag{28}$$

The solution has the form

$$u^* = \frac{q}{2\pi}\,\frac{1}{s^\alpha + a\xi^2}, \tag{29}$$

and

$$\bar{u} = \frac{1}{2\pi}\int_0^\infty E_{\alpha,\alpha}(-\eta^2)\, J_0(\bar{r}\eta)\,\eta\, d\eta. \tag{30}$$

Here $\bar{u} = atu/q$.

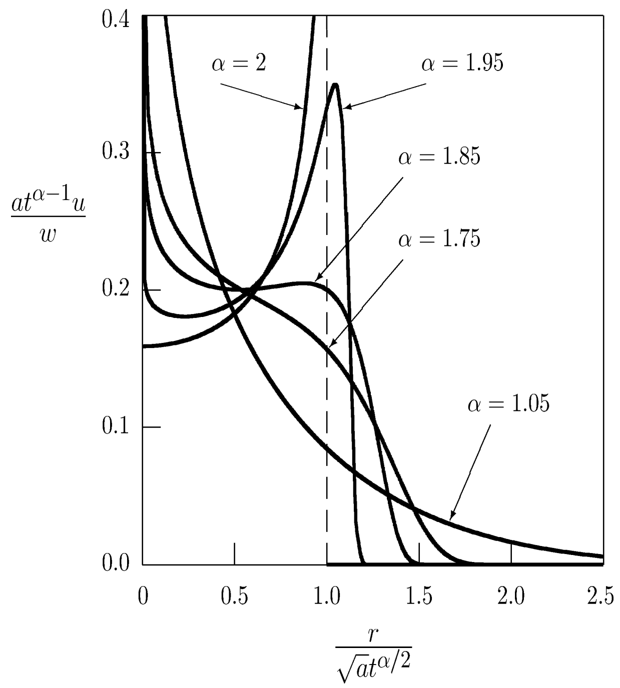

Fig. 2. Dependence of solution on the similarity variable (the second Cauchy problem with the delta pulse initial condition for the time derivative of a function).

Now let us discuss several particular cases of the obtained solution. In the case of the source problem the solution to the classical diffusion equation coincides with the corresponding solution to the first Cauchy problem (9). In the case of the source problem the solution to the wave equation coincides with the corresponding solution to the second Cauchy problem (22). For subdiffusion with $\alpha = 1/2$ we have

$$\bar{u} = \frac{1}{4\pi^{3/2}} \int_0^\infty \exp\left(-y^2 - \frac{\bar{r}^2}{8y}\right) dy. \tag{31}$$

It should be noted that

$$L^{-1}\left\{\frac{1}{s^\alpha + a\xi^2}\right\} \sim -\frac{1}{\Gamma(-\alpha)a^2\xi^4 t^{\alpha+1}} \qquad \text{for } \xi \to \infty, \quad 0 < \alpha < 2. \tag{32}$$

Hence, the solution has no singularity at the origin for all $0 < \alpha < 2$.

Dependence of nondimensional solution $\bar{u}$ on the similarity variable is depicted in Fig. 3. It should be emphasized that solution (22) the same both

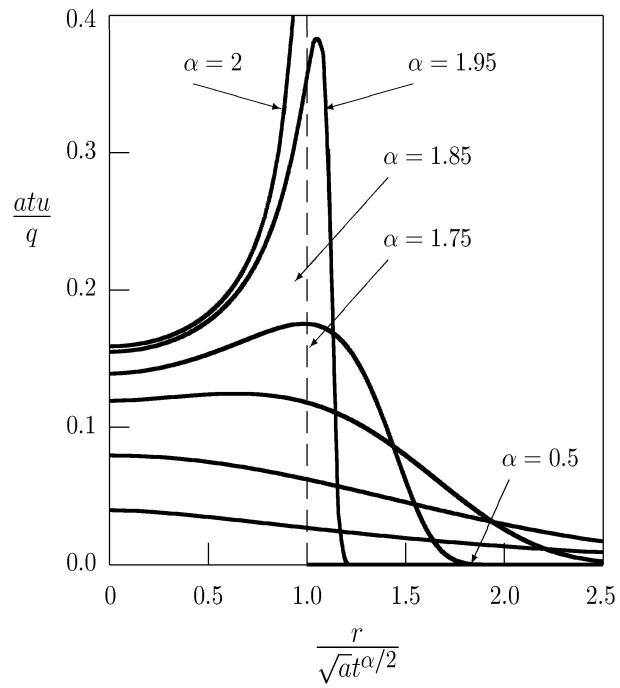

Fig. 3. Dependence of solution on the similarity variable (the delta pulse source problem with zero initial conditions).

for the second Cauchy problem and the source problem is approximated by solutions (21) and (30) with $\alpha \to 2$ in different ways, in particular the solution (21) has the ligarithmic singulariry at the origin, whereas the solution (30) has no singulariry.

References

[1] A. Pękalski, K. Sznajd-Weron (Eds.) *Anomalous Diffusion: From Basics to Applications*. Springer, Berlin 1999.

[2] R. Hilfer (Ed.) *Applications of Fractional Calculus in Physics*. World Scientific, Singapore 2000.

[3] R. Metzler, J. Klafter. The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.*, **339**, 1–77, 2000.

[4] G.M. Zaslavsky. Chaos, fractional kinetics, and anomalous transport. *Phys. Rep.*, **371**, 461–580, 2002.

[5] R. Metzler, J. Klafter. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *J. Phys. A: Math. Gen.*, **37**, R161–R208, 2004.

[6] F. Mainardi. The fundamental solutions for the fractional diffusion-wave equation. *Appl. Math. Lett.*, **9**, 23–28, 1996.

[7] W. Wyss. The fractional diffusion equation. *J. Math. Phys.*, **27**, 2782–2785, 1986.

[8] W.R. Schneider, W. Wyss. Fractional diffusion and wave equations. *J. Math. Phys.*, **30**, 134–144, 1989.

[9] A. Hanyga. Multidimensional solutions of time-fractional diffusion-wave equations. *Proc. R. Soc. Lond. A*, **458**, 933–957, 2002.

[10] Y.Z. Povstenko. Fractional heat conduction equation and associated thermal stress. *J. Thermal Stresses*, **28**, 83–102, 2005.

[11] Y.Z. Povstenko. Stresses exerted by a source of diffusion in a case of a non-parabolic diffusion equation. *Int. J. Eng. Sci.*, **43**, 977–991, 2005.

[12] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. *Theory and Applications of Fractional Differential Equations.* Elsevier, Amsterdam 2006.

[13] P.K. Kythe. *Fundamental Solutions for Differential Operators and Applications.* Birkhäuser, Boston 1996.

[14] W.R. Schneider, Fractional diffusion, in R. Lima, L. Streit and R. Viela Mendes (Eds.), *Dynamics and Stochastic Processes*, *Lecture Notes in Physics*, vol. 355, pp. 276–286, Springer, Berlin 1990.